Enumeration of unlabelled chordal graphs with bounded tree-width Discrete Mathematics Days 2024

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1 Introduction

For $k \geq 1$, a k-tree is defined recursively as either a complete graph on $k+1$ vertices or a graph obtained by adding a new vertex incident to a k-clique of a smaller k-tree. The class of k-trees plays an important rôle in graph theory as it allows for an alternative definition of the *tree-width* of a graph g as the minimum k such that g is a subgraph of a k-tree. In particular, k-trees are the maximal graphs with tree-width at most k. Tree-width is stable under taking minors, thus from [11] the number of graphs with n vertices and bounded tree-width grows like ρ^n, for some $\rho > 1$, up to some lower order terms and a factor $n!$ in the case of labelled graphs. However, determining the value of ρ is a notorious open problem.

An approach for the enumeration of constrained classes of graphs admitting some recursive decomposition is to derive a functional equation satisfied by the associated generating function, then compute asymptotic estimates of its coefficients using methods from complex analysis [8]. A natural operation to decompose graphs with bounded tree-width is known as the *clique-sum* of two graphs and consists in distinguishing a clique of the same size in each of the graphs and identifying together the vertices of the two cliques to obtain one unique graph, then removing any subset of the edges of the new clique. This latter step makes the clique-sum operation intractable in the setting of [8]. Note, however, that this new clique becomes a separator of the resulting graph.

A graph is chordal if every cycle of length greater than three contains at least one chord. Alternatively, Dirac proved in [6] that a graph is chordal if and only if every minimal separator is a clique. This characterisation makes the clique-sum operation amenable to recursive methods by restricting it to chordal graphs. Wormald first used it in [14] to obtain the generating function of labelled chordal graphs from a recursive system of equations; based on the fact that k-connected chordal graphs can be uniquely decomposed into their $(k+1)$-connected components, and by rooting the k-connected ones at k-cliques it is possible to derive an equation defining the generating function of k-connected graphs in terms of that of the $(k+1)$-connected ones. If we now consider chordal graphs with tree-width bounded by some $t \geq 1$, then one obtains a finite system of equations from the connected to the t-connected level, which is in fact composed of the class of t-trees. Thus one can see chordal graphs with tree-width at most t as a natural generalisation of t-trees. This work was continued in [3] to obtain an estimate for the number labelled chordal graphs with tree-width at most $t \geq 1$ and n vertices of the form

$$c\, n^{-5/2}\, \gamma^n \qquad \text{as } n \to \infty. \tag{1}$$

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for some $c > 0$ and $\gamma > 1$ depending on t and computable numerically up to any precision.

In his seminal work [13], Pólya developed a theory to encode symmetries of labelled combinatorial structures, and thus opened the way to enumerate their unlabelled counterparts. Using Pólya's theory, Otter [12] was able to enumerate unlabelled trees from the rooted ones. His method was generalised in what is known as the *dissymmetry theorem for tree-decomposable structures* [1]. Building on that theory, Gainer-Dewar managed to derive a system of equations from which the ordinary generating function of unlabelled k-trees can be computed [9]. An alternative derivation was later designed in [10], and was instrumental in obtaining in [7] an asymptotic estimate for the number of unlabelled k-trees with n vertices in the form of (1).

In this work, we generalise [9, 10] and derive a system of equations defining the ordinary generating function of unlabelled chordal graphs with bounded tree-width. Our method is based on the decomposition into $(k+1)$-connected components of chordal graphs rooted at k-cliques, similarly to [14] and [3], and requires a non-trivial extension of Pólya theory to rooted structures started in [2]. This decomposition is tree-like in the sense of [1], and we can apply a dissymmetry theorem to "unroot" our graphs analogous to the tools developed in [2].

Theorem 1. *Let $t \geq 1$ and $k \in [t]$. Then the class of unlabelled k-connected chordal graphs with tree-width at most t can be derived from a grammar. Furthermore, this grammar can be translated into a finite system of equations that completely defines the associated ordinary generating function.*

The derivation from Theorem1 implies an efficient algorithm to compute the number of unlabelled k-connected chordal graphs with tree-width at most t and n vertices. In a long version of this work we intend to prove an asymptotic estimate in the form of (1), following [7] and [3]. Using our grammar, one can also derive structural results on large random graphs in the class, similarly to what was done in [3] and [5], as well as design a Boltzmann sampler to generate large uniform random graphs.

2 An extension of Pólya theory

2.1 Extended cycle index sums

Let $\mathcal{A}$ be a class of labelled graphs. A *symmetry* of $\mathcal{A}$ is a tuple (a, σ) where $\mathsf{a} \in \mathcal{A}$ and σ is an automorphism of a. The set of all symmetries of $\mathcal{A}$ is denoted by $\mathcal{S}(\mathcal{A})$. For $(\mathsf{a}, \sigma) \in \mathcal{S}(\mathcal{A})$ and a k-clique $K = \{v_1, \dots, v_i\}$ of a, with $k \geq 1$, note that there exists a smallest positive integer j such that $\sigma^j(K) = K$. We then say that the k-cliques $K, \sigma(K), \dots, \sigma^{j-1}(K)$ form a cycle of length j. Note that σ^j restricted to the vertices of K may be different from the identity. And we say that the cycle of k-cliques has type μ, if $\mu \vdash k$ is the cycle structure of the permutation σ^j restricted to the vertices of K. If we now let $c_{\mu,j}(\mathsf{a}, \sigma)$ be the number of cycles of k-cliques of a of length j and type $\mu \vdash k$, then the *extended weight-monomial* of a symmetry (a, σ) of size n is defined as

$$w_{(\mathsf{a},\sigma)} := \frac{1}{n!} \prod_{k=1}^{n} \prod_{\mu \vdash k} \prod_{j \geq 1} s_{\mu,j}^{c_{\mu,j}(\mathsf{a},\sigma)}.$$

From there, we define the *extended cycle index sum* of $\mathcal{A}$ as the sum of extended weight-monomials of symmetries of $\mathcal{A}$

$$X_{\mathcal{A}} := \sum_{(\mathsf{a},\sigma) \in \mathcal{S}(\mathcal{A})} w_{(A,\sigma)}.$$

It is a formal power series and a refinement of the (classical) cycle index sum, as the latter can be recovered setting $s_{\lambda,j} = 1$, for all $\lambda \vdash k > 1$, and $s_{(1),j} = s_j$. In order to recover the (ordinary) generating function of an unlabelled class from its cycle index sum, we recall Pólya's classical result.

Proposition 2 (Pólya [13]). *Let $\mathcal{A}$ be a class of labelled graphs and $\mathcal{U}$ be the class obtained by unlabelling the graphs in $\mathcal{A}$. Then, if we denote by $Z_{\mathcal{A}}(s_1, s_2, s_3, \dots)$ the cycle index sum of $\mathcal{A}$ and by $U(x)$ the ordinary generating function of $\mathcal{U}$, we have $U(x) = Z_{\mathcal{A}}(s_i \to x^i)_{i \geq 1} = Z_{\mathcal{A}}(x, x^2, x^3, \dots)$.*

2.2 Symmetries of graphs rooted at cliques

For $k \geq 1$, a graph in $\mathcal{A}$ is said to be *rooted at a k-clique* if one of its k-cliques K is distinguished, and the vertices of K are ordered instead of labelled. We denote by $\mathcal{A}^{(k)}$ the class of graphs in $\mathcal{A}$ that are rooted at a k-clique, and for $\mathsf{a} \in \mathcal{A}^{(k)}$ we let $r(\mathsf{a})$ be its root clique. Then an automorphism σ of $\mathsf{a} \in \mathcal{A}^{(k)}$ is also required to map $r(\mathsf{a})$ to itself, maybe permuting its vertices.

We consider permutations with cycle type $\lambda = (\lambda_1^{n_1}, \dots, \lambda_k^{n_k}) \vdash k$ with a canonical ordering of their cycles, and recall that in that case there are $\alpha(\lambda) := k!/(\lambda_1^{n_1} \dots \lambda_k^{n_k} n_1! \dots n_k!)$ many permutations with cycle type λ. A symmetry $(\mathsf{a}, \sigma) \in \mathcal{S}(\mathcal{A}^{(k)})$ is said to be *λ-rooted* if $\sigma_{|r(\mathsf{a})}$ has cycle type λ and the order of the vertices of $r(\mathsf{a})$ respects the canonical ordering of λ. We denote by $\mathcal{S}_\lambda(\mathcal{A}^{(k)})$ the set of all λ-rooted symmetries of $\mathcal{A}^{(k)}$. And for $(\mathsf{a}, \sigma) \in \mathcal{S}_\lambda(\mathcal{A}^{(k)})$ and $i \in [n]$, we let $c^*_{\mu,j}(\mathsf{a}, \sigma)$ be the number of cycles of i-cliques of a under the action of σ with length j and type $\mu \vdash i$, but this time each one of those i-cliques is not entirely contained in $r(\mathsf{a})$.

Then the *λ-rooted cycle index sum* of $\mathcal{A}^{(k)}$ can be similarly defined as

$$X^\lambda_{\mathcal{A}^{(k)}} := \sum_{(\mathsf{a},\sigma) \in \mathcal{S}_\lambda(\mathcal{A}^{(k)})} \frac{1}{n!} \prod_{i=1}^{n} \prod_{\mu \vdash i} \prod_{j \geq 1} s_{\mu,j}^{c^*_{\mu,j}(\mathsf{a},\sigma)}.$$

In practice, the λ-rooted cycle index sum of $\mathcal{A}^{(k)}$ can be computed from that of $\mathcal{A}$ via a formal derivative:

$$X^\lambda_{\mathcal{A}^{(k)}} = \frac{k!}{\alpha(\lambda)\kappa(\lambda)} \frac{\partial}{\partial s_{\lambda,1}} X_{\mathcal{A}}, \qquad \text{with} \quad \kappa(\lambda) := \prod_{i=1}^{k-1} \prod_{\mu \vdash i} \prod_{j \geq 1} s_{\mu,j}^{c_{\mu,j}(K_k,\sigma)}, \tag{2}$$

where K_k is the complete graph on k vertices and σ is one of its automorphisms.

However, in order to reverse this operation, that is computing the extended cycle index sum of some class $\mathcal{A}$ of unrooted graphs from the λ-rooted cycle index sums of $\mathcal{A}^{(k)}$, λ-rooted symmetries do not carry enough information and one requires to point symmetries at cycles (see [2]).

2.3 Symmetries of cycle-pointed graphs

For a class of labelled graphs $\mathcal{A}$, rooted or not, a symmetry $(\mathsf{a}, \sigma) \in \mathcal{S}(\mathcal{A})$ is said to be *cycle-pointed* when one of the cycles C of cliques of σ is distinguished. If C is a cycle of k-cliques ($k \geq 1$), then the graph a is said to be *k-cycle-pointed*, or *k-pointed* for short. The symmetries of a k-pointed graph a are then the symmetries (a, σ) for which σ is pointed at a cycle of k-cliques of a, however if a is rooted at a k-clique then none of its symmetries can be pointed at a cycle of cliques totally contained in $r(\mathsf{a})$.

We denote by $\mathcal{A}^{\bullet_k}$ the class of k-pointed graphs in $\mathcal{A}$ and by $\mathcal{S}_p(\mathcal{A}^{\bullet_k})$ the class of its cycle-pointed symmetries. The extended cycle index sum of $\mathcal{A}^{\bullet_k}$ is defined as

$$X_{\mathcal{A}^{\bullet_k}} := \sum_{(\mathsf{a},\sigma) \in \mathcal{S}_p(\mathcal{A}^{\bullet_k})} \frac{\ell}{|\mathsf{a}|!} t_{\lambda,\ell} \prod_{i=1}^{|\mathsf{a}|} \prod_{\mu \vdash i} \prod_{j \geq 1} s_{\mu,j}^{c^\bullet_{\mu,j}(\mathsf{a},\sigma)},$$

where λ is the type of the pointed cycle of (a, σ), ℓ its length, and $c^\bullet_{\mu,j}(\mathsf{a}, \sigma)$ is the number of unpointed cycles of i-cliques of a with length j and type $\mu \vdash i$. In practice, and following [2], one can derive the cycle index sum of $\mathcal{A}^{\bullet_k}$ from that of $\mathcal{A}$

$$X_{\mathcal{A}^{\bullet_k}} = \sum_{\lambda \vdash k} \sum_{j \geq 1} j t_{\lambda,j} \frac{\partial}{\partial s_{\lambda,j}} X_{\mathcal{A}}. \tag{3}$$

Thus, if every graph in $\mathcal{A}$ has at least one k-clique then $X_{\mathcal{A}}$ is completely determined by $X_{\mathcal{A}^{\bullet_k}}$. Denote by Ψ the operator such that $X_{\mathcal{A}} = \Psi(X_{\mathcal{A}^{\bullet_k}})$. Then we finally have

$$X_{\mathcal{A}} = \Psi\left(X_{\mathcal{A}^{\bullet_k}}\right). \tag{4}$$

Finally, reproducing the proof methods developed in [13] and [1], combinatorial construction rules can be readily translated into extended cycle index sums. For instance, if we let $\mathcal{A}$ and $\mathcal{B}$ be classes of labelled graphs, rooted or not, and $k \geq 1$ then, using the language from [8], we have

$$X_{\mathcal{A}+\mathcal{B}} = X_{\mathcal{A}} + X_{\mathcal{B}} \qquad \text{and} \qquad X_{\mathcal{A}\times\mathcal{B}} = X_{\mathcal{A}} \cdot X_{\mathcal{B}}, \tag{5}$$

$$X_{\mathcal{A}^{\bullet_k}+\mathcal{B}^{\bullet_k}} = X_{\mathcal{A}^{\bullet_k}} + X_{\mathcal{B}^{\bullet_k}} \qquad \text{and} \qquad X_{\mathcal{A}^{\bullet_k}\times\mathcal{B}} = X_{\mathcal{B}\times\mathcal{A}^{\bullet_k}} = X_{\mathcal{B}} \cdot X_{\mathcal{A}^{\bullet_k}}, \tag{6}$$

$$(\mathcal{A}+\mathcal{B})^{\bullet_k} = \mathcal{A}^{\bullet_k} + \mathcal{B}^{\bullet_k} \qquad \text{and} \qquad (\mathcal{A}\times\mathcal{B})^{\bullet_k} = \mathcal{A}\times\mathcal{B}^{\bullet_k} + \mathcal{A}^{\bullet_k}\times\mathcal{B}. \tag{7}$$

Note that the same considerations apply to rooted or non-rooted graphs with a distinguished subgraph. Precisely, automorphisms have to preserve the root and/or the distinguished subgraph, though maybe permuting its vertices, and the cycles of cliques entirely contained in the root of a cycle-pointed symmetry are not taken into account in its extended weight-monommial. Furthemore, extended cycles index sum, λ-rooted symmetries and λ-cycle index sums are defined for rooted and/or k-pointed graphs in the same way.

3 Chordal clique-sums and symmetries

3.1 Substitutions of cliques

Let $k \geq 1$, $\mathcal{A}$ be a class of (possibly rooted) labelled graphs, and $\mathcal{B}$ be a class of graphs rooted at a k-clique. The *clique substitution* $\mathcal{A} \circ_k \mathcal{B}$ is the class obtained by identifying each k-clique of graphs $\mathsf{a} \in \mathcal{A}$ with the root of graphs in $\mathcal{B}$. This results in a graph for which the base graph a is now a distinguished subgraph.

If $C = (c_1, \ldots, c_\ell)$ is a cycle of cliques and $C_1, \ldots, C_k$ is a sequence of k copies of C, then the *composed cycle* of $C_1, \ldots, C_k$ is the cycle of cliques of length ℓk such that for $i \in [k-1]$ and $j \in [\ell]$, the clique coming after c_j in C_i is c_j in C_{i+1}, and the clique coming after c_j in C_k is $c_{j+1 \bmod \ell}$ in C_1. And if we denote by $(X_{\mathcal{A}})^{[j]}$ the cycle index sum resulting from multiplying the second subindex of all variables by j, that is, $s_{\lambda,i} \to s_{\lambda,ij}$, then the extended cycle index sum of the class $\mathcal{A} \circ_k \mathcal{B}$ is defined using the classical composition of cycle index sums

$$X_{\mathcal{A}\circ_k\mathcal{B}} = X_{\mathcal{A}} \left(s_{\lambda,j} \to s_{\lambda,j} \cdot \left(X_{\mathcal{B}}^{\lambda}\right)^{[j]} \right)_{\lambda\vdash k, j\geq 1}. \tag{8}$$

If additionally the graphs in both $\mathcal{A}$ and $\mathcal{B}$ are unpointed, then we also define the *k-pointed substitution* $\mathcal{A}^{\bullet_k} \odot_k \mathcal{B}$ as the class of all k-pointed graphs obtained by the following procedure: let first $(\mathsf{a}, \sigma) \in \mathcal{S}_p(\mathcal{A}^{\bullet_k})$ be a symmetry pointed at some cycle $C = (c_1, \ldots, c_k)$ with type λ, take k copies of a graph $\mathsf{p} \in \mathcal{B}^{\bullet_k}$ admiting a λ-rooted symmetry, and identify each clique in C with the root of one of the copies of p following the canonial order of λ. Second, for every unpointed cycle $D \neq C$ of (a, σ) with type μ, choose a graph $\mathsf{b} \in \mathcal{B}$ admitting a μ-rooted symmetry, take $|D|$ copies of b and identify each clique in D with the root of one of the copies of b. The base graph a is now a distinguished subgraph of the resulting graph and the vertices (except possibly the ones in the root) are assigned unique labels such that the relative order is preserved.

Furthermore, if $C_1, \ldots, C_k$ are the pointed cycles of the copies of p pasted respectively at $c_1, \ldots, c_k$ then the pointed cycle of the resulting graph is the composed cycle of $C_1, \ldots, C_k$. Note that this construction also works if p is a j-pointed graph with $j \neq k$. In that case the resulting graph is j-pointed. Adapting [2], it can then be shown that the k-pointed substitution $\mathcal{A}^{\bullet_k} \odot_k \mathcal{B}$ is a class of k-pointed graphs, that is, every graph admits a symmetry pointed at the cycle of k-cliques. The extended cycle index sum of $\mathcal{A}^{\bullet_k} \odot_k (\mathcal{B}, \mathcal{P})$ is then obtained via the *k-pointed plethystic composition* of their extended cycle index sums, whose definition is an extension of [2]

$$X_{\mathcal{A}^{\bullet_k}\odot_k\mathcal{B}} = X_{\mathcal{A}^{\bullet_k}} \odot_k X_{\mathcal{B}} := X_{\mathcal{A}^{\bullet_k}} \left(s_{\lambda,i} \to \left(X_{\mathcal{B}}^{\lambda}\right)^{[i]}, t_{\mu,j} \to \left(X_{\mathcal{B}^{\bullet_k}}^{\mu}\right)^{[j]} \right), \tag{9}$$

and where $i, j \geq 1$, and λ and μ both range over the partitions of k.

3.2 Starlike chordal clique-sum

Let $\mathcal{A}$ be a class of labelled graphs. The *starlike chordal clique-sum* of $\mathcal{A}$ is the class of rooted graphs $\mathsf{star}(\mathcal{A})$ obtained by taking a multiset of graphs from $\mathcal{A}^{(k)}$, identifying their rooted k-cliques together, and relabelling all the other vertices respecting their previous relative order. Then the λ-rooted extended cycle index sum of $\mathsf{star}(\mathcal{A})$ is

$$X^{\lambda}_{\mathsf{star}(\mathcal{A})} = \exp\left(\sum_{j\geq 1} \frac{1}{j} \left(X^{\lambda^j}_{\mathcal{A}^{(k)}}\right)^{[j]}\right), \tag{10}$$

where λ^j denotes the cycle type of σ^j if σ has cycle type λ.

Similarly, for $\ell \geq 1$ we define the *ℓ-pointed starlike chordal clique-sum* of $\mathcal{A}$ as the class $\mathsf{star}_\ell(\mathcal{A})$ of all the rooted and k-pointed graphs obtained by choosing some multiset S of elements of $\mathcal{A}^{(k)}$, considering $j \geq \ell$ disjoint copies $\mathsf{p}_1, \ldots, \mathsf{p}_j$ of some $\mathsf{p} \in \mathcal{A}^{\bullet_k}$, and then identifying together the rooted cliques of all the graphs in S and the rooted cliques of $\mathsf{p}_1, \ldots, \mathsf{p}_j$. Again, we relabel the non-root vertices to preserve the relative order. Furthermore, if $C_1, \ldots, C_j$ are the pointed cycles of $\mathsf{p}_1, \ldots, \mathsf{p}_j$, respectively, then the pointed cycle of the resulting graph is the composed cycle of $C_1, \ldots, C_j$. One can then show that every graph in $\mathsf{star}_\ell(\mathcal{A})$ admits a symmetry containing the pointed cycle of k-cliques as one of its cycles of cliques. In fact, the converse is also true and we have $\mathsf{star}(\mathcal{A})^{\bullet_k} = \mathsf{star}_1(\mathcal{A})$.

If, for $\ell \geq 1$, we now let

$$Z_{\mathsf{set}_\ell}(s_1, t_1, s_2, t_2, \ldots) = \sum_{j\geq \ell} j\, t_j \frac{\partial}{\partial s_j} \exp\left(\sum_{i\geq 1} \frac{s_i}{i}\right),$$

then the λ-cycle index sum of $\mathsf{star}_\ell(\mathcal{A})$ is given by

$$X^{\lambda}_{\mathsf{star}_\ell(\mathcal{A})} = Z_{\mathsf{set}_\ell}\left(s_i \to \left(X^{\lambda^i}_{\mathcal{A}^{(k)}}\right)^{[i]}, t_j \to \left(X^{\lambda^j}_{\mathcal{A}^{\bullet_k}}\right)^{[j]}\right)_{i\geq 1, j\geq \ell}. \tag{11}$$

4 Counting chordal graphs with bounded tree-width

Fix $t \geq 1$, and for any $k \in [t+1]$ let $\mathcal{G}_k$ (resp. $\mathcal{G}_k^{\bullet_k}$) be the class of labelled k-connected chordal graphs with tree-width at most t (resp. and that are k-pointed). Note that $\mathcal{G}_{t+1}$ is reduced to the $(t+1)$-clique, with cycle index sum

$$X_{\mathcal{G}_{t+1}} = \frac{1}{(t+1)!} \sum_{\lambda \vdash t+1} \alpha(\lambda)\kappa(\lambda). \tag{12}$$

From there, the relation (2) gives us the class $\mathcal{G}^{(t)}_{t+1}$. Fix now some $k \in [t]$ and some $\lambda \vdash k$. Then, adapting the scheme from [3] and provided we know $\mathcal{G}^{(k)}_{k+1}$, we can obtain by iteration the class $\mathcal{G}^{(k)}_k$ as a solution of the recursive equation

$$\mathcal{G}^{(k)}_k = \mathsf{star}\left(\mathcal{G}^{(k)}_{k+1} \circ_k \mathcal{G}^{(k)}_k\right). \tag{13}$$

To now obtain $\mathcal{G}_k$ from $\mathcal{G}^{(k)}_k$, we proceed following [2], by using the dissymmetry theorem for tree-decomposable classes [1] on $\mathcal{G}_k^{\bullet_k}$, which can be derived by adapting [3]. This gives

$$\mathcal{G}_k^{\bullet_k} \simeq \mathcal{G}^{(k)}_k + (\mathcal{G}_{k+1})^{\bullet_k}_{\geq 2} \odot_k \mathcal{G}^{(k)}_k + \mathsf{star}_2\left(\mathcal{G}^{(k)}_{k+1} \circ_k \mathcal{G}^{(k)}_k\right), \tag{14}$$

where the extended cycle index sum of $(\mathcal{G}_{k+1})^{\bullet_k}_{\geq 2}$ is defined as in (3) but without the terms with $j = 1$. Equations (13) and (14) relating combinatorial classes can then be translated into relations between extended cycle index sums using the various identities derived in Sections 2 and 3.

Finally, starting from (12) and by successive iterations of the recursive step (13), the cycle pointing step (3) and the unrooting step, composed of (14) together with (2) and (4), we can obtain $\mathcal{G}_k$ for any $k \in [t]$. In practice, we are able to compute its extended cycle index sum and the associated generating function, using Proposition 2, whose n-th coefficient is the number of unlabelled graphs with n vertices. We provide an effective implementation of the algorithm computing any term of the generating function on this repository, and as an example display next the first numbers of unlabelled chordal graphs with tree-width at most t, connectivity k and up to ten vertices.

	$t=1$	$t=2$	$t=3$	$t=4$
$k=1$	1 1 1 2 3 6 11 23 47 106	1 1 2 4 11 35 124 500 2224 10640	1 1 2 5 14 53 234 1265 8015 58490	1 1 2 5 15 57 266 1556 11187 97859
$k=2$	-	0 1 1 1 2 5 12 39 136 529	0 1 1 2 4 14 55 293 1842 13491	0 1 1 2 5 17 75 455 3486 32907
$k=3$	-	-	0 0 1 1 1 2 5 15 58 275	0 0 1 1 2 4 14 62 391 3182
$k=4$	-	-	-	0 0 0 1 1 1 2 5 15 64

The last non-empty line of column t corresponds to unlabelled t-trees, while the line $k = 1$ of the second column corresponds to connected chordal series-parallel graphs with OEIS sequence A243788. To the extent of our knowledges, the other sequences are new. Note that an algorithm was designed to compute, among others, the first numbers of unlabelled chordal planar graphs (OEIS sequence A243787). The first discrepancy between A243787 and the line $k = 1$ of the third column is given by the unique non-planar connected chordal graph with tree-width three and six vertices: it is the starlike chordal sum of three K_4's at a common triangle. By adapting [4] to the context of Pólya theory, we believe that a similar program could be developed in order to obtain additional terms of the ordinary generating function of unlabelled chordal planar gaphs with n vertices as well as an asymptotic estimate in the form of (1).

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