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Chordal graphs with bounded tree-width

(Extended abstract)

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Abstract

Given $t \geq 2$ and $0 \leq k \leq t$, we prove that the number of labelled k-connected chordal graphs with n vertices and tree-width at most t is asymptotically $cn^{-5/2}\gamma^n n!$, as $n \to \infty$, for some constants $c, \gamma > 0$ depending on t and k. Additionally, we show that the number of i-cliques ($2 \leq i \leq t$) in a uniform random k-connected chordal graph with tree-width at most t is normally distributed as $n \to \infty$.

The asymptotic enumeration of graphs of tree-width at most t is wide open for $t \geq 3$. To the best of our knowledge, this is the first non-trivial class of graphs with bounded tree-width where the asymptotic counting problem is solved. Our starting point is the work of Wormald [Counting Labelled Chordal Graphs, *Graphs and Combinatorics* (1985)], were an algorithm is developed to obtain the exact number of labelled chordal graphs on n vertices..

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1 Introduction

Tree-width is a fundamental parameter in structural and algorithmic graph theory, as illustrated for instance in [8]. It can be defined in terms of tree-decompositions or equivalently in terms of k-trees. A k-tree is defined recursively as either a complete graph on $k+1$ vertices or a graph obtained by adjoining a new vertex adjacent to a k-clique of a smaller k-tree. The tree-width of a graph Γ is then the minimum k such that Γ is a subgraph of a k-tree. In particular, k-trees are the maximal graphs with tree-width at most k. The number of k-trees on n labelled vertices was independently shown [3, 17, 15] to be

$$\binom{n}{k}(k(n-k)+1)^{n-k-2} = \frac{1}{\sqrt{2\pi}\,k!\,k^{k+2}}\,n^{-5/2}\,(ek)^n\,n!\,(1+o(1)), \tag{1}$$

where the estimate holds for k fixed and $n\to\infty$. However, there are relatively few results on the enumeration of graphs of given tree-width or on properties of random graphs with given tree-width. Graphs of tree-width one are forests (acyclic graphs) and their enumeration is a classical result, while graphs of tree-width at most two are series-parallel graphs and were first counted in [5]. The problem of counting graphs of tree-width three is still open. From now on, we will use t to denote the tree-width while k will denote the connectivity of a graph. All graphs considered in this work will be simple and labelled, that is, with vertex-set $[n]$.

Given that tree-width is non-increasing under taking minors, the class of graphs with tree-width at most t is 'small' when t is fixed, in the sense that the number $g_{n,t}$ of labelled graphs with n vertices and tree-width at most t grows at most like $c^n n!$ for some $c>0$ depending on t (see [18, 13]). The best known bounds for $g_{n,t}$ are, up to lower order terms,

$$\left(\frac{2^t tn}{\log t}\right)^n \le g_{n,t} \le (2^t tn)^n.$$

The upper bound follows by considering all possible subgraphs of t-trees, and the lower bound uses a suitable construction developed in [2]. In the present work we determine the asymptotic number of labelled chordal graphs with tree-width at most t, following the approach in [14] and [11], and based on the analysis of systems of equations satisfied by generating functions.

A graph is chordal if every cycle of length greater than three contains at least one chord, that is, an edge connecting non-consecutive vertices of the cycle. Chordal graphs have been extensively studied in structural graph theory and graph algorithms (see for instance [16]), but not so much from the point of view of enumeration. Wormald [20] used generating functions to develop a method for finding the exact number of chordal graphs with n vertices for a given value of n. It is based on decomposing chordal graphs into k-connected components for each $k\ge 1$. As remarked in [20], it is difficult to define the k-connected components of arbitrary graphs for $k>3$, but for chordal graphs they are well defined. It is a consequence of Dirac's characterisation [10]: *in a chordal graph every minimal separator is a clique.*

For fixed $n,t\ge 1$ and $0\le k\le t$, let $\mathcal{G}_{t,k,n}$ be the set of k-connected chordal graphs with n labelled vertices and tree-width at most t. Our two main results are the following.

Theorem 1.1. *For $t \geq 1$ and $0 \leq k \leq t$, there exist $c_{t,k} > 0$ and $\gamma_{t,k} > 1$ such that*

$$|\mathcal{G}_{t,k,n}| = c_{t,k}\, n^{-5/2}\, \gamma_{t,k}^n\, n!\, (1 + o(1)) \qquad \text{as } n \to \infty.$$

We remark that in principle, for fixed t and k the constants $c_{t,k}$ and $\gamma_{t,k}$ can be computed, at least approximately.

Theorem 1.2. *Let $t \geq 1$, $0 \leq k \leq t$. For $i \in \{2, \ldots, t\}$ let $X_{n,i}$ denote the number of i-cliques in a uniform random graph in $\mathcal{G}_{t,k,n}$, and set $\mathbf{X_n} = (X_{n,2}, \ldots, X_{n,t})$. Then $\mathbf{X_n}$ satisfies a multivariate central limit theorem, that is, as $n \to \infty$ we have*

$$\frac{1}{\sqrt{n}} (\mathbf{X}_n - \mathbb{E}\,\mathbf{X}_n) \xrightarrow{d} N(0, \Sigma), \qquad \text{with} \quad \mathbb{E}\,\mathbf{X}_n \sim \alpha n \quad \text{and} \quad \mathbb{C}\mathrm{ov}\,\mathbf{X}_n \sim \Sigma n,$$

and where α is a $(t-1)$-dimensional vector of positive numbers and Σ is a $(t-1) \times (t-1)$-dimensional positive semi-definite matrix.

Let us point that more structural asymptotic results can be expected. Notably, the class of chordal graphs with tree-width at most t is *subcritical* in the sense of [12]. It follows from [19] that the uniform random connected chordal graph with tree-width at most t with distances rescaled by $1/\sqrt{n}$ admits the *Continuum Random Tree* (CRT) [1] as a scaling limit, multiplied by a constant that depends on t.

A more complete version of the work presented here can be found in [6].

2 Decomposition of chordal graphs

Let $k \geq 1$. A k-separator of a graph Γ is a subset of k vertices whose removal disconnects Γ. And Γ is said to be k-connected if it contains no i-separator for $i \in [k-1]$. With this definition, we consider the complete graph on k vertices to be k-connected, for any $k \geq 1$, contrary to the usual definition of connectivity (see for instance [9]). A k-connected component of Γ is a k-connected subgraph that is maximal, in term of subgraph containment, with that property.

An essential consequence of chordality is that k-connected chordal graphs admit a unique decomposition into $(k+1)$-connected components through its k-separators. This is a generalisation of the well-known decomposition of a connected graph into so-called blocks, that are maximal 2-connected components. And it induces a system of functional equations satisfied by the generating function counting chordal graphs of tree-width at most k.

We now fix some $t \geq 1$ and let $\mathcal{G}$ be the family of chordal graphs with tree-width at most t. For a graph $\Gamma \in \mathcal{G}$ and $j \in [t]$, let us denote by $n_j(\Gamma)$ the number of j-cliques of Γ. In the rest of the paper, we will write $\mathbf{x}$ as a short-hand for $x_1, \ldots, x_t$, and define the multivariate (exponential) generating function associated to $\mathcal{G}$ to be

$$G(\mathbf{x}) = G(x_1, \ldots, x_t) = \sum_{\Gamma \in \mathcal{G}} \frac{1}{n_1(\Gamma)!} \prod_{j=1}^{t} x_j^{n_j(\Gamma)}.$$

Let g_n denote the number of chordal graphs with n vertices and tree-width at most t. Then,

$$G(x, 1, \ldots, 1) = \sum_{n \geq 1} \frac{g_n}{n!} x^n.$$

For $0 \leq k \leq t+1$, let $\mathcal{G}_k$ be the family of k-connected chordal graphs with tree-width at most t and $G_k(\mathbf{x})$ be the associated generating function. In particular, we have

$$G_{t+1}(\mathbf{x}) = \frac{1}{(t+1)!} \prod_{j \in [t]} x_j^{\binom{t+1}{j}}. \tag{2}$$

For other values of k, we need to consider graphs rooted at a clique. Rooting the graph $\Gamma \in \mathcal{G}_k$ at an i-clique means distinguishing one i-clique K of Γ and choosing an ordering of (the labels of) the vertices of K. In order to avoid over-counting, we will discount the subcliques of K. Let $i \in [k]$ and define $\mathcal{G}_k^{(i)}$ to be the family of k-connected chordal graphs with tree-width at most t and rooted at an i-clique. Let then $G_k^{(i)}(\mathbf{x})$ be the associated generating function, where now for $1 \leq j \leq i$ the variables x_j mark the number of j-cliques that are not subcliques of the root.

Lemma 2.1. *Let $k \in [t]$. Then the following equations hold:*

$$G_{k+1}^{(k)}(\mathbf{x}) = k! \prod_{j=1}^{k-1} x_j^{-\binom{k}{j}} \frac{\partial}{\partial x_k} G_{k+1}(\mathbf{x}), \tag{3}$$

$$G_k^{(k)}(\mathbf{x}) = \exp\left(G_{k+1}^{(k)}\big(x_1, \ldots, x_{k-1}, x_k G_k^{(k)}(\mathbf{x}), x_{k+1}, \ldots, x_t\big)\right), \tag{4}$$

$$G_k(\mathbf{x}) = \frac{1}{k!} \prod_{j=1}^{k-1} x_j^{\binom{k}{j}} \int G_k^{(k)}(\mathbf{x})\ dx_k. \tag{5}$$

Finally, the fact that a graph is the set of its connected components can be translated as $G(\mathbf{x}) = G_0(\mathbf{x}) = \exp(G_1(\mathbf{x}))$. Then, it is clear that one can derive $G_0(\mathbf{x})$ from $G_{t+1}(\mathbf{x})$ by successively using Identities (3), (4) and (5) from Lemma 2.1.

3 Asymptotic analysis

Fix $t \geq 1$. To prove Theorems 1.1 and 1.2, we use rather classical methods from [14] and [11, Chapter 2] which consist in deriving asymptotic estimates from local expansions of the generating functions from Section 2 at their singularities, typically by applying a *Transfer Theorem* (for instance [11, Lemma 2.18]).

However, the main difficulties here are the multivariate nature of Lemma 2.1, in particular the fact that the local expansions are with respect to different variables from one step to the next, and the fact that local expansions have to be "carried over" from $G_{t+1}(\mathbf{x})$ to $G_0(x_1, 1, \ldots, 1)$. To overcome this, we extend some of the tools and notions present in [11].

Sketch of the proofs of Theorems 1.1 and 1.2. Starting with G_{t+1} which is an explicit monomial, we recursively compute via Lemma 2.1 local representations of $G_t, G_{t-1}, \dots, G_1$ and finally of $G_0 = \exp(G_1)$.

The first step of the induction amounts to computing a multivariate local representation of the generating function of t-trees. Let $x_2, \dots, x_t \in \mathbb{R}_+$. Then there exist two functions $h_1(x)$ and $h_2(x)$, that are analytic and non-zero at $x_1 = 1/e$, such that for $x_1 \sim 1/e$ we have

$$G_t(\mathbf{x}) = \frac{\prod_{j=1}^t x_j^{\binom{t}{j}}}{t!} \left(h_1(tX) + h_2(tX)(1-etX)^{3/2}\right), \qquad \text{where } X = \prod_{j=1}^t x_j^{\binom{t}{j-1}}.$$

From there, one can prove that the above representation for $G_t(\mathbf{x})$ implies corresponding representation for $G_{t-1}(\mathbf{x}), G_{t-2}(\mathbf{x}), \dots, G_1(\mathbf{x})$, then $G_0(\mathbf{x})$. And the main counting result can be deduced by setting $x_2 = \cdots = x_t = 1$ then applying a *Transfer Theorem*.

Finally, the joint central limit theorem can be obtained in a similar manner: first showing that a local representation of $G_k(\mathbf{x})$ can be extended uniformly in a neighbourhood of $(1, \dots, 1) \in \mathbb{C}^{t-1}$, then concluding with the *Quasi-Powers Theorem* [11, Theorem 2.22].

4 Concluding remarks

Let us mention a recent result [7] giving an estimate $cn^{-5/2}\gamma^n n!$ for the number of labelled *planar* chordal graphs with $\gamma \approx 11.89$. It turns pout that the class of chordal graphs with tree-width at most three is exactly the same as the class of chordal graphs not containing K_5 as a minor, whose asymptotic growth is, according to Theorem 1.1 and some numerical computations of the form $cn^{-5/2}\delta^n n!$ with $\delta = 1/\rho_{3,1} \approx 12.98$.

Since the number of all chordal graphs grows like $2^{n^2/4}$ (see [4]), the singularity $\rho_t = \rho_{t,1} \to 0$ as $t \to \infty$. Concerning the rate of convergence, since the exponential growth of t-trees is $(etn)^n$, we have $\rho_t = O(1/t)$. And since the growth of all graphs of tree-width at most t is at most $(2^t tn)^n$, we also have $\rho_t = \Omega\left(1/(t2^t)\right)$. A remaining problem is to narrow the gap between the upper and lower bounds. Heuristic arguments suggest that ρ_t decreases exponentially in t.

As a final question, we consider letting $t = t(n)$ grow with n. Recall that a class of labelled graphs is small when the number of graphs in the class grows at most like $c^n n!$ for some $c > 0$, and large otherwise. One can prove that if $t = (1+\epsilon)\log n$ then the class of labelled chordal graphs of tree-width at most t is large for each $\epsilon > 0$. We leave as an open problem to determine at which order of magnitude between $t = O(1)$ and $t = \log n$ the class ceases to be small.

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