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# Bijective enumeration of constellations in higher genus

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## Abstract

Bousquet-Mélou and Schaeffer gave in 2000 a bijective enumeration of some planar maps called constellations. In 2019, Lepoutre described a bijection between bicolorable maps of arbitrary genus and some unicellular maps of the same genus.

We present a bijection between constellations of higher genus and some unicellular maps that generalizes both existing bijections at the same time.

Using this bijection, we manage to enumerate a subclass of constellations on the torus, proving that its generating function is a rational function of the generating function of some trees.

***keywords:*** combinatorics, maps, enumeration, bijection, blossoming map, constellation, rationality

**MSC2020:** 05C30, 05C10

## Resum

Bousquet-Mélou i Schaeffer donaren el 2000 una enumeració bijectiva de certs mapes anomenats constel·lacions. El 2019, Lepoutre va descriure una bijecció entre mapes biacoloribles de gènere arbitrari i certs mapes unicel·lulars del mateix gènere.

Presentem una bijecció entre constel·lacions de gènere superior i certs mapes unicel·lulars que generalitza les dues bijeccions existents alhora.

Fent servir aquesta bijecció, enumerem una subclasse de constel·lacions sobre el tor, demostrant que llur funció generadora és una funció racional de la funció generadora de certs arbres.

***paraules clau:*** combinatòria, mapes, enumeració, bijecció, mapa florit, constel·lació, racionalitat

**MSC2020:** 05C30, 05C10

# Resumen

Bousquet-Mélou y Schaeffer dieron en el año 2000 una enumeración biyectiva de ciertos mapas llamados constelaciones. En 2019, Lepoutre describió una biyección entre mapas bicoloreables de género arbitrario y ciertos mapas unicelulares del mismo género.

Presentamos una biyección entre constelaciones de género superior y ciertos mapas unicelulares que generaliza las dos biyecciones existentes a la vez.

Usando esta biyección, enumeramos una subclase de constelaciones sobre el toro, demostrando que su función generadora es una función racional de la función generadora de ciertos árboles.

***palabras clave:*** combinatoria, mapas, enumeración, biyección, mapa floreciente, constelación, racionalidad

**MSC2020:** 05C30, 05C10

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I would also like to thank Juanjo Rué for introducing me to the topic of maps, putting me in touch with Marie and Éric and following my progress during this time.

# Contents

# Chapter 1

# Introduction

Maps are embeddings of graphs in a surface. They are fundamental combinatorial objects that appear in many other fields of mathematics such as algebra and mathematical physics. The enumeration of planar maps began with the work of Tutte in the sixties [10]. In his work, Tutte enumerated a variety of families of maps, obtaining remarkably simple formulas. For example, he showed that the number of planar maps with $n$ edges is

$$\frac{2(2n)!3^n}{n!(n+2)!}.$$

His methods are based on the recursive combinatorial decomposition of maps and the equations obtained usually require the introduction of additional parameters called catalytic variables. In the late eighties, these techniques were extended to maps on surfaces of higher genus by Bender and Canfield [1], [2].

The simplicity of the formulas obtained by Tutte called for bijective demonstrations. Cori and Vauquelin gave the first bijective proof of the enumeration of planar maps in 1981 [7]. After them, many others continued this work, starting with Schaeffer, who gave numerous bijective constructions in the late nineties. In 1997, he introduced blossoming trees to formulate a new bijection for planar maps [9]. In 2000, Bousquet-Mélou and Schaeffer gave a bijection between planar constellations and some blossoming trees, which allowed them to prove enumerative formulas for constellations [3]. It should be mentioned that there is a second trend of bijections of maps based on trees decorated with some integers that encode metric properties of the maps. These bijections were applied to planar constellations in [4] and were later extended to higher genus in [5].

In positive genus, the natural equivalent of trees are unicellular maps. Chapuy, Marcus and Schaeffer introduced in [6] some techniques to analyse these unicellular maps by decomposing them into schemes with branches. In 2019, Lepoutre gave a blossoming bijection for bicolorable maps of any genus, which are a particular case of constellations [8].

The purpose of our work is to extend the bijection of [3] to higher genus. To do so, we reformulate their planar bijection inspired by the work of Lepoutre. Our main result is that there is a blossoming bijection between constellations and some blossoming unicellular maps that extends not only the planar blossoming bijection of [3], but the bijection of [8] too. We also enumerate a particular case of constellations on the torus using this bijection.

The report is structured in the following way. In Chapter 2, we give some elementary notions of maps. Chapters 3, 4 and 5 describe the objects that we use. Chapter 6 is dedicated to proving our main bijective result. In chapter 7, we show how our maps can be rerooted, which is needed to enumerate them. Finally, in Chapter 8 we give the already mentioned enumeration.

The results of these bachelor thesis will be exploited in a nearby future to get further results, to be published elsewhere.

# Chapter 2

# Preliminaries

## 2.1 Maps

**Definition 2.1.** A *map* $M$ of genus $g$ is a proper embedding of a graph in $\mathcal{S}_g$, the torus with $g$ holes, such that the maximal connected components of $\mathcal{S}_g \setminus M$ are contractible. These components are called *faces.* Multiple edges and loops are allowed. Maps are considered up to orientation preserving homeomorphisms.

This definition of maps is a very geometrical one. But maps are, in fact, purely combinatorial objects. Indeed, a map is completely determined by the cyclic order of edges around every vertex. This description allows us to encode maps using permutations, which leads to an algebraic approach to the subject. We will not consider this point of view here.

Maps of genus 0 are called *planar maps.* They receive this name because embedding graphs in the sphere or in the plane is essentially the same. The stereographical projection, for instance, can produce a plane embedding from a sphere embedding. All the faces of a planar map embedded in the plane are contractible except for one, the *exterior face*, which is homeomorphic to the complement of a disk.

The following is a classical result.

**Proposition 2.2** (Euler's formula)**.** *Maps of genus $g$ satisfy the relation*

$$v - e + f = 2 - 2g,$$

*where $v, e$ and $f$ denote the number of vertices, edges and faces, respectively.*

**Definition 2.3.** In a map, a *corner* is a couple of consecutive edges around a vertex. Equivalently, a corner can be seen as an incidence between a face and a vertex. The *degree* of a vertex or face is its number of corners.

**Definition 2.4.** A *rooted map* is a map with a marked corner, which is called the *root corner* (or, simply, *root*). This root corner naturally defines a *root vertex* and a *root face.*

**Definition 2.5.** In a rooted map, we have a notion of clockwise and counterclockwise when following contractible cycles. Precisely, we say that a tour around a contractible cycle is clockwise (resp. counterclockwise) if the root face lies on the left (resp. right) side of it. Tours around the root face (which is a contractible cycle) are an exception to this and follow the opposite convention.

**Definition 2.6.** In maps, edges join two (possibly equal) vertices and separate two (possibly equal) faces. Thus, given a map $M$, we can define its *dual map* $M^*$ in the following way (Figure 2.1). The faces (resp. vertices) of $M$ become the vertices (resp. faces) of $M^*$ and the dual of an edge $e$ joining vertices $v_1$ and $v_2$ and separating faces $f_1$ and $f_2$ is an edge $e^*$ joining vertices $f_1^*$ and $f_2^*$ and separating faces $v_1^*$ and $v_2^*$.

Note that the dual of a corner is "itself" (i.e., the same vertex-face incidence) and that dualization is involutive: $(M^*)^* = M$.

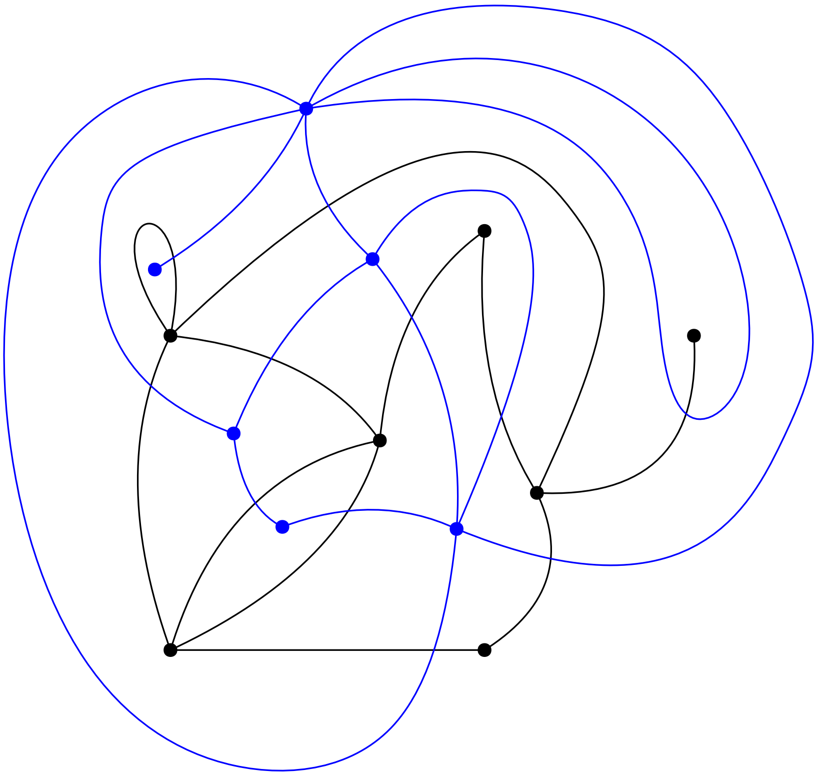


Figure 2.1: A planar map and its dual.

**Definition 2.7.** A *unicellular map* is a map with a single face.

## 2.2 Plane trees

**Definition 2.8.** A *plane tree* is a unicellular map of genus 0.

**Definition 2.9.** In plane trees, vertices of degree 1 are called *leaves*.

Since plane trees are maps, they can also be rooted on a corner.

**Definition 2.10.** In rooted plane trees, any vertex $u$ different from the root has a *parent* $v$, which is defined as the first vertex in its unique path towards the root. In this case, $u$ is said to be the *child* of $v$.

**Definition 2.11.** The *lexicographical order* of the vertices of a rooted plane tree is the order determined by the clockwise tour around its unique face, starting at the root. More specifically, we say that vertex $v_1$ precedes vertex $v_2$ in lexicographical order ($v_1 \prec_{\text{lex}} v_2$ or, simply, $v_1 \prec v_2$) if $v_1$ is encountered before $v_2$ in this tour.

Plane trees receive this name because they are embeddings of trees (connected graphs with no cycles) in the plane (or, equivalently, in the sphere). The cyclic order of the edges around any vertex defined by the embedding allows us define this lexicographical order. But trees could be embedded in surfaces of higher genus. In this case, they would not be maps, since not all faces would be contractible, but the embedding would still define a cyclic order of the edges around any vertex. Therefore, we can (and will) talk about the lexicographical order of the vertices of a rooted embedded tree even if it is not a plane tree.

## 2.3 Orientations

**Definition 2.12.** An *orientation* of a map is an orientation (or direction) of each of its edges. An *oriented map* is a map endowed with an orientation. An oriented edge has a *left side* and a *right side*, which are defined to coincide with the usual notions of left and right when the edge points upwards.

**Definition 2.13.** In a rooted map, we say that an orientation is *accessible* if there exists an oriented path from the root vertex to any other vertex.

**Definition 2.14.** Let $M$ be a rooted map endowed with an accessible orientation. The *leftmost Breadth-First Search (BFS) exploration tree* of $M$ is the one produced by Algorithm 1.

Figure 2.2 contains an example of the leftmost BFS exploration tree of a map of genus 1. When representing maps of genus 1, we draw them in a thick square, the parallel sides of which should be identified to obtain the torus.

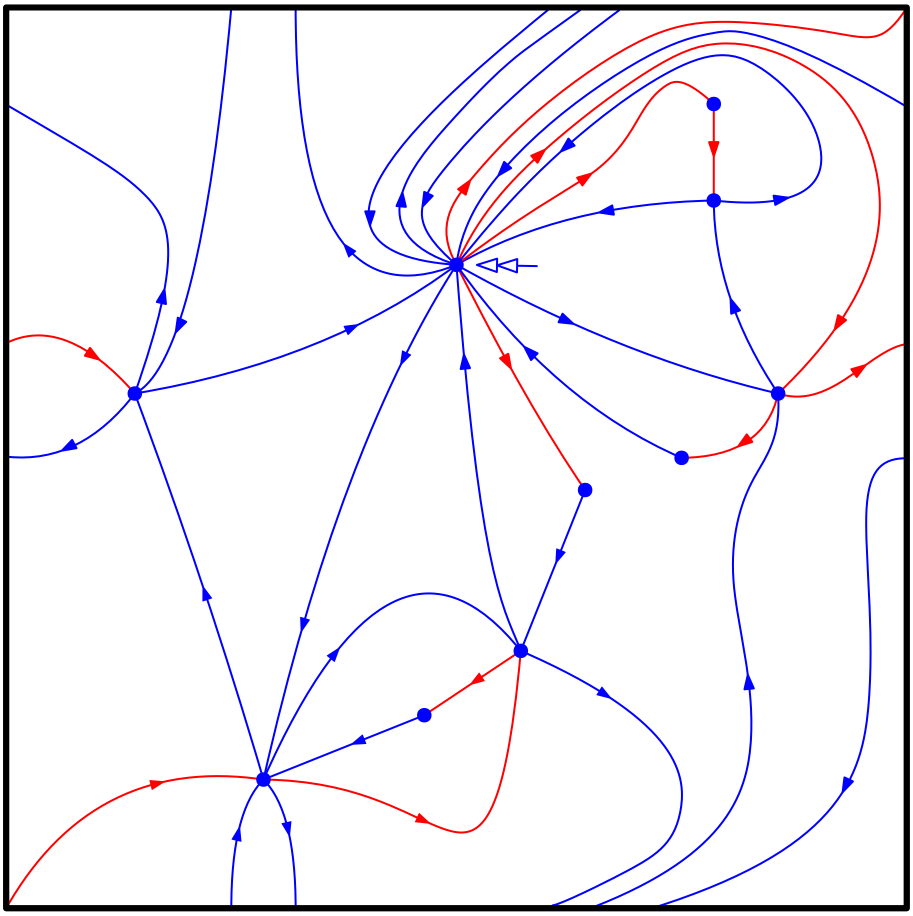


Figure 2.2: In red, the leftmost BFS exploration tree of a rooted, accessibly oriented map of genus 1. The root is pointed by the double arrow.

**Algorithm 1** Leftmost BFS traversal of a rooted planar map $M$

```
let $Q$ be a queue
let $r$ be the root vertex of $M$
mark $r$ as visited
$Q$.enqueue ($r$)
while $Q$ is not empty do
  $u = Q$.dequeue ()
  order the edges that depart from $u$ counterclockwise around $u$, starting at the root corner if $u$ is the root vertex, or starting at the edge travelled to visit $u$ otherwise. Name these edges $e_1, \dots, e_k$
  for i=1 to k do
    let $v$ be the destination of $e_i$
    if $v$ is not visited then
      mark $e_i$ as travelled
      mark $v$ as visited
      $Q$.enqueue ($v$)
    end if
  end for
end while
```

**Proposition 2.15.** *The leftmost BFS exploration tree of an accessibly oriented rooted map $M$ is a tree that visits all vertices.*

*Proof.* It has no cycles because every vertex is visited only once and every vertex is visited because the orientation is accessible. ☐

The following fact can be easily deduced from Algorithm 1.

**Proposition 2.16.** *In an accessibly oriented map $M$, if every vertex is labelled with the length of the shortest path from the root to it, the leftmost BFS exploration tree $T$ of $M$ can be characterized as the unique spanning tree of $M$ satisfying that*

*(i) labels increase by* 1 *along edges (i.e., $T$ is a BFS tree),*

*(ii) if $e$ is an edge in $M \setminus T$ going from a vertex $u$ to a vertex $v$ such that the label of $v$ is one unit larger than the label of $u$, then none of the following conditions hold:*

  *a) $u$ precedes the parent of $v$ in the lexicographical order of $T$ (Figure 2.3a),*

  *b) $u$ is the parent of $v$ and the edge connecting $u$ and $v$ in $T$ is found after $e$ when turning clockwise around $v$ starting from the edge connecting $v$ to its parent (or starting from the root corner in the case that $v$ is the root vertex) (Figure 2.3b).*

*If $T$ were a BFS exploration tree of $M$ but it were not* leftmost, *then there would be some edge breaking condition (ii). We say that these edges are* bad.

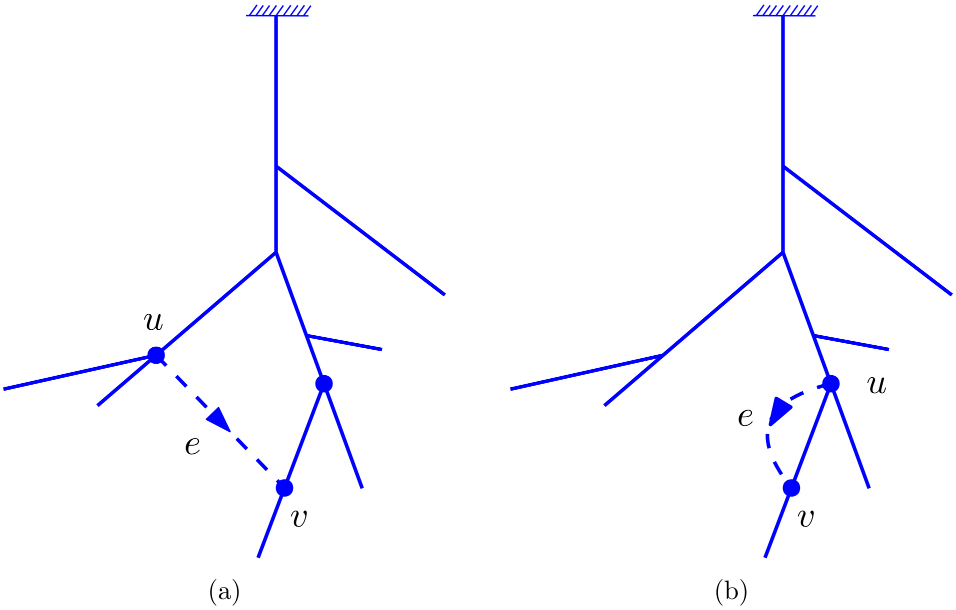


Figure 2.3: The two types of bad edge.

# Chapter 3

# Constellations

**Definition 3.1.** Let $m \geq 2$. We say that a map whose faces are bicolored (black and white) is an *$m$-constellation* if

(i) adjacent faces have different colors,

(ii) black faces have degree $m$ and white faces have degree $mi$ for some integer $i \geq 1$ (which can be different among white faces),

(iii) vertices can be labeled with integers in $\{1, 2, \dots, m\}$ in such a way that turning clockwise around any black face the labels read $1, 2, \dots, m$.

**Definition 3.2.** A *rooted constellation* (Figure 3.1) is a constellation that is rooted on a white corner. The first edge found when turning counterclockwise around the root vertex starting from the root corner is called the *root edge*. Note that the root corner can be recovered from the root edge, so it is equivalent to root a constellation on a white corner or on an edge.

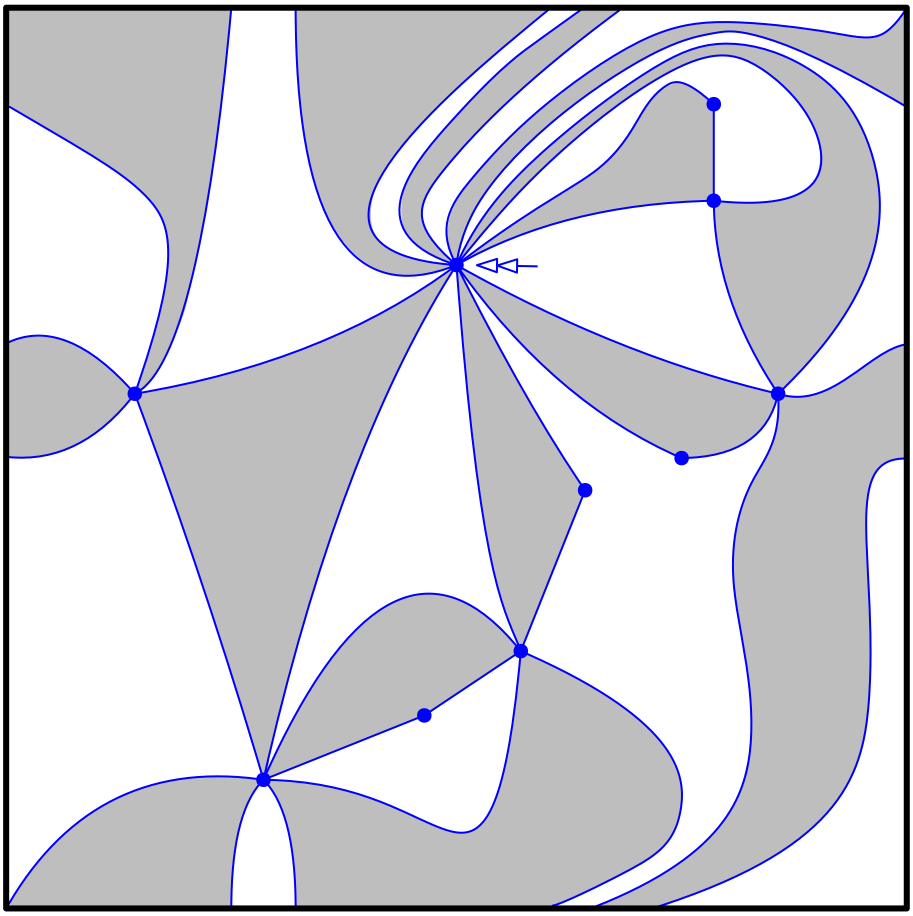


Figure 3.1: A rooted 3-constellation of genus 1. The root is pointed by the double arrow.

**Definition 3.3.** The dual of an $m$-constellation is called an *$m$-Eulerian map* (Figures 3.2 and 3.3). $m$-Eulerian maps can be rooted on a corner, in the same way as constellations. Their *root edge* is defined as the first edge found when turning clockwise around the root vertex, starting at the root corner.

The dual of a rooted $m$-constellation (resp. rooted $m$-Eulerian map) is a rooted $m$-Eulerian map (resp. rooted $m$-constellation) with the "same" root. In other words, the root vertex, the root face and the root edge become, respectively, the root face, the root vertex and the root edge through dualization.

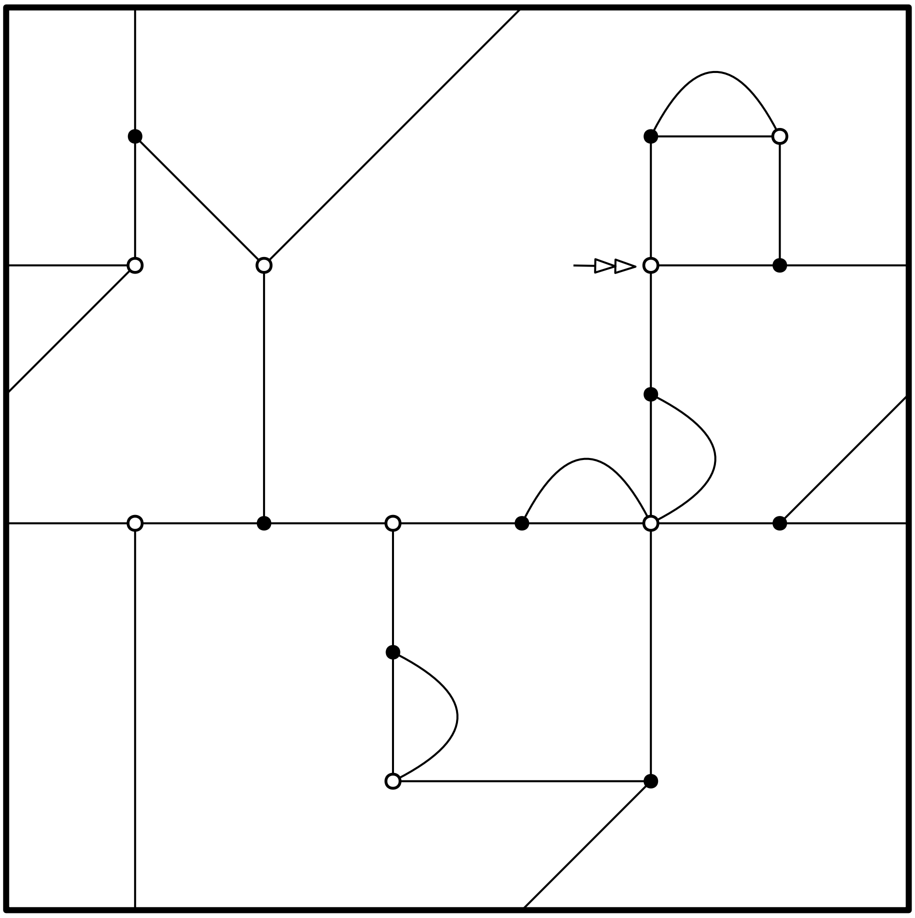


Figure 3.2: A rooted 3-Eulerian map of genus 1. The root corner is pointed by the double arrow.

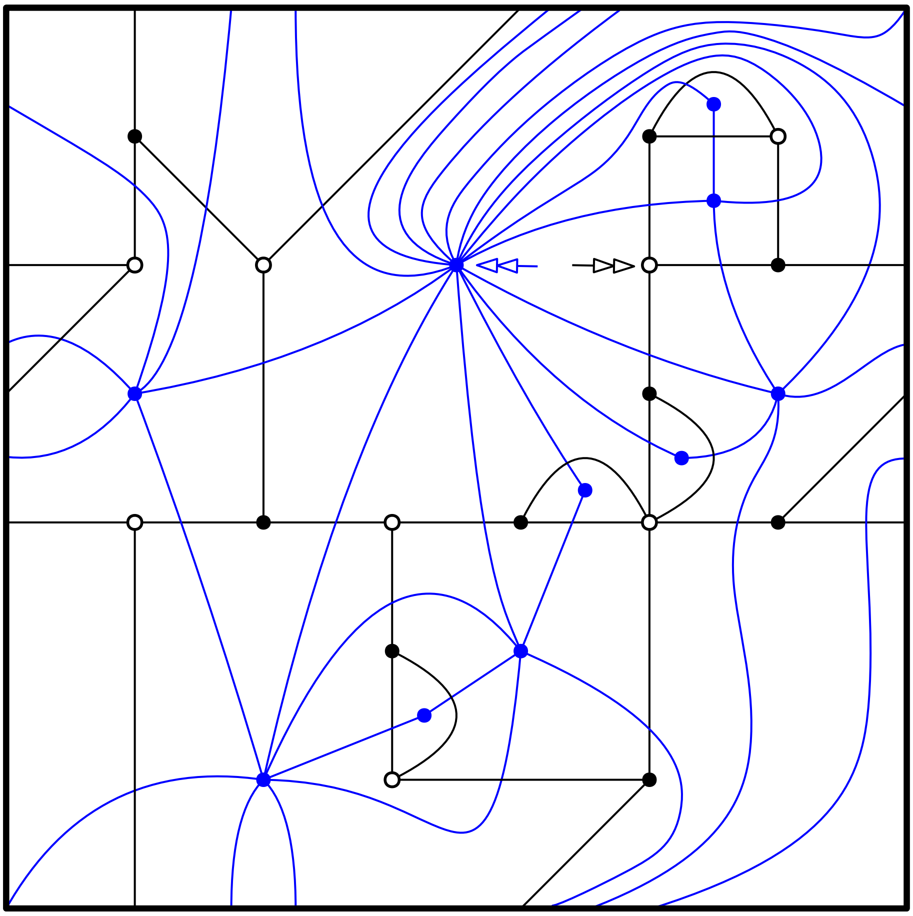


Figure 3.3: A rooted 3-constellation of genus 1 (blue) with its dual rooted 3-Eulerian map (black). Their roots are pointed by the double arrows.

If we only require conditions (i) and (ii) in Definition 3.1, we obtain the definition of an *$m$-hypermap*. The following is a classical result.

**Proposition 3.4.** *Planar $m$-hypermaps are planar $m$-constellations.*

*Proof.* Let $G$ be a planar $m$-hypermap and let us orient its edges by making them turn clockwise around black faces. In order to prove the statement it suffices to show that the lengh of any oriented simple cycle is a multiple of $m$, since then we can fix the label of any vertex and the other labels are deduced without contradicting condition (iii) in Definition 3.1.

Let us consider a oriented simple cycle of $G$. Since $G$ is planar, this cycle separates the sphere into two contractible connected components. In one of these connected components, all the faces adjacent to the cycle are black. Since every edge belongs to a black and a white face, the length of the cycle is equal to the sum of the degrees of black faces minus the sum of the degrees of the white faces in this component. This total is a multiple of $m$ because all faces have a degree multiple of $m$. ☐

**Definition 3.5.** Consider a rooted $m$-constellation. The *canonical orientation* of its edges is the orientation for which its edges turn clockwise around black faces. When endowed with this canonical orientation, the *canonical labelling* (Figure 3.4) of its vertices is obtained by labelling every vertex with the length of the shortest oriented path to it from the root vertex.

This orientation and labelling was introduced by Bouttier, Di Francesco and Guitter in [4] for planar constellations to define what is now known as the BGD bijection.

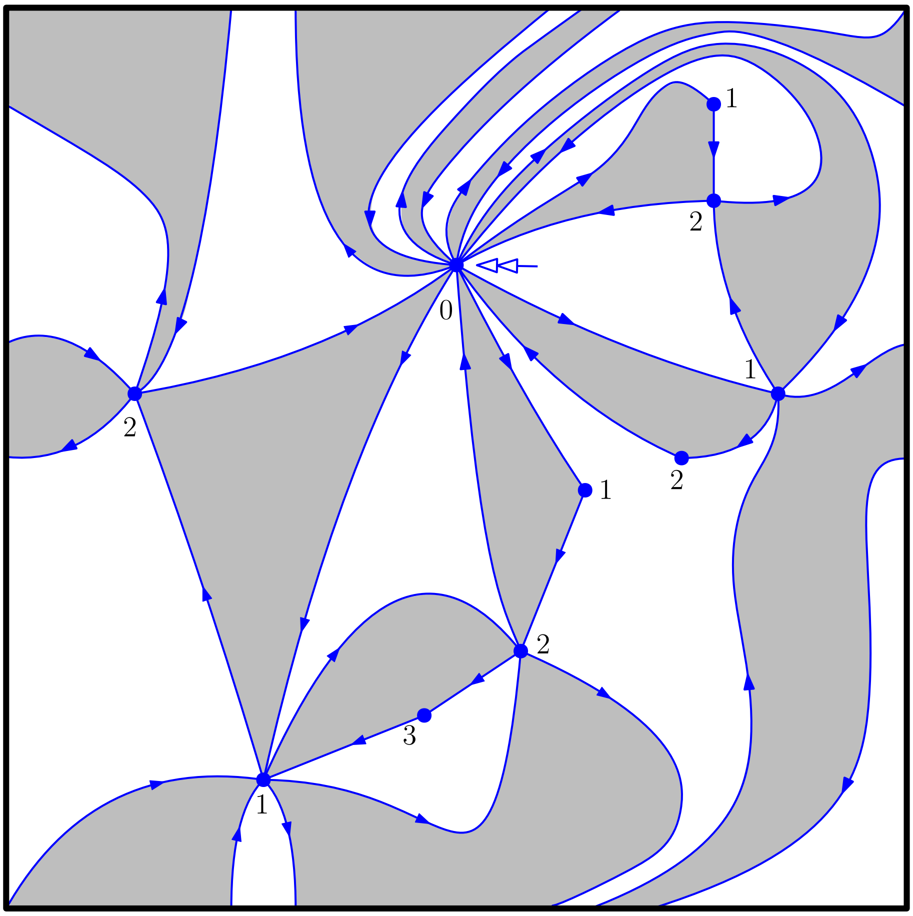


Figure 3.4: A rooted 3-constellation of genus 1 endowed with its canonical orientation and labelling. The root is pointed by the double arrow.

**Proposition 3.6.** *The canonical orientation of a rooted $m$-constellation is accessible and, thus, its canonical labelling is well defined.*

*Proof.* Consider any (non oriented) path on the map. Every backwards edge along the path can be replaced by the complementary edges on its black face so that the resulting path is oriented. ☐

The canonical labelling of the faces of an $m$-Eulerian map is the canonical labelling of the vertices of its dual constellation.

**Proposition 3.7.** *In a rooted $m$-constellation endowed with its canonical labelling, when turning clockwise around black faces, the labels read $i, i + 1, \ldots, i + m - 1$, for some $i \geq 0$.*

*Proof.* Condition (iii) in Definition 3.1 implies that, if $l_1$ and $l_2$ are the canonical labels of two vertices $v_1$ and $v_2$ joined by an edge oriented towards $v_2$, then $l_1 + 1 \equiv l_2 \pmod{m}$. Therefore, if the smallest canonical label in a black face is $i$, the remaining ones are, at least, $i + 1, i + 2, \ldots, i + m - 1$ when read clockwise. But these values are attained by following the edges around the face and the result follows. ☐

# Chapter 4

# Blossoming unicellular maps

Blossoming bijections were introduced by Schaeffer in [9] to put some classes of maps in bijection with decorated trees. These bijections consist in selecting a canonical spanning tree (or, more generally, a canonical spanning submap) and cut into two half-edges the edges not belonging to it. The resulting map is said to be a blossoming map, which can be *closed* back into the original map. The following definition is similar to the one given by Lepoutre in [8], but not identical.

**Definition 4.1.** A *blossoming map* is a map with *stems* (that can be viewed as half-edges) attached to its vertices. There are two types of stems: *outstems*, which are outgoing stems, and *instems*, which are ingoing stems.

Stems separate corners as if they were edges, which means that they count towards the (total) degree of their vertex. We will use the term *inner degree* when we want to ignore stems, i.e., when we only count the number of incident edges to a vertex (loops are counted twice). Stems also have a right and left side, defined in the same way as we did for edges.

**Definition 4.2.** A *rooted blossoming map* is a blossoming map with a marked instem, which is called the *root (instem)*. The vertex to which the root is attached is called the *root vertex* and the face incident to the root is called the *root face*. The corner on the right side of the root is called the *root corner*.

From now on we focus on blossoming maps that are unicellular. The two following definitions also have similar counterparts in [8].

**Definition 4.3.** The *contour order* of the corners or stems of a rooted unicellular blossoming map is the order determined by the counterclockwise tour around its unique face, starting at the root corner. More specifically, we say that corner (or stem) $k_1$ precedes corner (or stem) $k_2$ in contour order ($k_1 \prec k_2$) if $k_1$ is encountered before $k_2$ in this tour.

Observe that the root corner and the root instem are, respectively, the first corner and the first stem in contour order.

**Definition 4.4.** The *good orientation* of a unicellular blossoming map is the orientation for which every edge is, first, followed backwards and, then, forwards in a tour around the unique face starting at the root corner. Note that it does not matter whether the face is followed clockwise or counterclockwise.

Observe that in the good orientation of a rooted unicellular blossoming map of genus 0 (which is a tree), edges are oriented from a child to its parent.

**Definition 4.5.** Given a rooted unicellular blossoming map which has $m$ more instems than outstems, we can label its corners in the following way.

We make a counterclockwise tour around its unique face starting at the first corner after the root corner. Along this tour, we will visit every corner once and we will label it with the value of a counter that starts at $m-1$, increases by 1 every time we encounter an outstem and decreases by 1 every time we encounter an instem. The result of this procedure is called the *good labelling* of the unicellular blossoming map.

When endowed with its good orientation, the edges of the map have a right and a left side. Stems have one label on their left and one on their right. Edges have two labels on either of their sides, and along each side, both labels are equal. Hence, it makes sense to speak about *the* left and *the* right label of an edge. We say that an edge or stem *increases* (resp. *decreases*) by $d$ if the value of its left label minus the value of its right label is $d$ (resp. $-d$).

Observe that, since there are $m$ more instems than outstems, the last corner to label, which is the root corner, has good label 0. Also note that all stems decrease by 1 except for the root instem, which increases by $m-1$.

It is clear that the good labelling satisfies that, for any vertex, the total increment of the ingoing edges and stems is equal to the total increment of the outgoing ones.

# Chapter 5

# $m$-bipartite unicellular maps

**Definition 5.1.** Let $m \geq 2$. We say that a rooted blossoming plane tree whose vertices are bicolored (black and white) is *$m$-bipartite* (Figure 5.1) if

(i) neighbouring vertices have different colors, instems are attached to white vertices and outstems are attached to black vertices,

(ii) black vertices have total degree $m$,

(iii) white vertices have total degree $mi$ for some integer $i \geq 1$ (which can be different among white vertices),

(iv) black vertices have inner degree 1 or 2,

(v) white vertices of degree $mi$ have exactly $i-1$ neighbours of inner degree 1.

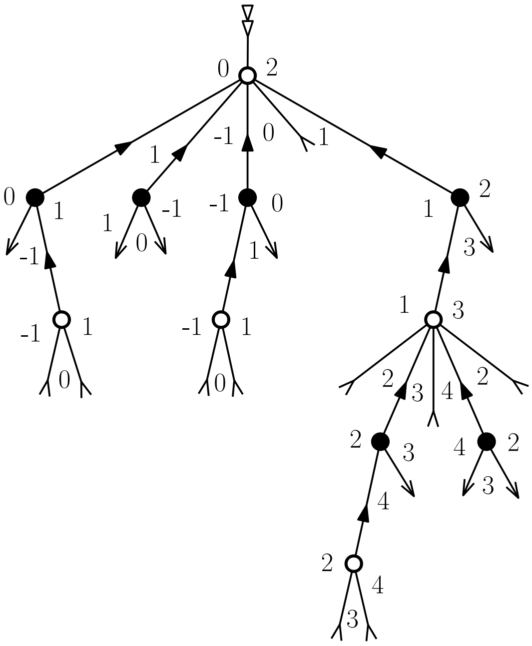


Figure 5.1: A 3-bipartite tree endowed with its good orientation and good labelling. Its root instem is marked with a double arrow.

These objects are defined in [3] by Bousquet-Mélou and Schaeffer with a slightly different terminology. In their article, they name these objects $m$-Eulerian trees, and they construct a bijection between them and planar constellations. In the next definition, we give a generalization of $m$-bipartite trees to higher genus ($m$-bipartite unicellular maps) that we will show to be in bijection with constellations of higher genus. The change of nomenclature is made to avoid name collisions between our objects and the $m$-Eulerian maps defined in Chapter 3.

**Definition 5.2.** Let $m \geq 2$. We say that a rooted unicellular blossoming map with $m$ more instems than outstems and whose vertices are bicolored (black and white) is an *$m$-bipartite unicellular map* (Figure 5.2) if

(i) neighbouring vertices have different colors, instems are attached to white vertices and outstems are attached to black vertices,

(ii) black vertices have degree $m$,

(iii) white vertices have degree $mi$ for some integer $i \geq 1$ (which can be different among white vertices),

and, when endowed with its good labelling,

(iv) the edges whose origin is a black vertex either decrease by 1 or increase by $m-1$,

(v) the edges whose origin is a white vertex decrease by $m-1$.

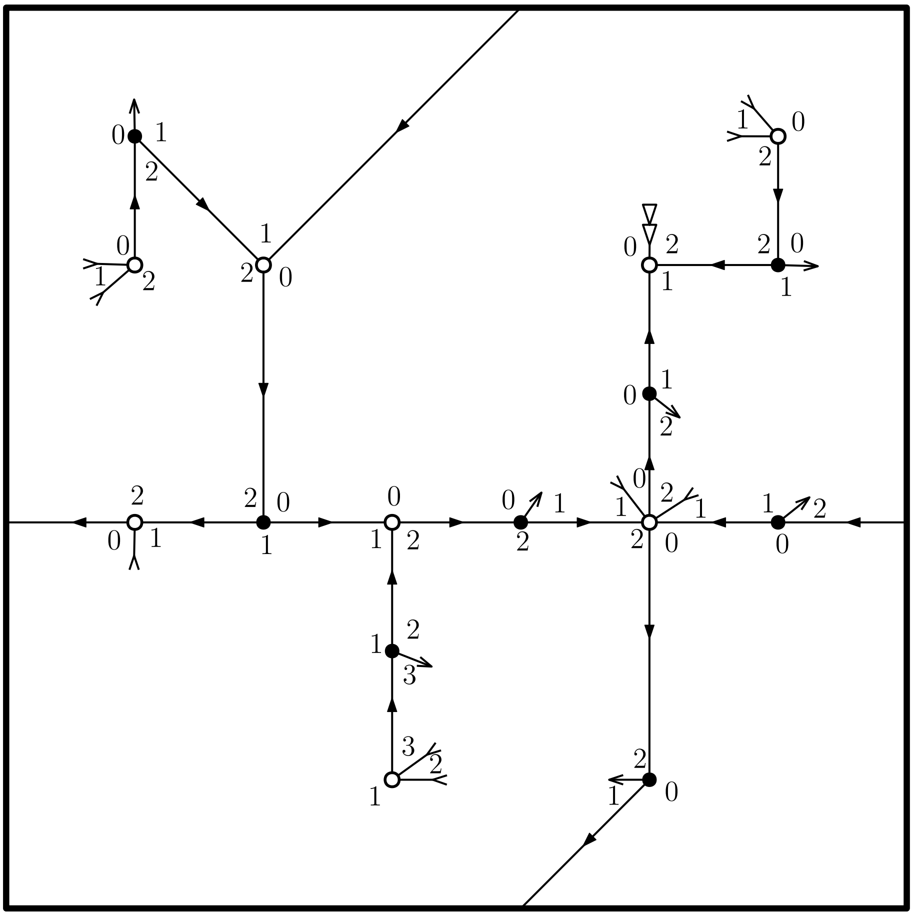


Figure 5.2: A rooted 3-bipartite unicellular map of genus 1 with its good orientation and good labelling. The root is marked with a double arrow.

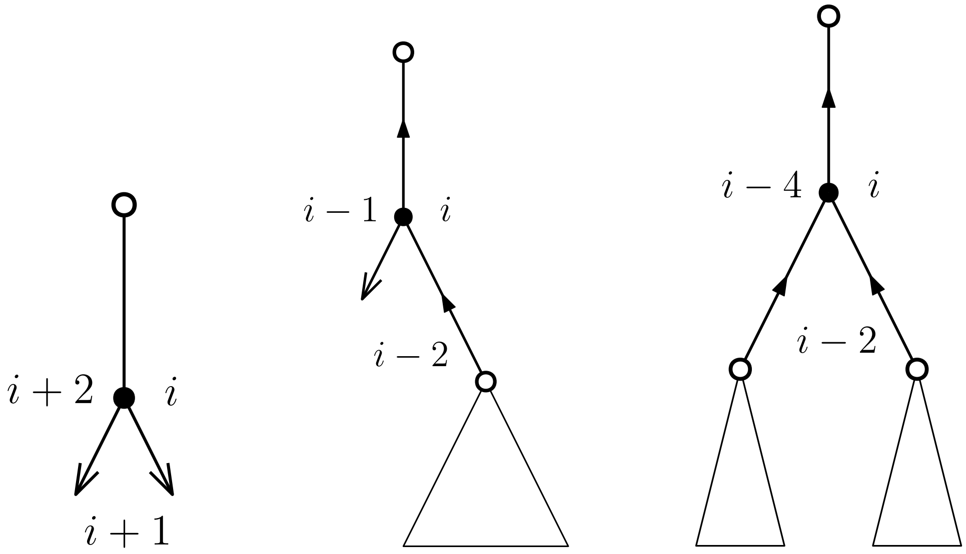
**Proposition 5.3.** *In $m$-bipartite unicellular maps, when turning clockwise around black vertices, their good labels read $i, i+1, \dots, i+m-1$.*

*Proof.* From Definition 5.2 we can see that, if label $l_1$ is followed by label $l_2$ in clockwise order around a black vertex, either $l_2 = l_1 + 1$ or $l_2 = l_1 - m + 1$. Thus, there has to be exactly one jump of $m-1$ and the result follows. □

**Proposition 5.4.** *$m$-bipartite unicellular maps of genus 0 are $m$-bipartite trees and vice-versa.*

*Proof.* Suppose that $U$ is an $m$-bipartite unicellular map of genus 0. Endow it with its good orientation. We have to prove that (iv) and (v) in Definition 5.1 hold.

Consider a black vertex. If its inner degree is $k$, it has $k-1$ white children and $m-k$ outstems. Since the edges oriented from white to black decrease by $m-1$ (condition (v) in Definition 5.2), the single outgoing edge increases by $d=-(k-1)(m-1)+(m-k)=2m-km-1$. Taking into account condition (iv) in Definition 5.2, the only possible values of $k$ are 1 (in which case $d=m-1$) or 2 (in which case $d=-1$), because if $k\geq 3$, then $d\leq -m-1$. Thus, condition (iv) in Definition 5.1 is satisfied. Figure 5.3 displays the three cases.



(a) Case $k=1$. (b) Case $k=2$. (c) Case $k>2$.

Figure 5.3: The three cases for a black vertex when $m=3$.

Now, consider a white vertex of degree $mi$ (Figure 5.4). We know that its single outgoing edge decreases by $m-1$, that its instems decrease by 1 and that every ingoing edge either increases by $m-1$ or decreases by 1. This can only happen if exactly $i-1$ of its ingoing edges increase by $m-1$. As we have just seen, an edge oriented from black to white increases by $m-1$ if and only if the black vertex has inner degree 1. Thus, condition (v) in Definition 5.1 is also satisfied.

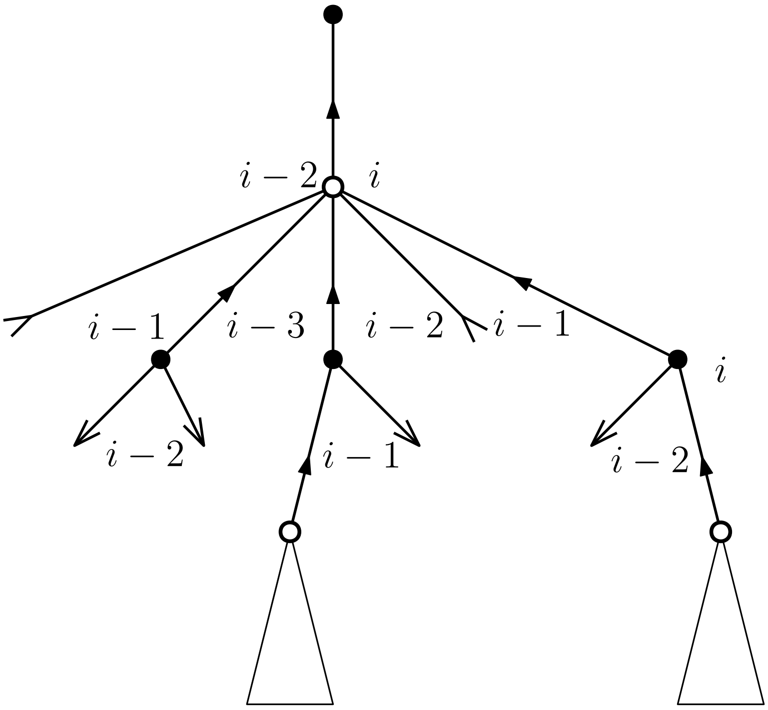


Figure 5.4: The situation of a white vertex of degree 6 ($m = 3$, $i = 2$).

Conversely, suppose that $T$ is an $m$-bipartite tree. Conditions (iv) and (v) in Definition 5.2 have to be proven. $T$ is a unicellular map of genus 0, so we can endow it with its good labelling. We will proceed by induction, from the leaves upwards.

Consider a black vertex. If its inner degree is 1, it has $m-1$ outstems, so its outgoing edge increases by $m-1$. If its inner degree is 2, it has $m-2$ outstems and one ingoing edge that decreases by $m-1$ (by the hypothesis of induction). Therefore, its outgoing edge decreases by 1. Condition (iv) in Definition 5.2 is thus satisfied for this vertex.

Now, consider a white vertex of degree $mi$. By definition, $i-1$ of its children have inner degree 1 and increase by $m-1$ by the hypothesis of induction. Its remaining children have inner degree 2 and decrease by 1 by the hypothesis of induction. Its instems also decrease by 1. Therefore, its outgoing edge decreases by $(m-1)\,i-(i-1)\,(m-1) = m-1$ and condition (v) in Definition 5.2 is satisfied for this vertex. ☐

**Definition 5.5.** Given an $m$-bipartite unicellular blossoming map, consider the cyclic word formed by its stems in their contour order. Outstems are represented by the letter $o$ and instems are represented by the letter $i$. Now we match letters $o$ and $i$ as if they were opening and closing parentheses, respectively. First, every letter $o$ immediately followed by a letter $i$ is *matched* with it. Then, all matched letters are removed and this procedure is repeated until no more matchings are possible. Since there are exactly $m$ more instems than outstems, $m$ instems remain unmatched. We call these instems *single*.

Note that the matching described is the only possible one, since, in a correct parenthesis word, an opening parentheses next to a closing one always have to be matched and can be ignored from that point on.

**Definition 5.6.** An $m$-bipartite unicellular map is *well-rooted* if its root instem is single (Figure 5.5). Equivalently, this means that, if $o$ and $i$ are matched, then $o \prec_{\text{lex}} i$.

This notion of well-rootedness is the same as in [8], and it is analogous to the notion of balance described in [3] for the planar case.

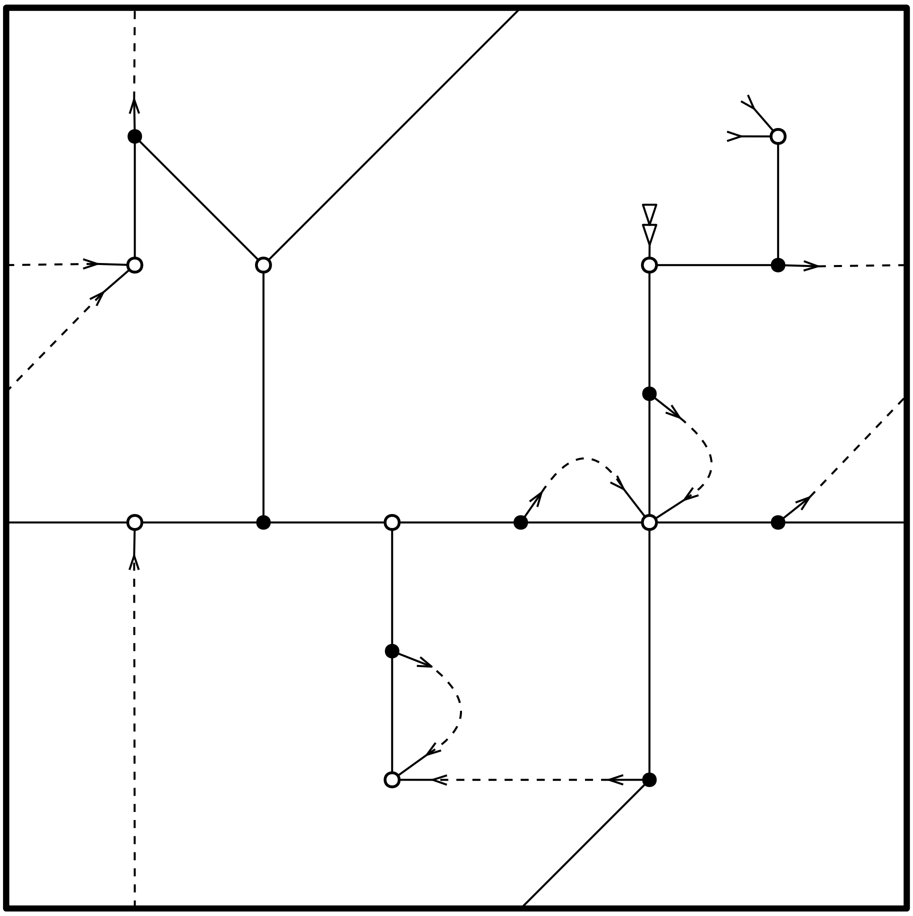


Figure 5.5: A well-rooted 3-bipartite unicellular map of genus 1 with its matching (dashed lines). The root is marked with a double arrow.

**Proposition 5.7.** *An $m$-bipartite unicellular map $U$ is well-rooted if and only if its good labels are non negative.*

*Proof.* Suppose that $U$ is well-rooted. In order for a corner $k$ to have a negative label, there must be, at least, $m$ single non-root instems between the root corner and $k$. But this is impossible because there are only $m - 1$ single instems apart from the root instem.

Conversely, suppose that $U$ is not well-rooted. Then, the root instem is matched with some outstem $b$. Since the label of the root corner is 0, the labels around $b$ are $-1$ and 0. ☐

# Chapter 6

# The bijection between $m$-constellations and $m$-bipartite unicellular maps

This chapter is dedicated to proving our main result:

**Theorem 6.1.** *Rooted $m$-constellations of genus $g$ with $d_i$ white faces of degree $mi$ are in bijection with well-rooted $m$-bipartite unicellular maps of genus $g$ with $d_i$ white vertices of degree $mi$.*

## 6.1 The closure $\Phi$

In this section we describe how a well-rooted $m$-bipartite unicellular map can be closed to obtain an $m$-Eulerian map.

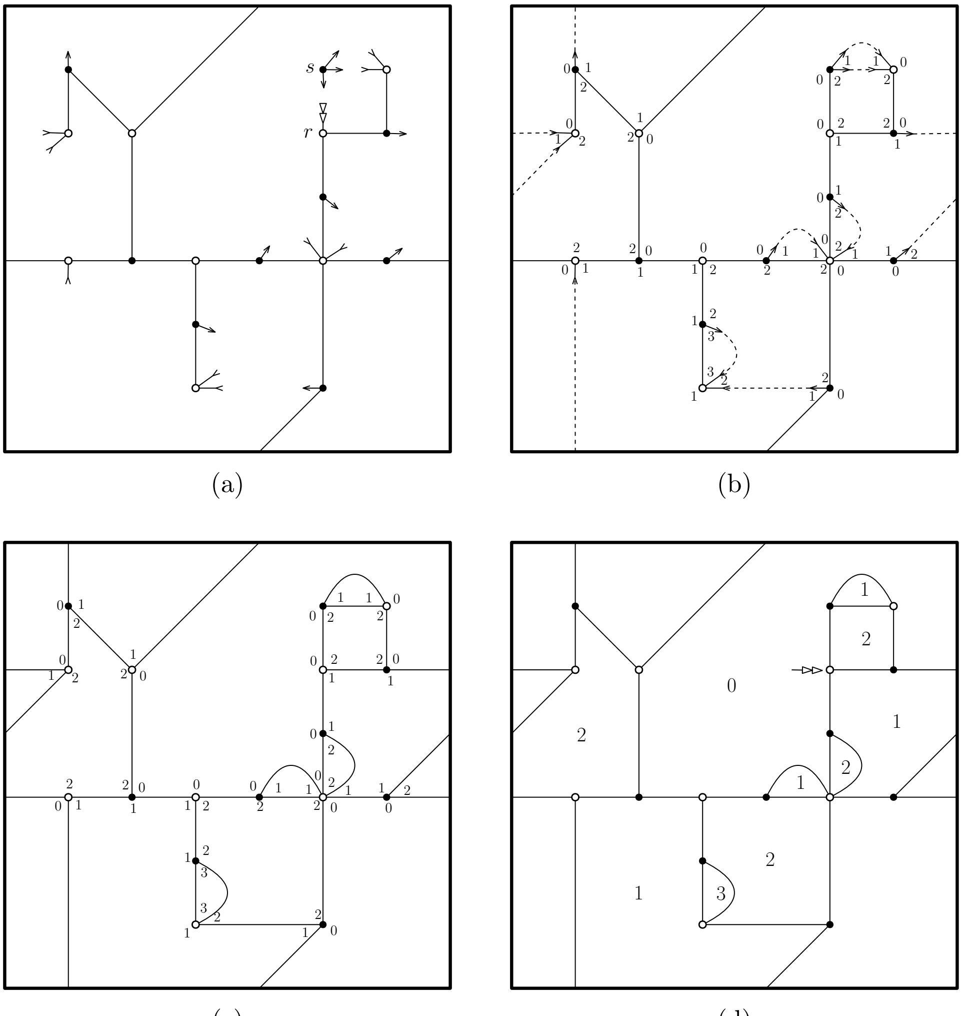
**Definition 6.2.** Let $U$ be a well-rooted $m$-bipartite unicellular map. Let $r$ be its root vertex. We define the *closure* $\Phi(U)$ of $U$ in the following way.

First, place a black vertex $s$ with $m$ outstems attached to it in the unique face and connect one of the outstems to the root instem. The new edge is oriented towards $r$. This intermediate map will be called $O$. The good labelling of the corners of $U$ can be extended to include the newly created corners. This yields a labelling for which all stems decrease by 1.

Now, every pair of matched stems $b, l$ is connected to form a complete edge. The fact that the matched stems of $U$ form a valid parentheses word ensures that these new edges can be drawn without intersections. Moreover, since $U$ was well-rooted, $s$ has not been enclosed by any of these edges.

After this, there are $m-1$ unmatched instems (the original single instems except for the root instem, which no longer exists) and $m-1$ unmatched outstems (the ones attached to $s$). We match them in the only possible way and connect them to form complete edges as we did for the rest. It is clear that this can also be done without intersections, so the final result, $\Phi(U)$, is a map. In fact, the following lemma shows that it is an $m$-Eulerian map. We choose to root it on the same corner as $U$ or, equivalently, on the edge joining $r$ and $s$.

The closure procedure is illustrated in Figure 6.1.



(a) (b)

(c) (d)

Figure 6.1: The closure of a well-rooted 3-bipartite unicellular map.

**Lemma 6.3.** *The closure* $\Phi(U)$ *of a well-rooted* $m$*-bipartite unicellular map* $U$ *of genus* $g$ *with* $d_i$ *white vertices of degree* $mi$ *is a rooted* $m$*-Eulerian map of genus* $g$ *with* $d_i$ *white vertices of degree* $mi$*. Moreover, the good labelling of the corners of* $U$ *corresponds to the canonical labelling of the faces of* $\Phi(U)$*.*

*Proof.* It is clear that the white vertices in $\Phi(U)$ correspond to the white vertices in $U$ and that their degrees are preserved. The black vertices in

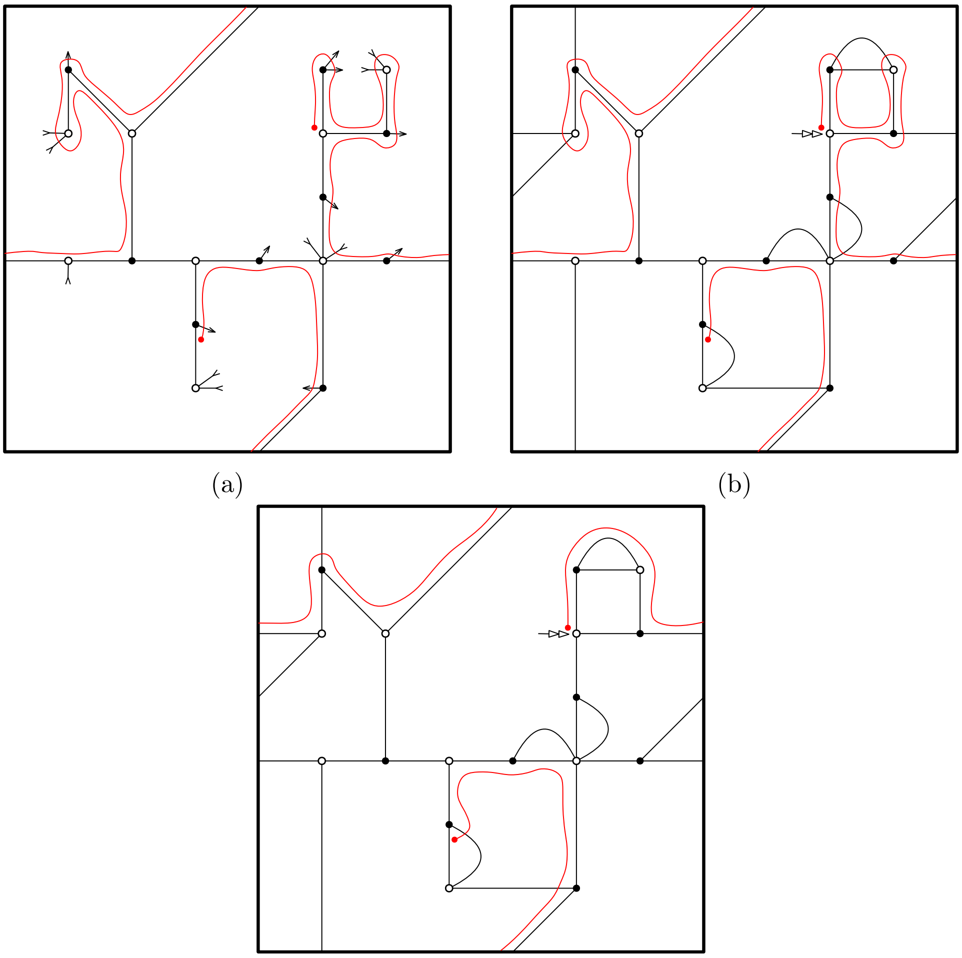
$\Phi(U)$ correspond to either the black vertices in $U$ or to $s$, and all of them have degree $m$. It is also clear that neighbouring vertices in $\Phi(U)$ have different colors. The genus of the map is preserved because all we do is divide a contractible face into multiple faces.

In $U$, the good labels of two corners adjacent along an edge coincide and this still holds after connecting two matched stems. Hence, in any face of $\Phi(U)$, all corners have the same good label. Proposition 5.3 implies that (the dual of) condition (iii) in Definition 3.1 holds. We conclude that $\Phi(U)$ is a rooted $m$-Eulerian map.

We now prove the second part of the lemma. Let $f$ be a face of $\Phi(U)$. We claim that the canonical label of $f$ in $\Phi(U)$ is larger or equal to the label of $f$ induced by the good labelling of $U$. Consider any copath in $\Phi(U)$ from the root face to $f$ that only crosses edges in clockwise sense around black vertices. If the faces of $\Phi(U)$ have the induced good labelling, then, after crossing an edge, we either increase the label by 1 or decrease it by $m-1$. Thus, the canonical label of $f$ is at least as large as its good label.

Let $k$ be a corner of $f$ in $U$. We now claim that the canonical label of $f$ in $\Phi(U)$ is not greater than the good label of $k$ in $U$. Consider in $O$ a path that follows the face counterclockwise from the root corner of $U$ to $k$ (Figure 6.2a) and imagine this path drawn on $\Phi(U)$ (Figure 6.2b). Along this path in $O$, we may encounter some outstems and instems. Since $U$ is well-rooted, it is clear the we will encounter at least as many outstems as instems. In particular, the instems we encounter are matched with previous outstems along the path. Crossing an outstem and, then, its matched instem corresponds to entering and exiting a face of $\Phi(U)$ through the same edge. Therefore, the path drawn on $\Phi(U)$ can be modified not to enter the faces that we would exit afterwards (Figure 6.2c). The resulting path crosses as many edges in $\Phi(U)$ as the difference between crossed outstems and instems in $U$. These edges are crossed clockwise around their black vertex, so the claim follows.

□



(a)

(b)

(c)

Figure 6.2

## 6.2 The opening $\Psi$

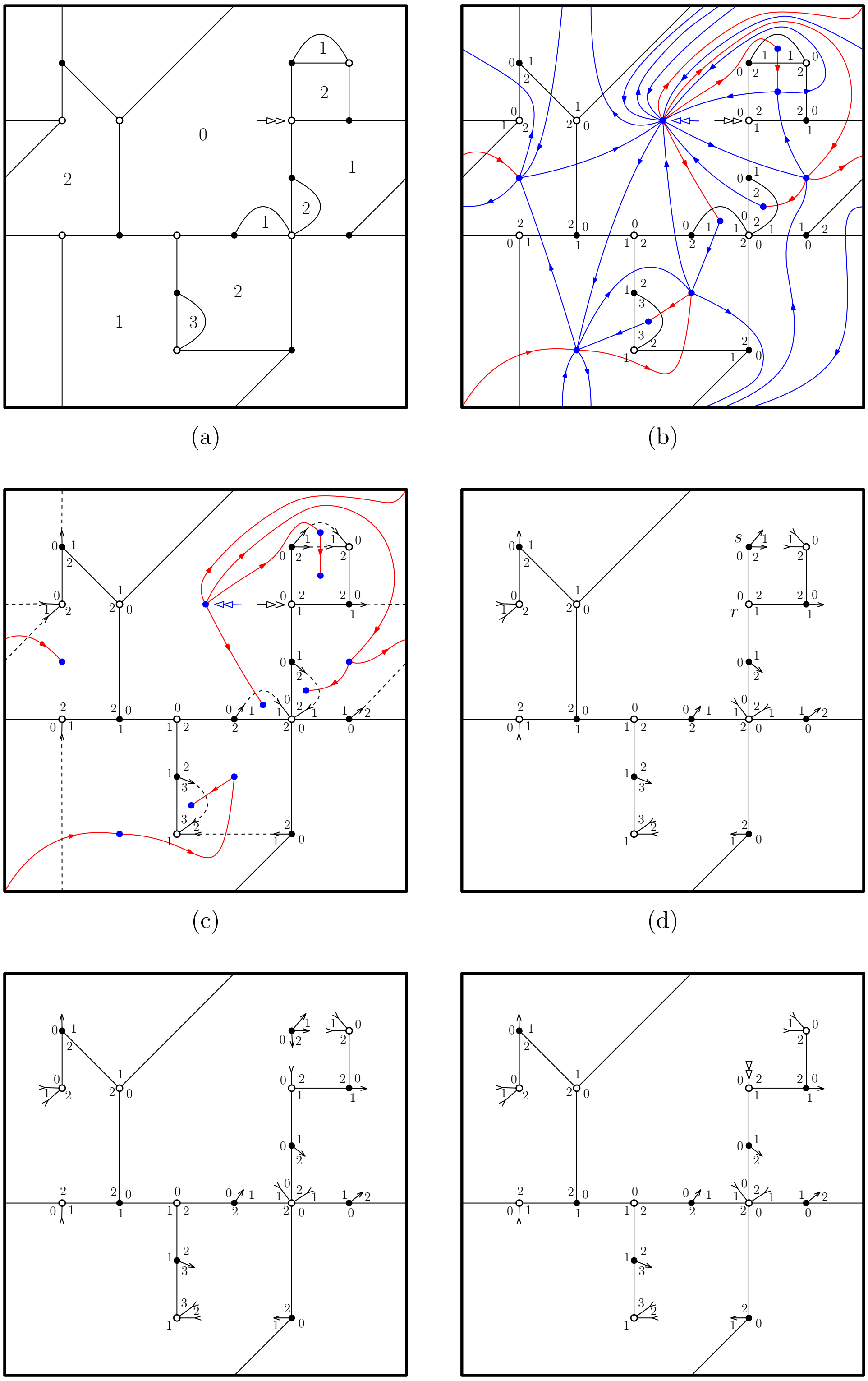
In the previous section we obtained a rooted $m$-Eulerian map starting from a well-rooted $m$-bipartite unicellular map by adding a black vertex with $m$ outstems and connecting pairs of stems. Here, we would like to do the inverse transformation, that is, starting from a rooted $m$-Eulerian map, cut some of its edges into stems so that the result is a well-rooted $m$-bipartite unicellular map and a disconnected black vertex with $m$ outstems.

**Definition 6.4.** Let $M$ be a rooted $m$-Eulerian map. We define its *opening* $\Psi(M)$ in the following way. First, consider the dual map $C$ of $M$, which is a rooted $m$-constellation. Endow $C$ with its canonical orientation and labelling and take its leftmost BFS exploration tree, $T$. For every edge of $M$

whose dual belongs to $T$, cut it into two stems: an instem attached to the white vertex and an outstem attached to the black one.

Observe that the dual of the root edge of $M$ is oriented towards the root vertex of $C$. This implies that its origin has label $m-1$ and that its destination has label 0. Therefore, after this procedure, the root edge remains uncut and, as we prove in the following lemma, it joins the root vertex and a black vertex $s$ with $m-1$ outstems attached to it. The last step is to cut the root edge and remove $s$.

We root the result of this, $\Psi(M)$, at the instem created when cutting the root edge.



(a) (b) (c) (d) (e) (f)

Figure 6.3: The opening of a rooted 3-Eulerian map.

**Lemma 6.5.** *The opening* $\Psi(M)$ *of a rooted* $m$*-Eulerian map* $M$ *of genus* $g$ *with* $d_i$ *white vertices of degree* $mi$ *is a well-rooted* $m$*-bipartite unicellular map of genus* $g$ *with* $d_i$ *white vertices of degree* $mi$*. Moreover, the canonical labelling of* $M$ *corresponds to the good labelling of* $\Psi(M)$ *and*

$$\Phi(\Psi(M)) = M.$$

*Proof.* Let $T$ be the leftmost BFS exploration tree of $C = M^*$ endowed with its canonical orientation. We call $O$ the result of opening $M$ through $T$, that is, cutting every edge of $M$ whose dual belongs to $T$ into two stems. We claim that $O$ is a unicellular map. Indeed, it has a single face because $T$ is connected and visits all vertices, and this face is contractible because $T$ is a tree.

From now on, let $r$ be the root vertex of $M$ and let $s$ be the vertex connected to the root through the root edge. As $T$ is a leftmost BFS tree, it cuts all edges incident to $s$ in $M$ except for the root edge. Thus, in $O$, $s$ is only incident to the root edge and we can cut it to obtain $\Psi(M)$ and a disconnected black vertex with $m$ outstems. $\Psi(M)$ is a unicellular blossoming map rooted on the instem created when cutting the rood edge of $M$. We now show that it is $m$-bipartite.

The canonical labelling of vertices in $C$ gives a canonical labelling of faces in $M$. We can label every corner of $\Psi(M)$ with the label of its face in $M$. We claim that this labelling coincides with the good labelling of $\Psi(M)$. Indeed, every edge in $T$ goes from a vertex with label $i$ to a vertex with label $i+1$ and has a black face on its right side. Thus, its dual in $M$ has a face with label $i$ on its left side and a face with label $i+1$ on its right side when oriented towards its black end. This is why all stems different from the root decrease by 1. Moreover, the root corner of $\Psi(M)$ was the root corner of $M$ and, thus, has label 0.

We now claim that $\Psi(M)$ satisfies conditions (iv) and (v) in Definition 5.2. Indeed, (iv) holds due to Proposition 3.7 and Lemma 6.6 ensures that (v) holds too.

$\Psi(M)$ has genus $g$ because it is unicellular on the same surface as $M$, the degrees of white vertices are preserved, two adjacent vertices have different colors and outstems are attached to black vertices, while instems are attached to white ones. Therefore, $\Psi(M)$ is an $m$-bipartite unicellular map of genus $g$ with $d_i$ white vertices of degree $mi$. Moreover, since the canonical labelling of $C$ is non negative, it follows from Proposition 5.7 that $\Psi(M)$ is well-rooted.

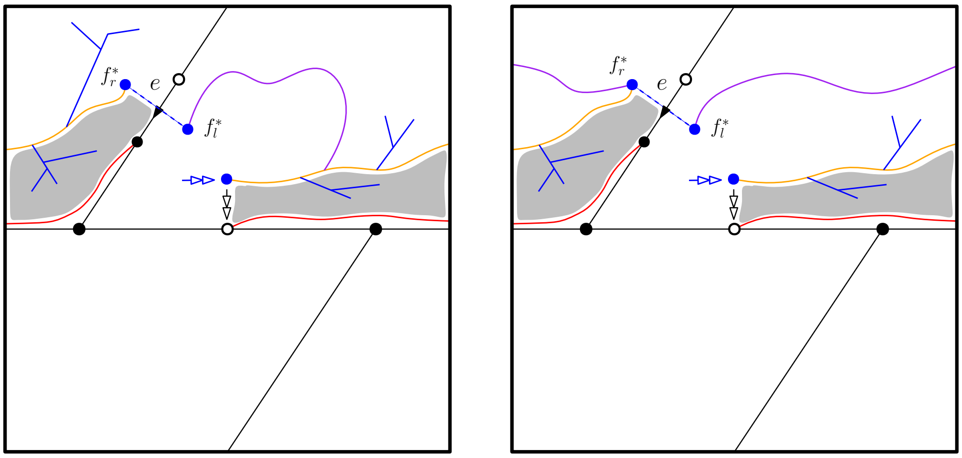
Finally, the fact that the stems of $\Psi(M)$ have only one possible matching, which must correspond to the edges in $M$ from which they were created, implies that $\Phi(\Psi(M)) = M$. $\square$

**Lemma 6.6.** *Let* $M$ *be a rooted* $m$*-Eulerian map, let* $M^*$ *be its dual* $m$*-constellation endowed with its canonical orientation and canonical labelling,* *let $T$ be any BFS exploration tree of $M^*$ and let $O$ be the unicellular blossoming map obtained by cutting the edges of $M$ whose dual belongs to $T$. We root $O$ in the same corner as $M$ and endow it with its good orientation. Then, $T$ is the leftmost BFS exploration tree of $M^*$ if, and only if $O$ has no edges that increase by* 1.

*Proof.* Suppose that there is an edge $e^*$ in $O$ that increases by 1. Let $f_l$ and $f_r$ be the faces on the left and right side of $e^*$ in $M$, respectively. Since $e^*$ has a good orientation, a counterclockwise tour around the face of $O$ starting from its root corner visits, first, the right side of $e^*$ and, then, its left side. Let $Q$ be the section of this counterclockwise tour that ends in the right side of $e^*$. Let $P_l$ and $P_r$ be the paths from the root of $T$ to $f_l^*$ and $f_r^*$, respectively. $P_r$ and $Q$ form a "closed region", the interior of which we define to be on the right side of $e^*$ (Figure 6.4). Since $f_l$ is outside this region, $P_l$ cannot go inside of it. If it did, it would have to cross $Q$ to get out of it, but this is impossible because the duals of the edges of $O$ do not belong to $T$. Since the length of $P_l$ is one unit larger than the one of $P_r$, there are two possibilities:

a) Either $P_l$ deviates to the right of $P_r$ at some point (Figure 6.4a), or

b) $P_l$ is the result of appending an extra edge to the end of $P_r$ (Figure 6.4b).



(a)

(b)

Figure 6.4: In dashed, $e$; in red, $Q$; in orange, $P_r$; in purple, $P_l$; in blue, the rest of $T$. The shaded area is the "closed region".

In both cases, $e$ is a bad edge and we conclude, by proposition 2.16, that $T$ is not the leftmost BFS tree.

Conversely, suppose that $T$ is not the leftmost BFS exploration tree of $M^*$. From Proposition 2.16, we know that there exists a bad edge $e$ going from a vertex $u$ to a vertex $v$. Let $Q$ be the section of the counterclockwise tour around the unique face of $O$ that ends in the side of $e^*$ that belongs to $u^*$. Let $P_u$ and $P_v$ be the paths from the root of $T$ to $u$ and $v$, respectively. By the same argument as before, $P_u$ and $Q$ form a "closed region" in which $P_v$ cannot enter. Therefore, the tour around the unique face of $O$ visits, first, the side of $e^*$ that belongs to $u^*$ and, then, the side that belongs to $v^*$. We conclude $e^*$ is an edge oriented from white to black that increases by 1.

□

## 6.3 The complete bijection

So far we have shown that the closure of a well-rooted $m$-bipartite unicellular map is a rooted $m$-Eulerian map and that the opening of a rooted $m$-Eulerian map is a well-rooted $m$-bipartite unicellular map whose closure is the original map. To show that $\Phi$ and $\Psi$ are inverse operations and, thus, to prove that well-rooted $m$-bipartite unicellular maps and rooted $m$-Eulerian maps are in bijection, we just need the following lemma.

**Lemma 6.7.** *Let $U$ be a well-rooted $m$-bipartite unicellular map. Then,*

$$\Psi\left(\Phi\left(U\right)\right) = U.$$

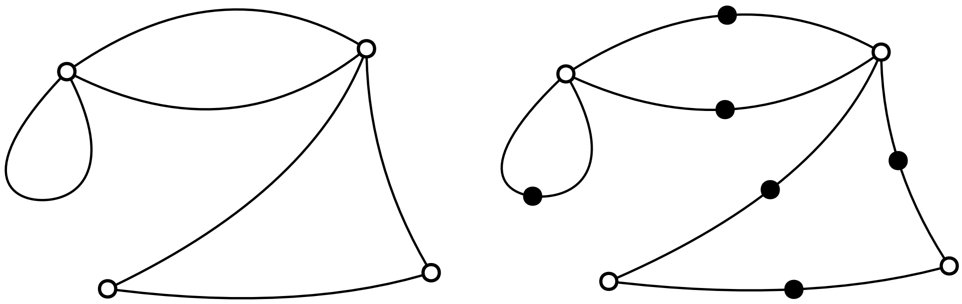
*Proof.* Let $M = \Phi\left(U\right)$, let $C = M^*$ and let $T$ be the tree formed by the edges of $M$ that are not in $U$. Endow $C$ with its canonical labelling and its canonical orientation. Lemma 6.3 tells us that the canonical labelling of $M$ corresponds to the good labelling of $U$. Therefore, $T$ is a BFS exploration tree of $C$. To prove that $\Psi\left(M\right) = U$, we need to show that it is the leftmost BFS tree. By definition, $U$ has no edges that increase by 1, so the result follows from Lemma 6.6. □

We can finally assemble our theorem.

*Proof of Theorem 6.1.* Lemmas 6.3, 6.5 and 6.7 prove the bijection between rooted $m$-Eulerian maps of genus $g$ with $d_i$ white vertices of degree $mi$ and well-rooted $m$-bipartite unicellular maps of genus $g$ with $d_i$ white vertices of degree $mi$. The duality relation between rooted $m$-constellations and rooted $m$-Eulerian maps concludes the proof. □

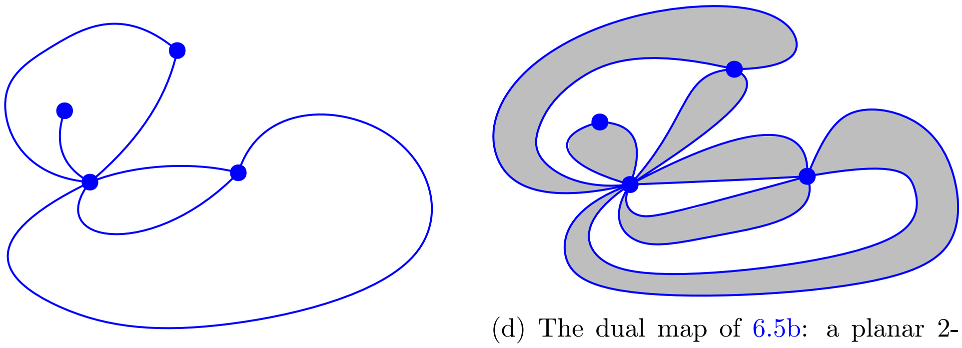
**Remark 6.1.** As shown in Proposition 5.4, the $m$-Eulerian trees described in [3] by Bousquet-Mélou and Schaeffer are the planar instances of the $m$-bipartite unicellular trees we have introduced here. We use the same closing operation as they do, but flipping the orientation of the surface, which amounts to swapping the notions of left-right and clockwise-counterclockwise. Thus, when we restrict our bijection to the sphere, we recover their bijection.

**Remark 6.2.** In [8], Lepoutre gives a bijection between bicolorable maps of arbitrary genus and an adequate family of blossoming unicellular maps. It is easy to convince oneself that bicolorable maps are, in fact, 2-Eulerian maps whose black vertices have been replaced by a single edge connecting their two white neighbours (Figure 6.5). In his bijection, Lepoutre opens bicolorable maps in the following way[1]. First, he takes the dual of the bicolorable map, which is a bipartite map, and endows it with its geodesic orientation. Then, he considers the leftmost BFS exploration tree of this oriented bipartite map and, finally, he cuts all the edges of the bicolorable map whose dual does not belong to the tree. This is essentially the same way in which we open 2-Eulerian maps: the difference between our canonical orientation of 2-constellations (we orient vertices clockwise around black faces) and Lepoutre's geodesic orientation of the bipartite map is explained by the fact that he has collapsed the black faces of the 2-constellation into edges to obtain the bipartite map. Therefore, we can say that our bijection also generalizes the one given by Lepoutre in [8].



(a) A planar Eulerian map.

(b) A planar 2-Eulerian map.



(c) The dual map of 6.5a.

(d) The dual map of 6.5b: a planar 2-constellation.

Figure 6.5: The relation between bicolorable maps and 2-Eulerian maps.

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[1] In his article, the opening is actually defined in a different way, but he shows that ultimately they are the same operation.

# Chapter 7

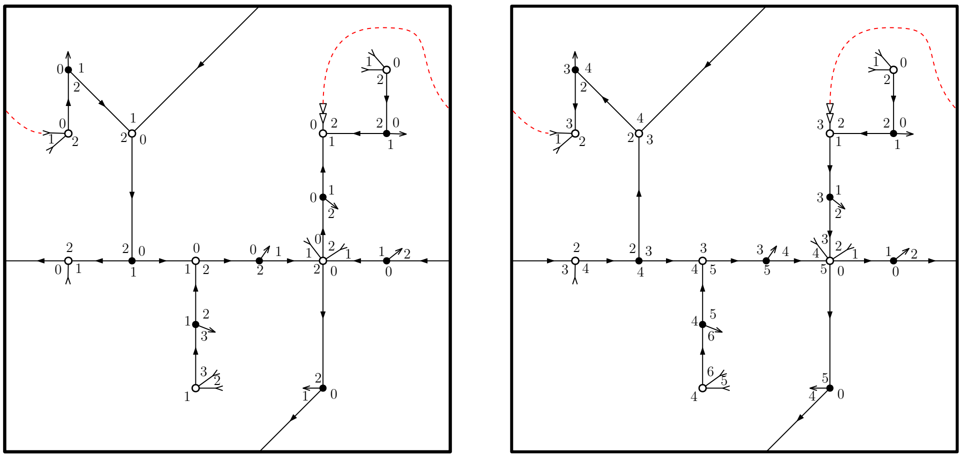
# Rerooting an $m$-bipartite unicellular map

After Theorem 6.1 has been established, one can try to enumerate rooted constellations by enumerating well-rooted $m$-bipartite unicellular maps, which will hopefully be easier to study.

The first problem we run into when trying to count well-rooted $m$-bipartite unicellular maps is precisely the fact that they are well-rooted. As Lepoutre explains in [8], well-rootedness is a global notion, since it requires the positivity of all the good labels of a map. This complicates the task of counting these objects and, thus, we would like to get rid of it. In order to do so, we use the technique of rerooting first introduced in [9] and which was successfully used in [3] and [8]. Specifically, we provide an algorithm to reroot a well-rooted map on any instem, which will later yield an enumerative relation between $m$-bipartite unicellular maps and well-rooted $m$-bipartite unicellular maps. Lemma 7.3 and Proposition 7.2 have analogous counterparts in [8].

**Definition 7.1.** Let $U$ be a well-rooted $m$-bipartite unicellular map, let $r$ be its root and let $t$ be a distinguished instem of $U$. We endow $U$ with its good orientation and its good labelling.

The *rerooting* algorithm is defined as follows (Figure 7.1). If $t = r$, we do nothing. Otherwise, we first join $r$ and $t$ to create an edge. This divides the single face of $U$ into faces $f_L$ and $f_R$, where $f_L$ is the one containing the root corner of $U$. We then add $m$ to all labels of $f_L$ and we reverse the orientation of all the edges that separate $f_L$ and $f_R$. Finally, we cut the edge joining $r$ and $t$ back into two instems and we swap the roles of $r$ and $t$: $t$ becomes the root and $r$ becomes the distinguished instem.



(a) (b)

Figure 7.1: The rerooting algorithm of a 3-regular bipartite unicellular map. $U$ on the left and $U'$ on the right.

We will use this rerooting procedure to prove the following result.

**Proposition 7.2.** *$m$-bipartite unicellular maps with a distinguished single instem are in bijection with well-rooted $m$-bipartite unicellular maps with a distinguished instem.*

**Lemma 7.3.** *Let $U$ be a well-rooted $m$-bipartite unicellular map, let $t$ be an instem of $U$ different from the root $r$ and let $U'$ be the result of rerooting $U$ on $t$. Consider the faces $f_1$ and $f_2$ obtained after joining $r$ and $t$ to create an edge. The good orientation of an edge $e$ is different in $U$ and $U'$ if, and only if, $e$ separates faces $f_1$ and $f_2$.*

*Proof.* The good orientation of an edge $e$ is determined by the order in which a clockwise tour around the unique face visits its sides. Observe that one such tour starting at $r$ visits, first, all the sides of edges belonging to $f_1$ (WLOG) and, then, all the sides of edges belonging to $f_2$. If we start the tour at $t$, it visits, first, all the sides of edges belonging to $f_2$ (but in the same order!) and, then, all the sides of edges belonging to $f_1$ (in the same order as well).

We now distinguish two cases:

- If both sides of $e$ belong to the same face, then they are visited in the same order in both $U$ and $U'$.
- If the sides of $e$ do not belong to the same face, then they are visited in the opposite order in $U$ and $U'$.

□

**Lemma 7.4.** *Let $U$ be a well-rooted $m$-bipartite unicellular map endowed with its good orientation. An edge $e$ oriented from black to white increases by $m-1$ if, and only if the black vertex has $m-1$ outstems.*

*Proof.* If the black vertex has $m-1$ outstems, the edge clearly increases by $m-1$. Conversely, if the black vertex has at most $m-2$ outstems, then it has at least two incident edges. Among these edges, the one travelled to leave the vertex after the first time it is visited in the tour around the unique face is an ingoing edge. Therefore, it decreases by $m-1$ and the remaining edges (including $e$) must decrease by 1. ☐

*Proof of Proposition 7.2.* Let $U$ be a well-rooted $m$-bipartite unicellular map, let $t$ be an instem of $U$ different from the root $r$ and let $U'$ be the result of rerooting $U$ on $t$. We claim that $U'$ is an $m$-bipartite unicellular map with a distinguished single instem, endowed with its good orientation and its (possibly translated[1]) good labelling.

The orientation of $U'$ is good by Lemma 7.3. Around stems, either both labels have increased or remained unchanged, except for $r$, which now decreases by 1, and $t$, which now increases by $m-1$. The only edges for which a single side has changed its label are the ones separating faces $f_L$ and $f_R$. Let $e$ be any such edge and let us study two cases depending on its orientation.

- If, in $U$, $e$ was oriented from white to black, it decreased by $m-1$ by definition. We also know that its left side belonged to $f_L$. Therefore, in $U'$, $e$ is oriented from black to white and decreases by 1.

- If, in $U$, $e$ was oriented from black to white, we claim that it decreased by 1. Indeed, in $U$ there are no black vertices that belong to both faces $f_L$ and $f_R$ with $m-1$ outstems, and by Lemma 7.4 the claim follows. We also know that its left side belonged to $f_L$. Therefore, in $U'$, $e$ is oriented from white to black and decreases by $m-1$.

We conclude that the labelling is the (possibly translated) good labelling. Finally, $r$ is single because $U$ was well-rooted. ☐

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[1] By this we mean that all labels differ from the good ones by a fixed amount.

# Chapter 8

# Enumeration of bipartite 3-face-colorable cubic maps on the torus

In this chapter, we focus our study on bipartite 3-face-colorable cubic maps of genus 1, which are 3-Eulerian maps of genus 1 whose white vertices all have degree 3. This is a very particular case compared to the general $m$-constellations of arbitrary genus for which we have built a bijection, but it allows for relatively simple calculations that can be done by hand.

## 8.1 $m$-Eulerian $m$-regular maps

Before fixing $m = 3$, let us study rooted $m$-Eulerian $m$-regular maps of arbitrary genus whose white vertices have the smallest possible degree. These maps can be equivalently described as rooted bipartite $m$-regular maps whose faces can be labelled with integers in $\{1, 2, \ldots, m\}$ in such a way that, when turning clockwise around any black vertex, the labels of the faces read $1, 2, \ldots, m$. In the case $m = 3$, this last condition can be formulated by saying that the map is 3-face-colorable.

The image of this family of maps through the opening algorithm is the family of well-rooted $m$-bipartite unicellular map of arbitrary genus whose white vertices have degree $m$. Consider one of these maps and endow it with its good orientation and good labelling. It is clear that every vertex has, at least, one edge departing from it, the edge travelled to visit it for the first time in the face tour. In particular, white vertices have, at least, one such edge. But all edges departing from a white vertex decrease by $m - 1$, which implies that their remaining $m - 1$ edges are oriented towards it and decrease by 1. This means, in fact, that all edges oriented from black to white decrease by 1. Therefore, taking into account the stems, all black vertices have outdegree $m-1$ and indegree 1 [1], while all white vertices have outdegree 1 and indegree $m-1$, except for the (white) root vertex, which has indegree $m$ because the root instem increases by $m-1$.

When this map is closed, we recover our rooted $m$-Eulerian $m$-regular map. Every edge in the closed map inherits an orientation from the unicellular map, either its good orientation if already existed or the orientation of the stems that were united to create it in the other case. All white vertices have outdegree 1 and all black vertices have indegree 1 except for the (white) root vertex, which has outdegree 0, and the black star that was added during the closing algorithm, which has indegree 0. If we swap the edge between the two, which is the edge created from the root instem in the unicellular map, we obtain an $\alpha$-orientation of the map: all white vertices have outdegree 1 and all black vertices have indegree 1.

## 8.2 Opening and rerooting

From here on, we fix $m=3$ and we restrict ourselves to the torus.

Let $\mathcal{C}$ be the class of rooted bipartite 3-face-colorable cubic maps of genus 1 and let $\mathcal{O}$ be the class of well-rooted 3-bipartite unicellular maps of genus 1 whose white vertices have degree 3 . The maps in $\mathcal{C}$ are counted by their number of white vertices and the maps in $\mathcal{O}$ are counted by their number of instems.

**Lemma 8.1.** *The number of instems of a map $o \in \mathcal{O}$ is equal to the number of white vertices of its closure $c \in \mathcal{C}$.*

*Proof.* Let $v, e$ and $f$ be the number of vertices, edges and faces of $c$, respectively. By Euler's formula, we know that

$$v - e + f = 2 - 2g = 0.$$

Since $c$ is a cubic map, $e = \frac{3}{2}v$. By substitution we get that

$$f = \frac{3}{2}v - v = \frac{1}{2}v.$$

As $c$ is also bipartite, it has as many white vertices as black ones. Therefore, it also has as many white vertices as faces.

When closing $o$, a new face will be created for each of its instems different from the root instem. The result follows. $\square$

An immediate consequence of this is that $\mathcal{C}$ and $\mathcal{O}$ have the same generating functions: $C(z) = O(z)$.

---

[1] By outdegree (resp. indegree) we mean the number of outgoing (resp. ingoing) stems and edges connected to a vertex

Let $\mathcal{U}$ be the class of 3-bipartite unicellular maps of genus 1 whose white vertices have degree 3 counted by their number of instems different from the root. The rerooting procedure between $\mathcal{O}$ and $\mathcal{U}$ yields the following result.

**Lemma 8.2.** *The generating functions of $\mathcal{O}$ and $\mathcal{U}$ satisfy the relation*

$$O(t) = 3\int_0^t U(z)\,dz.$$

*Proof.* This is direct from 7.2. ☐

## 8.3 The pruned maps and their enumeration

We follow the framework introduced by Chapuy, Marcus and Schaeffer in [6] to study unicellular maps.

**Definition 8.3.** The *extended scheme* of a map $u \in \mathcal{U}$ is the map obtained by, first, removing all its stems and, then, iteratively removing all its vertices of degree 1.

This procedure only removes stems and treelike parts from the map, so an extended map is also a unicellular map. In fact, any map $u \in \mathcal{U}$ can be decomposed into an extended scheme and some attached stems and treelike parts.

An extended scheme can only have vertices of degree 2, which we call *branch vertices*, and vertices of degree 3, which we call *scheme vertices*. The branch vertex to which the root or the treelike part containing the root is attached is called *root branch vertex*. Let $v_2$ and $v_3$ be the number of branch vertices and scheme vertices of any extended scheme, respectively. Euler's formula yields

$$\begin{gathered} v_2 + v_3 - e + f = 2 - 2g \\ v_2 + v_3 - \frac{1}{2}(2v_2 + 3v_3) + 1 = 0 \\ v_3 = 2. \end{gathered}$$

Therefore, these extended schemes are formed by two scheme vertices between which there are three branches that can contain many branch vertices. The treelike parts can only be attached to white branch vertices. Indeed, in the original map $u$ endowed with its good orientation, a vertex with an attached tree always has indegree 2, regardless of whether the tree contains the root or not. Moreover, if we keep the good orientation of $u$ on its extended scheme, all branch vertices must have indegree and outdegree 1 except for the root branch vertex, which has indegree 2 and outdegree 0. Since the sum of indegrees and the sum of outdegrees of the map are equal, the two scheme vertices have to be black.

Let $\mathcal{T}$ be the class of these attachable treelike parts, counted by their number of instems. For the sake of simplicity, we will consider that a single instem is a treelike part and belongs to $\mathcal{T}$.

**Lemma 8.4.** *The generating function of $\mathcal{T}$ satisfies the following recursive relation:*

$$T(z) = z + 2T(z)^2 .$$

*Proof.* Since a map $u \in \mathcal{U}$ can be rerooted and this only changes its good orientation and good labels, we can assume without loss of generality that the treelike parts do not contain the root. The following diagram illustrates the three possibilities that produce the recurrence relation.

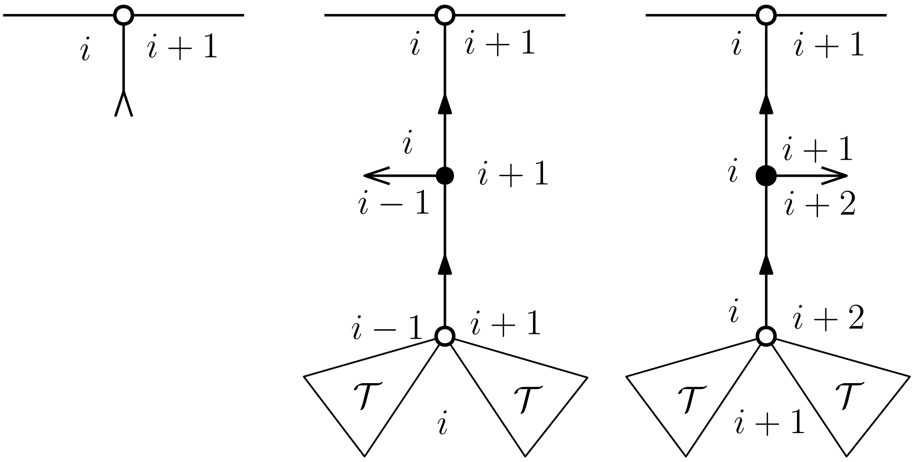

Figure 8.1: Structure of a treelike part.

The treelike part either is an instem or has a black vertex with an outstem attached to it (two possibilities) followed by a white vertex with two treelike parts attached to it. ☐

**Definition 8.5.** Let $u \in \mathcal{U}$ be a 3-bipartite unicellular map whose white vertices have degree 3. Its *pruned map $p$* is obtained by replacing all its treelike parts by stems. The treelike part containing the root is replaced by a root stem. Let $\mathcal{P}$ be the class obtained by pruning every map in $\mathcal{U}$. The pruned maps of $\mathcal{P}$ are counted by their number of instems different from the root.

Observe that, if we keep the good labels on the pruned map, the rules of the labelling still aply. In other words, $\mathcal{P} \subset \mathcal{U}$.

**Lemma 8.6.** *The generating functions of $\mathcal{U}$, $\mathcal{P}$ and $\mathcal{T}$ satisfy the relation*

$$U(z) = \frac{\partial T}{\partial z} P(T(z)) .$$

*Proof.* To recover a map $u \in \mathcal{U}$ from its pruned map $p \in \mathcal{P}$, two steps are needed. First, we replace all instems by treelike parts, obtaining the factor $P(T(z))$. Then, an additional treelike part is attached to the root stem and one of its instems is selected to be the root. This gives the factor $\frac{\partial T}{\partial z}$. □

We would now like to enumerate $\mathcal{P}$.

**Definition 8.7.** The *labelled scheme* of a pruned map $p \in \mathcal{P}$ is obtained by removing all its branch vertices except for the root branch vertex. The good labels of the remaining corners are kept.

It is clear that labelled schemes are uniquely determined by the lowest label on each of its scheme vertices (Figure 8.2). There is, thus, a correspondance between labelled schemes and pairs $(i, j) \in \mathbb{Z}^2$.

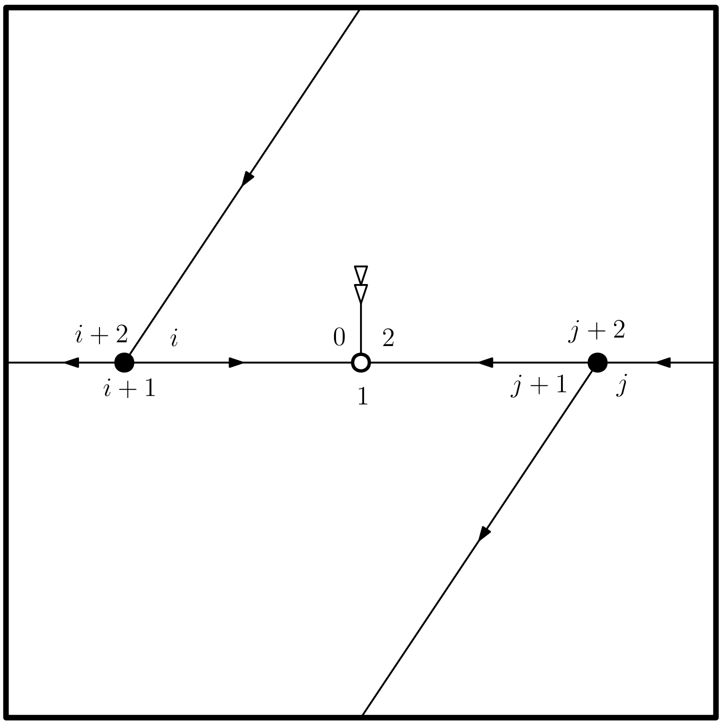

Figure 8.2: A generic labelled scheme.

The labelled scheme associated to the pair $(i, j)$ will be denoted $l_{i,j}$, and the subclass of pruned maps that have $l_{i,j}$ as labelled scheme will be denoted $\mathcal{P}_{i,j}$. Given $(i, j) \in \mathbb{Z}^2$, we want to compute $P_{i,j}(z)$. To do so, we replace every edge of $l_{i,j}$ by a valid branch whose labels agree with the labels of $l_{i,j}$. A branch starts at a black vertex. There are 4 ways to place the stems of the first two vertices, as shown in Figure 8.3.

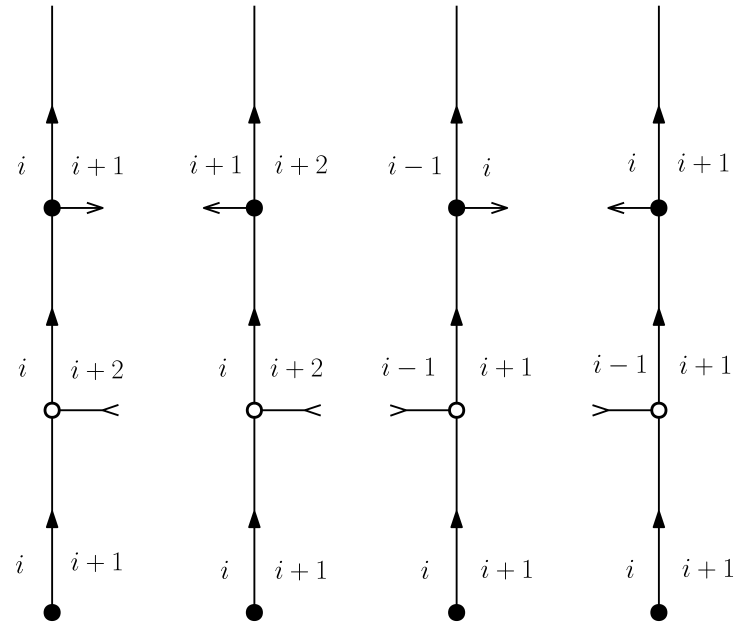

Figure 8.3: The first step of a branch.

The branches that go from a black vertex to a white vertex are made of steps like these. The *height* variation of a step is the difference between its initial left label and its final left label. As shown in Figure 8.3, 2 of these steps remain in the same height, one of them increases its height by 1 and the remaining one decreases it by 1. These steps are known as *weighted Motzkin steps*, and they form *weighted Motzkin paths* when put one after the other. The *height* variation of a weighted Motzkin path is the sum of the variations of its weighted Motzkin steps. A black-to-white edge of the labelled scheme whose initial labels are $(i, i+1)$ and whose final labels are $(j, j+1)$ can be replaced by a weighted Motzkin path that goes from height $i$ to height $j$.

Let $\mathcal{A}$ be the class of weighted Motzkin paths that go from height 0 to height $-1$ and remain non-negative until the last step, counted by their number of instems. Their generating function $A\left(z\right)$ satisfies the following recurrence relation.

$$A = z\left(1 + 2A + A^2\right).$$

Indeed, there are three possibilities:

- We start with a descending step and we are done.
- We start by a step that remains in the same height and then we have an element of $\mathcal{A}$.
- We start with an ascending step and then we have two consecutive elements of $\mathcal{A}$.

We can also write

$$z^{-1} = A^{-1} + 2 + A.$$

Let $\mathcal{B}$ be the class of weighted Motzkin paths that go from height 0 to height 0, counted by their number of instems. Their generating function $B(z)$ satisfies the following recurrence relation.

$$B = 1 + 2zB + 2zAB.$$

This can be shown by a similar argument as before. By isolating $B$ and substituting $z$ we get

$$B = \frac{1}{1 - 2z - 2zA} = \frac{1}{1 - \frac{2(1+A)}{A^{-1}+2+A}} = \frac{A^{-1} + 2 + A}{A^{-1} - A} = \frac{1+A}{1-A}.$$

Let $\mathcal{C}_{i,j}$ be the class of weighted Motzkin paths that go from height $i$ to height $j$, counted by their number of instems. It is clear that their generating function $C_{i,j}(z)$ can be expressed in the following way:

$$C_{i,j} = BA^{|j-i|}.$$

A black-to-black branch in a pruned map is just a black-to-white branch with an additional edge.

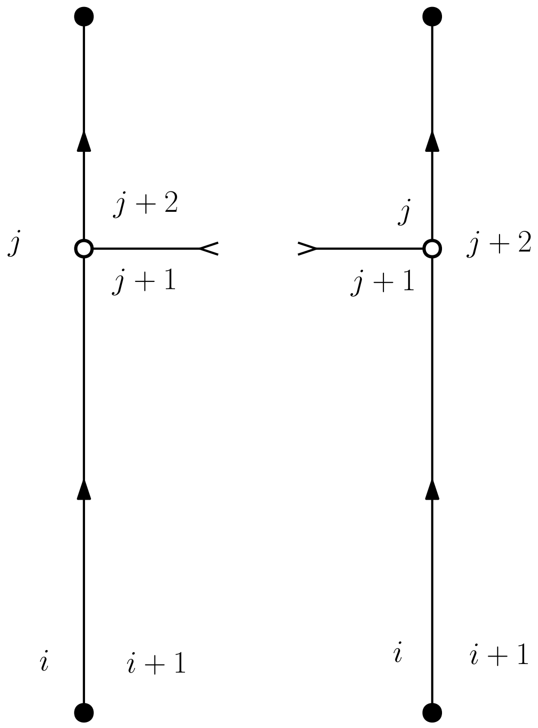

Figure 8.4: A black-to-black branch.

Let $\mathcal{D}_{i,j}$ be the class of black-to-black branches whose initial left label is $i$ and whose final left label is $j$, counted by their number of instems. In the light of Figure 8.4, their generating function $D_{i,j}(z)$ can be expressed in the following way:

$$D_{i,j} = C_{i,j}z + C_{i,j+1}z.$$

Taking into account Figure 8.2, we can now enumerate $\mathcal{P}_{i,j}$. We have that

$$\begin{aligned}
P_{i,j} &= C_{i,0} \cdot C_{j+1,1} \cdot D_{i+1,j} \cdot D_{j,i} \\
&= z^2 B^4 A^{|i|} A^{|j|} \left(A^{|i-j+1|} + A^{|i-j|}\right)\left(A^{|i-j|} + A^{|i-j+1|}\right) \\
&= z^2 B^4 A^{|i|+|j|} \left(A^{|i-j+1|} + A^{|i-j|}\right)^2.
\end{aligned}$$

To enumerate $\mathcal{P}$, we just have to sum for all $i, j \in \mathbb{Z}$.

$$\begin{aligned}
P &= \sum_{i,j\in\mathbb{Z}} P_{i,j} \\
&= \sum_{i,j\in\mathbb{Z}} z^2 B^4 A^{|i|+|j|} \left(A^{|i-j+1|} + A^{|i-j|}\right)^2 \\
&= z^2 B^4 \sum_{i,j\in\mathbb{Z}} A^{|i|+|j|} \left(A^{|i-j+1|} + A^{|i-j|}\right)^2 \\
&\dots \\
&= z^2 B^4 \frac{\left(A^2+1\right)(A+1)^4}{\left(A^3-1\right)^2} \\
&= z^2 \left(\frac{1+A}{1-A}\right)^4 \frac{\left(A^2+1\right)(A+1)^4}{\left(A^2+A+1\right)^2 (A-1)^2} \\
&= z^2 \frac{\left(A^2+1\right)(A+1)^8}{\left(A^2+A+1\right)^2 (A-1)^6} \\
&= z^2 \frac{\left(A^2+1\right)\left(A^2+2A+1\right)^4}{\left(A^2+A+1\right)^2 \left(A^2-2A+1\right)^3} \\
&= z^2 \frac{\left(A+A^{-1}\right)\left(A+2+A^{-1}\right)^4}{\left(A+1+A^{-1}\right)^2 \left(A-2+A^{-1}\right)^3} \\
&= z^2 \frac{\left(z^{-1}-2\right) z^{-4}}{\left(z^{-1}-1\right)^2 \left(z^{-1}-4\right)^3} \\
&= \frac{z^2 (2z-1)}{(z-1)^2 (4z-1)^3}.
\end{aligned}$$

We can finally enumerate $\mathcal{O}$.

$$\begin{aligned}
O\left(t\right) &= 3\int_0^t U\left(z\right)dz && \text{by Lemma 8.2}\\
&= 3\int_0^t \frac{\partial T}{\partial z} P\left(T\left(z\right)\right)dz && \text{by Lemma 8.6}\\
&= 3\left(\int_0^z P\left(\eta\right)d\eta\right)\left(T\left(t\right)\right)\\
&= \left(\frac{z^3}{\left(1-z\right)\left(1-4z\right)^2}\right)\left(T\left(t\right)\right).
\end{aligned}$$

Putting everything together, we get the following theorem.

**Theorem 8.8.** *Bipartite 3-face-colorable cubic maps of genus 1 are enumerated by*

$$C\left(z\right) = \frac{T\left(z\right)^3}{\left(1-T\left(z\right)\right)\left(1-4T\left(z\right)\right)^2},$$

*where $T\left(z\right)$ is the unique generating function satisfying $T\left(z\right) = z + 2T\left(z\right)^2$. In particular, $C\left(z\right)$ is a rational function of $T\left(z\right)$.*

# Chapter 9

# Future work

In view of Theorem 8.8, we can formulate the following conjecture.

**Conjecture.** Bipartite 3-face-colorable cubic maps of arbitrary genus are enumerated by a generating function which is a rational function of $T(z)$.

Since blossoming bijections in [8] produce enumerative results in which there is scheme by scheme rationality, we hope that will also be the case for these maps.

Besides extending Theorem 8.8 to higher genus, another line of work is to look for similar enumerative results for $m > 3$.

# Bibliography

[1] Edward A Bender and E.Rodney Canfield. “The asymptotic number of rooted maps on a surface”. In: *Journal of Combinatorial Theory, Series A* 43.2 (1986), pp. 244–257.

[2] Edward A Bender and E.Rodney Canfield. “The number of rooted maps on an orientable surface”. In: *Journal of Combinatorial Theory, Series B* 53.2 (1991), pp. 293–299.

[3] Mireille Bousquet-Mélou and Gilles Schaeffer. “Enumeration of Planar Constellations”. In: *Advances in Applied Mathematics* 24.4 (2000), pp. 337–368.

[4] Jérémie Bouttier, Philippe Di Francesco, and Emmanuel Guitter. “Planar maps as labeled mobiles”. In: *The Electronic Journal of Combinatorics* 11 (2004), R69.

[5] Guillaume Chapuy. “Asymptotic enumeration of constellations and related families of maps on orientable surfaces”. In: *Combinatorics, Probability and Computing* 18.4 (2009), pp. 477–516.

[6] Guillaume Chapuy, Michel Marcus, and Gilles Schaeffer. “A bijection for rooted maps on orientable surfaces”. In: *SIAM Journal on Discrete Mathematics* 33.3 (2009), pp. 1587–1611.

[7] Robert Cori and Bernard Vauquelin. “Planar Maps are Well Labeled Trees”. In: *Canadian Journal of Mathematics* 33.5 (1981), pp. 1023–1042.

[8] Mathias Lepoutre. “Blossoming bijection for higher-genus maps”. In: *Journal of Combinatorial Theory, Series A* 165 (2019), pp. 187–224.

[9] Gilles Schaeffer. “Bijective census and random generation of Eulerian planar maps with prescribed vertex degrees”. In: *The Electronic Journal of Combinatorics [electronic only]* 4.1 (1997), R20.

[10] William Tutte. “A Census of Planar Maps”. In: *Canadian Journal of Mathematics* 15 (1963), pp. 249–271.