UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Facultat de Matemàtiques i Estadística

MASTER THESIS

Enumeration of chordal planar graphs and maps

Jordi Castellví

Advisors
Clément Requilé
Marc Noy

In partial fulfillment of the requirements for the
Master's Degree in Advanced Mathematics and Mathematical Engineering

May 2022

This thesis is based on the research conducted under a grant of the *Beques d'Iniciació a la Recerca (INIREC)* program together with professors Marc Noy and Clément Requilé. The result of this research is an article that has already been submitted for publication, the preprint of which can be found in https://arxiv.org/abs/2202.13340.

Abstract

We determine the number of labelled chordal planar graphs with n vertices, which is asymptotically $c_1 \cdot n^{-5/2}\gamma^n n!$ for a constant $c_1 > 0$ and $\gamma \approx 11.89235$. We also determine the number of rooted simple chordal planar maps with n edges, which is asymptotically $c_2 n^{-3/2}\delta^n$, where $\delta = 1/\sigma \approx 6.40375$, and σ is an algebraic number of degree 12. The proofs are based on combinatorial decompositions and singularity analysis. Chordal planar graphs (or maps) are a natural example of a subcritical class of graphs in which the class of 3-connected graphs is relatively rich. The 3-connected members are precisely chordal triangulations, those obtained starting from K_4 by repeatedly adding vertices adjacent to an existing triangular face.

Acknowledgements

I would like to express my gratitude to Marc Noy and Clément Requilé for their time and support during these months. In particular, I really appreciate the knowledge they have shared with me about analytic combinatorics, a subject I was new to. They have been very kind to me and I am looking forward to keep working with them.

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Chapter 1

Introduction

Chordal graphs have been much studied in structural graph theory and graph algorithms (see for instance [11]), but much less from the point of view of enumeration. It is known that the asymptotic number of labelled chordal graphs with n vertices is $\binom{n}{n/2}2^{n^2/4}$; an explanation for this estimate is that as n goes to infinity almost all chordal graphs with n vertices are split, that is, the vertex set can be partitioned into a clique and an independent set [2]. See also [18] for results on the exact counting of chordal labelled graphs.

On the other hand, there has been much work on counting planar graphs and related classes of graphs since the seminal work by Giménez and Noy [9]. Here we focus on planar graphs that are at the same time chordal. To count them we use, as in [9], the canonical decomposition of graphs into k-connected components for $k = 1, 2, 3$. The starting point is the enumeration of 3-connected chordal planar graphs: these are precisely the chordal triangulations, which when suitably rooted are in bijection with ternary trees. Then we use the decomposition of 2-connected graphs into 3-connected components. An important difference with the class of all planar graphs is that one cannot compose more than two graphs in series since otherwise a chordless cycle is created. A more significant difference is that the class of chordal planar graphs is *subcritical*, instead of being *critical* as the class of all planar graphs: this is reflected by the polynomial term $n^{-5/2}$ of the asymptotic estimates for the number of graphs in the class [6], as opposed to $n^{-7/2}$ for all planar graphs [9]. Thus we have a natural example in which the class of 3-connected graphs is relatively rich, yet the class is subcritical.

Our first result is the following.

Theorem 1. *Let g_n be the number of labelled chordal planar graphs with n vertices, c_n those which are connected, and b_n those which are 2-connected. Then, as $n \to \infty$, we have*

1. $g_n \sim g \cdot n^{-5/2}\gamma^n n!, \qquad \gamma \approx 11.89235,\ g \approx 0.00027205$
2. $c_n \sim c \cdot n^{-5/2}\gamma^n n!, \qquad c \approx 0.00027194,$

3. $b_n \sim b \cdot n^{-5/2} \gamma_b^n n!, \qquad \gamma_b \approx 10.76897, \; b \approx 0.00016215.$

We can add to the previous estimates the formula (see [1]) for the number t_n of 3-connected labelled chordal graphs

$$t_n = \binom{n}{3} \frac{(3n-9)!}{(2n-4)!} \approx t \cdot n^{-5/2} (27/4)^n n!, \qquad t = \frac{4\sqrt{3}}{3^{10}\sqrt{\pi}}. \tag{1.1}$$

As a corollary of Theorem 1, the limiting probability that a random labelled planar chordal graph (with the uniform distribution on graphs with n vertices) is connected tends to $p = c/g \approx 0.99963$ as $n \to \infty$. In fact it is straightforward to show [10] that the number of connected components is asymptotically distributed as $1+X$, where X is a Poisson law with parameter $C_0 \approx 0.00037470$, a value computed at the end of Chapter 4, so that $p = e^{-C_0}$.

Our second result is about rooted maps. A rooted map is a connected planar multigraph with a fixed embedding in the plane in which an edge (the *root edge*) is distinguished and directed. Rooted maps where first enumerated by Tutte [16] and have been since then the object of much study (see [15] for definitions on maps and an overview on their enumeration). We only consider simple maps (those with no loop or multiple edge) since they are the natural objects with respect to the property of being chordal.

Theorem 2. *Let M_n be the number of rooted chordal simple planar maps with n edges, and B_n those which are 2-connected. Then, as $n \to \infty$, we have*

1. $B_n \sim b \cdot n^{-3/2} \cdot \sigma_b^{-n}$, *with* $b \approx 0.071674$ *and* $\sigma_b^{-1} \approx 3.65370$,
2. $M_n \sim m \cdot n^{-3/2} \cdot \sigma^{-n}$, *with* $m \approx 0.12596$ *and* $\sigma^{-1} \approx 6.40375$.

The proof is again based on the structure of 3-connected chordal maps. As opposed to the class of general maps, the class of simple chordal maps is again subcritical. This is reflected in the term $n^{-3/2}$ instead of the usual $n^{-5/2}$ for classes of planar maps. Other natural subcritical classes are outerplanar maps [8] and series-parallel maps [4], but these two classes do not contain 3-connected graphs.

All the numerical computations were done using `Maple 2021`.

Chapter 2 contains all the definitions, notions and tools we need to prove our results. In Chapter 3 we analyse the combinatorial structure of chordal planar graphs according to their connectivity and deduce functional equations satisfied by the associated generating functions. Chapter 4 is devoted to the singularity analysis of said generating functions and concludes the proof of Theorem 1. Chapter 5 is the proof of Theorem 2.

Chapter 2

Preliminaries

In Sections 2.1 and 2.2 we present the basic tools we use through this thesis: generating functions, the symbolic method and the singularity analysis of generating functions. The main reference in these sections is the book *Analytic Combinatorics* [7], by Philippe Flajolet and Robert Sedgewick, though we also draw heavily on the book *Random Trees* [3], by Michael Drmota.

In Section 2.3 we introduce chordal graphs and characterize their 3-connected planar instances.

Section 2.4 is dedicated to the so-called dissymmetry theorem, a useful result that we use a couple of times.

2.1 Generating functions and the symbolic method

Definition 2.1. A *combinatorial class* is a set $\mathcal{A}$ together with a size function $|\cdot|\colon \mathcal{A} \to \mathbb{N}$ such that the number of objects in $\mathcal{A}$ with any given size is finite. We usually denote by A_n or a_n the number of objects of size n in $\mathcal{A}$.

In our setting, combinatorial classes will be families of graphs or maps, and the size function will be either the number of vertices (in the case of graphs) or edges (in the case of maps). There are two basic classes: the *neutral* class $\mathcal{E}$, consisting of a single element ε of size 0, and the *atomic* class $\mathcal{Z}$, consisting of a single element of size 1, called the *atom*.

Definition 2.2. The *ordinary generating function* of a combinatorial class $\mathcal{A}$ is the formal power series

$$A(x) = \sum_{n \geq 0} A_n x^n.$$

Observe that we can also write

$$A(x) = \sum_{\alpha \in \mathcal{A}} x^{|\alpha|}.$$

It is also common to denote $[x^n]A(x) = A_n$. This is called *coefficient extraction*.

The *symbolic method* is a method for building combinatorial classes by combining simpler classes using some constructions. The point of doing this is that the combinatorial constructions correspond to some operations between the generating functions of the classes involved, so that the enumerative properties of one class can be deduced from the enumerative properties of the simpler ones. We present the most basic constructions, which are the ones we will use. We do not give the proof of the operation between generating functions that corresponds to the each construction, they are not hard and can be found in [7]. In the following lines, $\mathcal{B}$ and $\mathcal{C}$ are combinatorial classes.

1. **Combinatorial sum (disjoint union)**. The sum $\mathcal{A} = \mathcal{B} + \mathcal{C}$ is the class that has the elements of both $\mathcal{B}$ and $\mathcal{C}$ and preserves the original size of each element. If $\mathcal{B}$ and $\mathcal{C}$ are not disjoint, we "artificially" distinguish their elements to make them disjoint. One has that

$$A(x) = B(x) + C(x).$$

2. **Cartesian product**. The product $\mathcal{A} = \mathcal{B} \times \mathcal{C}$ is defined by

$$\mathcal{A} = \{(\beta, \gamma) \mid \beta \in \mathcal{B}, \gamma \in \mathcal{C}\},$$

with size function $|(\beta, \gamma)| = |\beta| + |\gamma|$. One has that

$$A(x) = B(x) \cdot C(x).$$

3. **Sequence**. Suppose that $\mathcal{B}$ contains no object of size zero. Then, the sequence $\mathcal{A} = \textsc{Seq}(\mathcal{B})$ is defined as

$$\mathcal{A} = \mathcal{E} + \mathcal{B} + (\mathcal{B} \times \mathcal{B}) + (\mathcal{B} \times \mathcal{B} \times \mathcal{B}) + \cdots .$$

Intuitively, it is the class of finite sequences of elements in $\mathcal{B}$. The size function is given by the sums and products. One has that

$$A(x) = 1 + B(x) + B(x)^2 + \cdots = \frac{1}{1 - B(x)}.$$

4. **Substitution or composition**. The substitution of $\mathcal{C}$ into $\mathcal{B}$, denoted by $\mathcal{A} = \mathcal{B} \circ \mathcal{C}$ is the result of replacing each object β in $\mathcal{B}$ of size n by n elements of $\mathcal{C}$, while preserving the underlying structure of β. Formally,

$$\mathcal{A} = \sum_{\beta \in \mathcal{B}} \mathcal{E}_\beta \times \mathcal{C}^{|\beta|},$$

where each $\mathcal{E}_\beta$ is a neutral class and the summatory and power are given, respectively, by the combinatorial sum and the cartesian product defined above. One has that

$$A(x) = B(C(x)).$$

5. **Pointing**. The pointing of $\mathcal{B}$, denoted by $\mathcal{A} = \mathcal{B}^\bullet$ is the result of replacing each object β in $\mathcal{B}$ of size n by n distinct objects, while preserving the underlying structure of β. Formally,

$$\mathcal{A} = \sum_{\beta \in \mathcal{B}} \beta \times \left(\bigcup_{i=1}^{|\beta|} \mathcal{E}_i \right),$$

where each $\mathcal{E}_i$ is a neutral class. Intuitively, this corresponds to distinguishing all possible atoms. For example, from graphs we obtain rooted graphs. One has that

$$A(x) = xB'(x).$$

If, instead, one wishes to distinguish an atom and remove it, it suffices to differenciate:

$$A(x) = B'(x).$$

Often, the combinatorial objects we deal with are labelled. For instance, labelled graphs are graphs for which every vertex has a distinct integer label and the set of labels is a set of consecutive integers, starting at 1. A class containing objects with such labellings (no repetitions and consecutive labels starting at 1) is called a *labelled combinatorial class*. For reasons we will explain shortly, it is convenient to use *exponential generating functions* instead of ordinary ones for labelled classes.

Definition 2.3. The *exponential generating function* of a class $\mathcal{A}$ is the formal power series

$$A(x) = \sum_{n \geq 0} A_n \frac{x^n}{n!}.$$

The cartesian product of combinatorial classes defined above does not behave well with labelled classes, since a pair of labelled objects will have repeated labels. Thus, we redefine the product for labelled classes.

6. **Labelled product**. Suppose that $\mathcal{B}$ and $\mathcal{C}$ are labelled combinatorial classes. The *labelled product* $\beta * \gamma$ of two labelled objects $\beta \in \mathcal{B}$ and $\gamma \in \mathcal{C}$ is the set of all possible labelled pairs (β', γ') that preserve the relative order of the labels of β and γ in their original labellings.

The *labelled product* $\mathcal{A} = \mathcal{B} * \mathcal{C}$ is obtained by taking the ordered pairs from $\mathcal{B} \times \mathcal{C}$ and performing all order-consistent relabellings:

$$\mathcal{A} = \bigcup_{\substack{\beta \in \mathcal{B} \\ \gamma \in \mathcal{C}}} \beta * \gamma.$$

One has that

$$A(x) = B(x) \cdot C(x),$$

which is only true because we use exponential generating functions for labelled classes instead of ordinary ones.

The other constructions that use the cartesian product in the case of ordinary combinatorial classes are defined using the labelled product for labelled combinatorial classes, so that the relations between the generating functions are the same.

We define one last construction, which only makes sense for labelled classes.

7. **Set**. Suppose that $\mathcal{B}$ contains no object of size zero. Then, the set $\mathcal{A} = \textsc{Set}(\mathcal{B})$ is defined as

$$\mathcal{A} = \textsc{Seq}(\mathcal{B}) / \sim,$$

where $\sim$ is an equivalence relation identifying sequences that are permutations of each other. Intuitively, it is the class of finite sets of elements in $\mathcal{B}$. One has that

$$A(x) = 1 + \frac{B(x)}{1!} + \frac{B(x)^2}{2!} + \frac{B(x)^3}{3!} + \cdots = e^{B(x)} \equiv \exp\left(B(x)\right).$$

Sometimes we will want to mark some parameters in our combinatorial classes. For example, if we are dealing with a class $\mathcal{A}$ of graphs, we may want to keep track of edges in addition to vertices. If $A_{n,m}$ is the number of graphs in $\mathcal{A}$ with n vertices and m edges, we define the *bivariate generating function of $\mathcal{A}$ with parameter edges* as

$$A(x, y) = \sum_{n,m \geq 0} A_{n,m} x^n y^m = \sum_{\alpha \in \mathcal{A}} x^{|\alpha|} y^{e(\alpha)},$$

where $e(\alpha)$ is the number of edges of α. If the class is labelled, then we use exponential bivariate generating functions:

$$A(x, y) = \sum_{n,m \geq 0} A_{n,m} \frac{x^n}{n!} y^m = \sum_{\alpha \in \mathcal{A}} \frac{x^{|\alpha|}}{n!} y^{e(\alpha)}.$$

2.2 Singularity analysis of generating functions

In the previous section we defined generating functions as formal power series, formal objects that admit some algebraic operations. Now we will consider them as analytic objects, specifically complex functions. This allows us to use some powerful tools to extract information about their coefficients. The exact values of these coefficients are usually hard to compute, so we try to find their asymptotic behaviour. Precisely, for a given combinatorial class $\mathcal{A}$, we want to find a function $f\colon \mathbb{N} \to \mathbb{N}$ such that $\lim_{n\to\infty} A_n/f(n) = 1$. We say that f is the asymptotic estimate of $\mathcal{A}$ and write $A_n \sim f(n)$. For many classes of combinatorial objects, the typical asymptotic growth is of the form $A_n \sim K^n\theta(n)$, where $\theta(n)$ is subexponential, meaning that $\lim_{n\to\infty}(\theta(n))^{\frac{1}{n}} = 1$. To calculate K and $\theta(n)$, we rely on the two principles stated in [7, Chapter IV]:

> **First Principle of Coefficient Asymptotics**. The *location* of a function's singularities dictates the *exponential growth* (K^n) of its coefficients.
>
> **Second Principle of Coefficient Asymptotics**. The *nature* of a function's singularities determines the associated *subexponential factor* ($\theta(n)$).

The precise formulation of the first principle is given by the following theorem, which is taken from [7, Theorem IV.7].

Lemma 2.1 (location of singularities)**.** *Let $A(x)$ be a generating function analytic at 0 and suppose that ρ is its smallest singularity in modulus. Then, ρ is a positive real number and*

$$\sqrt[n]{[x^n]A(x)} \sim \rho^{-1}.$$

It should be noted that the fact that ρ is a positive real number is a direct consequence of Pringsheim's theorem.

To compute the subexponential growth, we can use the so-called transfer theorem.

Lemma 2.2 (transfer theorem)**.** *Assume that $f(x)$ has radius of convergence $\rho > 0$ and admits an analytic continuation to an open domain of the form*

$$\Delta(R,\phi) = \{x\colon |x| < R, x \neq \rho, |\arg(x-\rho)| > \phi\},$$

for some $R > \rho$ and $0 < \phi < \pi/2$. Further assume that $f(x)$ verifies, when $x \sim \rho$ such that $x \in \Delta(\phi, R)$,

$$f(x) \sim c \cdot \left(1 - \frac{x}{\rho}\right)^{-\alpha},$$

for some $c > 0$ *and* $\alpha \notin \{0, -1, -2, \dots\}$. *Then, the coefficients of* $f(x)$ *satisfy*

$$[x^n]f(x) \sim \frac{c}{\Gamma(\alpha)} n^{\alpha-1} \rho^{-n} \qquad \text{as } n \to \infty.$$

Another important result that we use is the so-called Drmota-Lalley-Woods' theorem, which deals with generating functions that are given by systems of equations. We state here the version in [3, Theorem 2.33].

Lemma 2.3 (Drmota-Lalley-Woods' theorem)**.** *Let* $\mathbf{F}(x, \mathbf{y}, \mathbf{u}) = (F_1(x, \mathbf{y}, \mathbf{u}), \dots, F_N(x, \mathbf{y}, \mathbf{u}))$ *be a non-linear system of functions analytic around* $x = 0$, $\mathbf{y} = (y_1, \dots, y_N) = \mathbf{0}$, $\mathbf{u} = (u_1, \dots, u_k) = \mathbf{0}$, *whose Taylor coefficients are all non-negative, such that* $\mathbf{F}(0, \mathbf{y}, \mathbf{u}) = \mathbf{0}$, $\mathbf{F}(x, \mathbf{0}, \mathbf{u}) \neq 0$, $\mathbf{F}_x(x, \mathbf{y}, \mathbf{u}) \neq 0$. *Furthermore, assume that the dependency graph of* $\mathbf{F}$ *is strongly connected and that the region of convergence of* $\mathbf{F}$ *is large enough so that there exists a complex neighbourhood* U *of* $\mathbf{u} = \mathbf{1} = (1, \dots, 1)$, *where the system*

$$\begin{aligned} y &= \mathbf{F}(x, \mathbf{y}, \mathbf{u}), \\ 0 &= \det(\mathbf{I} - \mathbf{F}_{\mathbf{y}}(x, \mathbf{y}, \mathbf{u})), \end{aligned}$$

has solutions $x = x_0(\mathbf{u}$ *and* $\mathbf{y} = \mathbf{y}_0(\mathbf{u})$ *that are real, positive and minimal for positive real* $\mathbf{u} \in U$.

Let

$$\mathbf{y} = \mathbf{y}(x, \mathbf{u}) = (y_1(x, \mathbf{u}), \dots, y_N(x, \mathbf{u}))$$

denote the analytic solutions of the system

$$\mathbf{y} = \mathbf{F}(x, \mathbf{y}, \mathbf{u})$$

with $\mathbf{y}(0, \mathbf{u}) = \mathbf{0}$.

Then there exist $\varepsilon > 0$ *such that* $y_j(x, \mathbf{u})$ *admit a representation of the form*

$$y_j(x, \mathbf{u}) = g_j(x, \mathbf{u}) - h_j(x, \mathbf{u}) \sqrt{1 - \frac{x}{x_0(\mathbf{u})}}$$

for $\mathbf{u} \in U$, $|x - x_0(\mathbf{u}| < \varepsilon$ *and* $|\arg(x - x_0(\mathbf{u})| \neq 0$, *where* $g_j(x, \mathbf{u}) \neq 0$ *and* $h_j(x, \mathbf{u}) \neq 0$ *are analytic functions with* $(g_j(x, \mathbf{u}))_j = (y_j(x, \mathbf{u}))_j = \mathbf{y}_0(\mathbf{u})$.

Furthermore, if $[x^n]y_j(x, \mathbf{1}) > 0$ *for* $1 \le j \le N$ *and for sufficiently large* $n \ge n_1$, *then there exists* $0 < \delta < \varepsilon$ *such that* $y_j(x, \mathbf{u})$ *is analytic in* $(x, \mathbf{u})$ *for* $\mathbf{u} \in U$ *and* $|x - x_0(\mathbf{u})| \ge \varepsilon$ *but* $|x| \le |x_0(\mathbf{u})| + \delta$ *(this condition guarantees that* $\mathbf{y}(x, \mathbf{u}))$ *has a unique smallest singularity with* $|x| = |x_0(\mathbf{u})|$*).*

The other analytic results that we use will be cited when needed.

2.3 Chordal graphs

Definition 2.4. A graph is *chordal* (or *triangulated*) if it has no induced cycles of length greater than 3.

Chordality can be defined in a variety of equivalent ways. In the following proposition, we present three alternative characterizations of chordal graphs. These well-known results first appeared in [14].

Proposition 2.1. *A graph is chordal if and only if any of the following conditions hold.*

(i) All cycles of length greater than 3 have a chord*, which is an edge not in the cycle connecting two of the vertices in the cycle.*

(ii) Every minimal separator is a clique.

(iii) It admits a perfect elimination ordering*, i.e., an ordering of the vertices such that the neighbours of any vertex v that occur after v in the order form a clique.*

Proof. It is clear that (i) is equivalent to the notion of chordality given in Definition 2.4.

Let us show that (i) implies (ii). Consider a minimal separator S and suppose that $u, v \in S$ are not connected by an edge. Let A, B be two connected components of $V \setminus S$. Since S is minimal, both u and v have some neighbour in A and B. Now consider the shortest paths of the form $ua_1a_2 \dots a_kv$ and $ub_1b_2 \dots b_lv$, where $a_i \in A$ and $b_i \in B$. Together they form a cycle of length at least 4 with no chords, which is a contradiction.

We now prove that (ii) implies (iii). We proceed by induction on the number of vertices. If our graph is complete, any ordering is a perfect elimination ordering. Otherwise, let v be a vertex that is not connected to every other vertex. Then, the set of its neighbours, $N(v)$, is a separating set, and thus it forms a clique. We set v as the first vertex in the ordering and the remaining vertices can be ordered by virtue of the hypothesis of induction.

Finally, (iii) implies (i) because in any cycle, the neighbours of the first vertex in the ordering are connected, which means that there is a chord unless the length of the cycle is 3. □

Definition 2.5. *Chordal* (or *stacked*) *triangulations* are the maps obtained from a K_4 by repeatedly adding a vertex in the interior of a triangular face and making it adjacent to the three vertices of the face.

Observe that this is equivalent to gluing copies of K_4 through triangles without gluing more than 2 through any triangle.

Proposition 2.2. *3-connected chordal planar graphs are chordal triangulations.*

Before going to the proof, we should make a couple of remarks. Observe that we are establishing a correspondance between a class of graphs and a class of maps. If we wanted to be precise, we should say that the class of graphs is the same as the underlying graphs of the class of maps. However, it makes sense to write the statement in this way because 3-connected planar chordal graphs admit a unique embedding on S^2. This can be seen by following the proof, but it is also a particular case of Whitney's theorem, which says that every 3-connected planar graph admits a unique embedding on the sphere.

Proof. We proceed by induction on the number of vertices. The smallest 3-connected graph is K_4, which is chordal and planar. Suppose that the number of vertices is greater than 4 and consider the first vertex in the perfect elimination ordering, v. v has at most 3 neighbours because they form a clique and the graph is planar. But it also has at least 3 neighbours because the graph is 3-connected. Therefore, v has exactly 3 neighbours, which form a triangle. Note that v belongs to no separating set of size 3, because it should be adjacent to, at least, the 2 other vertices in the separating set and 2 other vertices in the components that become separated, but it has only 3 neighbours. The graph obtained by deleting v is, thus, 3-connected, chordal and planar, and the hypothesis of induction concludes the proof. □

2.4 The dissymmetry theorem

The dissymmetry theorem is a useful result to un root certain structures.

Lemma 2.4 (dissymmetry theorem)**.** *Let $\mathcal{A}$ be a class of trees. Then, there is a bijection*

$$\mathcal{A} + \mathcal{A}_{\bullet\to\bullet} \simeq \mathcal{A}_{\bullet} + \mathcal{A}_{\bullet-\bullet},$$

where $\mathcal{A}_{\bullet}, \mathcal{A}_{\bullet-\bullet}$ and $\mathcal{A}_{\bullet\to\bullet}$ are the generating functions of trees in $\mathcal{A}$ rooted at vertices, edges and directed edges, respectively.

Proof. For a given tree $T \in \mathcal{A}$, we distinguish two cases.

- If the center of T is a vertex, we associate the tree rooted at the center to the unrooted tree, a tree rooted at any other vertex v to the tree rooted at a directed edge pointing towards the center whose starting point is v and a tree rooted at an undirected edge to the tree rooted at that edge directed away from the center.

- If the center of T is an edge, we associate the tree rooted at the center to the unrooted tree, a tree rooted at any other undirected edge to the tree rooted at the same edge directed away from the center and a tree rooted at a vertex v to the tree rooted at a directed edge pointing towards the center whose starting point is v (in case v belongs to the center, the edge is directed towards the other vertex of the center).

□

We will apply this theorem not to trees, but to *tree-decomposable* classes. A tree-decomposable class is a class for which every object has a different associated tree. These trees encode in some way the structure of the objects in the class. In particular, the vertices and edges of the trees correspond to some parts of the object, as we will see in the following chapter. Therefore, rooting the encoding trees in a vertex or edge translates to distinguishing these corresponding parts in the encoded object. This allows us to express the generating function of our class in terms of the generating function of the same class of objects but with some parts distinguished.

Since the classes of objects we are interested in are graphs and maps, we will use the words *vertex* and *edge* when refering to these objects and we will use the words *node* and *link* to talk about the vertices and edges of the encoding trees, in order to avoid confusions.

Chapter 3

Generating functions of chordal planar graphs

This chapter is devoted to derivating the equations that define the generating function of chordal planar graphs. In order to do so, we use, as in [7], the canonical decomposition of graphs into k-connected components, for $k = 1, 2, 3$.

The decomposition of a graph into connected and 2-conencted components is quite simple and is explained in Section 3.3. The decomposition of 2-connected graphs into 3-connected components is more involved; the full details can be found in [16] or [17]. For our purposes, all we need to know is that networks (a concept defined later) are parallel compositions of series compositions and 3-connected components.

In Section 3.1 we obtain the generating functions of 3-connected chordal planar graphs, in Section 3.2 we go to the 2-connected level and in Section 3.3 we finish with connected and general chordal planar graphs.

3.1 3-connected chordal planar graphs

Let $T(x)$ be the generating function of labelled 3-connected chordal planar graphs rooted at a directed edge, counted by the number of vertices minus two.

Proposition 3.1. *We have that*

$$T(x) = \frac{xS(x)}{2}, \tag{3.1}$$

where $S(x)$ is the generating function of labelled ternary trees counted by the number of internal vertices given by

$$S(x) = x\,(1 + S(x))^3\,. \tag{3.2}$$

Proof. Chordal triangulations with a marked face and a marked directed edge in this face are in bijection with ternary trees. Indeed, take the marked face to be the external face of the initial K_4. The three children of the root of the ternary tree correspond to the other faces of the initial K_4. Subdividing a face corresponds to giving three chindren to its associated leaf in the ternary tree. The faces of the triangulation are always distinguishable thanks to the marked directed edge.

Since there are two possible faces to mark in a chordal triangulation rooted at a directed edge and the number of vertices in a chordal triangulation is the number of internal vertices in its corresponding ternary tree plus three, $2T(x) = xS(x)$.

Moreover, (3.2) follows from the fact that a ternary tree is a root with three children whose subtrees are either empty or ternary trees. □

It is well known that $[x^n]S(x) = \frac{1}{2n+1}\binom{3n}{n}$, from which (1.1) follows.

We will later need the generating function counting labelled unrooted chordal triangulations, let us denote it by $U(x)$. One way to compute it would be to introduce the variable y in $T(x)$ and $U(x)$ counting edges, in such a way that $i![x^i y^j]T(x,y)$ is the number of labelled chordal triangulations rooted at a directed edge with $i+2$ vertices (the 2 from the root edge being unlabelled) and $j+1$ edges; and that $i![x^i y^j]U(x,y)$ is the number of labelled chordal triangulations with i vertices and y edges. These generating functions satisfy the relation

$$\frac{x^2}{2}T(x,y) = \frac{\partial}{\partial y}U(x,y),$$

and therefore $U(x)$ can be obtained by algebraic integration:

$$U(x) = \left(\frac{x^2}{2}\int T(x,y)dy\right)(x,1).$$

Alternatively, one can use the dissymmetry theorem to avoid this integration and keep the proof purely combinatorial.

Lemma 3.1. *$U(x)$ is given by*

$$U(x) = \frac{x^3}{24}\left(S(x) - S(x)^2\right).$$

Proof. Labelled chordal triangulations with n vertices are in bijection with the class of trees with $n-3$ nodes endowed with a labelling that satisfies the following conditions.

1. Every node is labelled with a subset of size 4 of $\{1,\ldots,n\}$.

2. The intersection of the label of a node with the labels of its neighbours has size 3 and this intersection is different for each neighbour. In particular, every node has degree at most 4.

3. The graph induced by the nodes whose label contains a given $i \in \{1, \dots, n\}$ is connected.

Indeed, the nodes of the tree correspond to the 4-cliques in the triangulation, their labels correspond to the labels of the 4 vertices of the 4-clique, and two nodes are adjacent if the associated 4-cliques are glued through a triangle.

Therefore, the generating function of these encoding trees, counted by their number of nodes plus 3, is the generating function of labelled chordal triangulations. We use the dissymmetry theorem on the trees. We denote by $A, A_{\bullet}, A_{\bullet-\bullet}$ and $A_{\bullet\to\bullet}$ the generating functions of $\mathcal{A}, \mathcal{A}_{\bullet}, \mathcal{A}_{\bullet-\bullet}$ and $\mathcal{A}_{\bullet\to\bullet}$, respectively. Note that, since all nodes have different labels, they are distinguishable and hence $A_{\bullet\to\bullet} = 2A_{\bullet-\bullet}$.

Rooting a tree at a node corresponds to rooting a chordal triangulation at a 4-clique. We fix the 4 vertices of the clique, and then at each triangle we attach a (possibly empty) chordal triangulation. This gives

$$A_{\bullet} = \frac{x^4}{24}\left(1+S(x)\right)^4 = \frac{x^3}{24}S(x)\left(1+S(x)\right).$$

Rooting at a link corresponds to rooting a chordal triangulation at a triangle shared by two 4-cliques. We fix the 3 vertices of the clique and then attach two chordal triangulations rooted at it. Taking into account symmetries this gives

$$A_{\bullet-\bullet} = \frac{x^3}{12}S(x)^2.$$

Finally, we have

$$U(x) = A = A_{\bullet} + A_{\bullet-\bullet} - A_{\bullet\to\bullet} = A_{\bullet} - A_{\bullet-\bullet} = \frac{x^3}{24}S(x)\left(1+S(x)\right) - \frac{x^3}{12}S(x)^2$$
$$= \frac{x^3}{24}(S(x) - S(x)^2).$$

□

3.2 2-connected chordal planar graphs

Definition 3.1. *Networks* are 2-connected labelled chordal graphs rooted at a directed edge, the endpoints of which are not marked.

Let $B(x,y)$ and $E(x,y)$ be the generating functions of 2-connected chordal planar graps and networks, respectively, where x marks vertices and y marks edges. The relation between these generating functions is

$$E(x,y) = \frac{2y}{x^2}\frac{\partial}{\partial y}B(x,y).$$

Lemma 3.2. *The generating function of networks is given by*

$$E(x, y) = y \exp\left(xE(x, y)^2 + \frac{T\left(xE(x, y)^3\right)}{E(x, y)}\right). \tag{3.3}$$

Proof. This equation reflects the fact that networks are parallel compositions of series compositions and 3-connected components. Since our graphs are chordal, they cannot have induced cycles of length greater than 3. Therefore, at most 2 networks can be composed in series, forming a triangle. The factor y encodes the root edge, the exponential encodes a (possibly empty) set of parallel networks, the term $xE(x, y)^2$ encodes the series composition of exactly two networks, and the term $T\left(xE(x, y)^3\right)/E(x, y)$ encodes chordal triangulations whose non-root edges have been replaced by networks (note that the number of non-root edges in a triangulation with $n - 2$ vertices is $3n - 1$). □

As before, one could obtain $B(x, y)$ from $E(x, y)$ by algebraic integration:

$$B(x, y) = \frac{x^2}{2} \int \frac{1}{y} E(x, y) dy. \tag{3.4}$$

But again, we do it using the dissymmetry theorem.

Lemma 3.3. *One has that*

$$B(x) = \frac{x^2}{2}\left(E(x) - \frac{xE(x)^3}{12}\left(S\left(xE(x)^3\right)^2 + 5S\left(xE(x)^3\right) + 8\right)\right), \tag{3.5}$$

where $B(x) = B(x, 1)$ *and* $E(x) = E(x, 1)$.

Proof. 2-connected chordal planar graphs can be encoded by trees with nodes of three types: e (edge), s (series, i.e., a triangle) and t (triangulation). Indeed, as we saw, 2-connected chordal planar graphs are the result of gluing triangles and chordal triangulations through edges. There are only links of type $s-e$ and $t-e$, which represent the fact that a triangle or a triangulation is glued through that edge to something else. Thus, nodes of type e always have degree at least 2. Figure 3.1 contains an example of a 2-connected graph and its encoding tree. The nodes of the tree contain all the information about the subgraph they represent: the labels of its vertices and its structure in the case of triangulations. For these encoding trees to be in bijection with 2-connected chordal planar graphs, they must satisfy the following conditions.

1. The edge represented by a node of type e appears in all its neighbours.

2. The graph induced by the nodes that contain some vertex with a given label is connected.

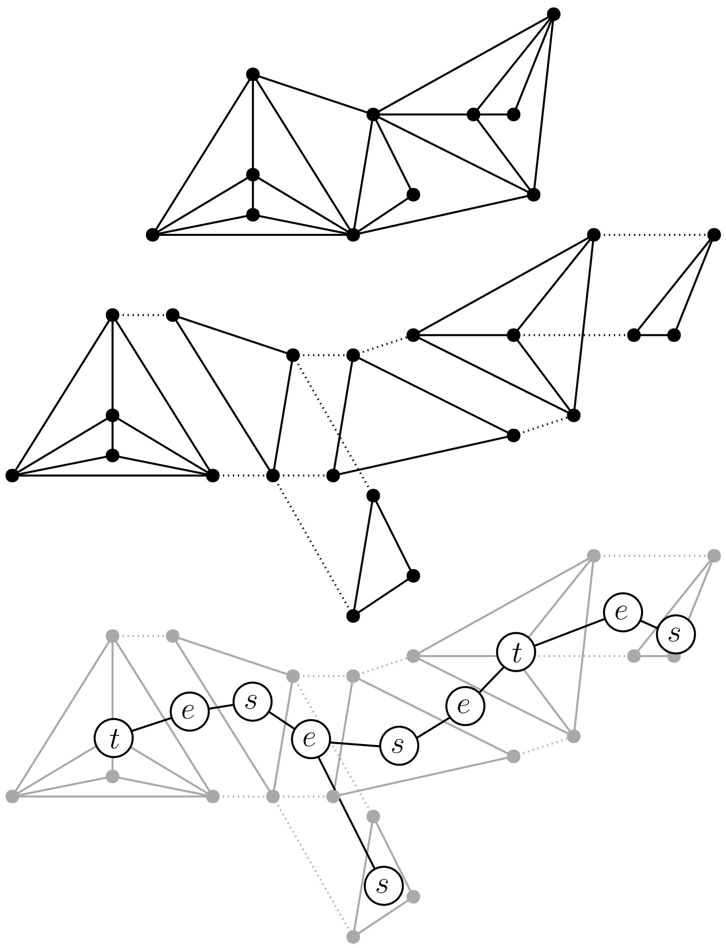

Figure 3.1: The tree decomposition of a 2-connected chordal planar graph.

3. The set of labels that appear in some node is of the form $\{1, \ldots, n\}$, for some n that will correspond to the number of vertices of the encoded 2-connected chordal planar graph.

The lines above are correct if the graph consisting of a single edge is not considered a 2-connected graph. However, for reasons we will explain later, it is convenient to count this single edge as a valid 2-connected chordal planar graph. We will have to add it to $B(x)$ individually.

To use the dissymmetry theorem, we need to compute the generating functions $R_s(x)$, $R_t(x)$, $R_e(x)$, $R_{s-e}(x)$ and $R_{t-e}(x)$.

Rooting the tree at a node of type s corresponds to fixing three unordered vertices of a triangle and attaching a (possibly empty) network at each of the three edges. Thus, $R_s(x) = x^3E(x)^3/6$.

Rooting at a node of type t corresponds to attaching a network at each of the edges of an unrooted chordal triangulation. Since the number of vertices and edges in a chordal triangulation are related by $e = 3(v-2)$, we have that $R_t(x) = U(xE(x)^3)/E(x)^6$.

Rooting at a node of type e corresponds to fixing the two unordered vertices of the edge and attaching a network with at least two parallel components. This is encoded by substracting the first two terms of the exponential in (3.3), which gives $R_e(x) = \frac{x^2}{2}\left(E(x) - 1 - xE(x)^2 - T(xE(x)^3)/E(x)\right)$.

Rooting at a link of type $s-e$ corresponds to rooting a graph at a triangle with one of its sides distinguished. Therefore, we fix the three vertices of the triangle (the two vertices on the root edge are unordered) and we attach a non-empty network to the root edge and any network to each of the two remaining edges. This yields $R_{e-s}(x) = \frac{x^3}{2}E(x)^2\left(E(x) - 1\right)$.

Finally, rooting at a link of type $t - e$ corresponds to rooting at a triangulation with a distinguished edge and attaching a network to each of its edges. The term $T\left(xE(x)^3\right)$ is explained as before and this gives $R_{e-t}(x) = \frac{x^2}{2}T(xE(x)^3)(E(x) - 1)/E(x)$.

Putting everything together, we obtain

$$\begin{aligned}
B(x) &= \frac{x^2}{2} + R_\bullet(x) + R_{\bullet-\bullet} - 2R_{\bullet\to\bullet} \\
&= \frac{x^2}{2} + R_\bullet(x) - R_{\bullet-\bullet} \\
&= \frac{x^2}{2} + R_s(x) + R_t(x) + R_e(x) - R_{s-e}(x) - R_{t-e}(x) \\
&= \frac{x^2}{2}\left(E(x) - \frac{xE(x)^3}{12}\left(S\left(xE(x)^3\right)^2 + 5S\left(xE(x)^3\right) + 8\right)\right).
\end{aligned}$$

Note that the first summand, $x^2/2$, corresponds to the single edge □

3.3 Connected and arbitrary chordal planar graphs

If one takes a connected graph and cuts it through all of its cut vertices, the resulting components are obviously 2-connected. They are called its *2-connected components* or *blocks*. Note that here single edges count as 2-connected graphs, since they have no cut vertex, which is why we chose to count them in $B(x)$. In our context, the 2-connected components are planar and chordal and, conversely, the result of glowing 2-connected planar chordal graphs through vertices is always a 2-connected planar chordal graph. In other words, a connected graph is planar and chordal if and only if all its 2-connected components are also planar and chordal. This is often stated saying that the class of chordal planar maps is *block-stable*.

Here we should introduce a relevant notion:

Definition 3.2. Suppose that $\mathcal{G}$ is a block-stable class of labelled graphs and denote the subclasses of connected graphs and 2-connected graphs by $\mathcal{C}$ and $\mathcal{B}$, their associated exponential generating functions by $C(x)$ and $B(x)$ with radius of convergence ρ_b and ρ_c, respectively. The class $\mathcal{G}$ is said to be *subcritical* [6] if

$$\rho_c C'(\rho_c) < \rho_b. \tag{3.6}$$

This condition has important implications on the structure of a class of graphs. Intuitively, subcritical classes are "tree-like" in some sense [10] exhibited for instance by the fact that their scaling limit is the continuum random tree [12], which means that the global structure is essentially determined by the block-decomposition tree, while the size of the blocks is bounded in expectation and at most logarithmic.

Let $C(x)$ denote the generating function of connected chordal planar graphs. We have the following lemma, which is valid for all *block-stable* classes of graphs.

Lemma 3.4. *The relation between $C(x)$ and $B(x)$ is given by*

$$C^\bullet(x) = xe^{B'(C^\bullet(x))}, \tag{3.7}$$

where $C^\bullet(x) = xC'(x)$ is the generating function of connected chordal planar graphs rooted at a vertex.

Proof. Indeed, the root vertex, encoded by the first instance of x, belongs to a (possibly empty) set of 2-connected components, which is encoded by the exponential. The other vertices of each of these 2-connected components may be cut vertices, which is why they are substituted by $C^\bullet(x)$. □

Observe from (3.7) that in the subcritical case B' is not a source of singularities. Instead, they have to come from a branch point.

Finally, an arbitrary chordal planar graph is a set of connected components. Thus, if we denote by $G(x)$ the generating function of chordal planar graphs, we have that

$$G(x) = e^{C(x)}. \tag{3.8}$$

Chapter 4

Asymptotic analysis of chordal planar graphs

In this chapter, we do the singularity analysis of the generating functions obtained in the previous one, completing the proof of Theorem 1.

Section 4.1 is devoted to the analysis of the genertating functions of 2-connected chordal planar graphs and Section 4.2 deals with the generating functions of connected and general chordal planar graphs.

4.1 2-connected graphs

Using (3.1) with $x = x(1+F)^3$ and setting $y = 1$, $F = F(x) = E(x) - 1$ and $S = S(x(1+F)^3)$, we transform Equations (3.2) and (3.3) into a system amenable to Lemma 2.3, with $u = 1$, as follows:

$$\begin{aligned} F &= \exp\left(x(1+F)^2 + \frac{x(1+F)^2 S}{2}\right) - 1, \\ S &= x(1+F)^3(1+S)^3. \end{aligned} \tag{4.1}$$

Let $\Phi(x,S,F)$ and $\Psi(x,S,F)$ be the right hand-side of the first and second equation in (4.1), respectively. Those functions are entire and define a system with a strongly connected dependency graph between variables S and F. Furthermore, both have non-negative coefficients and vanish at $x = 0$, while they satisfy $\Phi(x,0,0) \neq 0$ and $\Psi(x,0,0) \neq 0$, but also $\Phi_x(x,S,F) \neq 0$ (where $\Phi_x = \partial\Phi/\partial x$) and $\Psi_x(x,S,F) \neq 0$. Finally, sytem (4.1) extended by its Jacobian admits a solution that is non-zero. It is given by the following approximations:

$$\begin{aligned} \rho_b &\approx 0.092859, \\ E_0 &= E(\rho_b) = 1 + F(\rho_b) \approx 1.16454, \\ S_0 &= S(\rho_b E_0^3) \approx 0.41919. \end{aligned}$$

Thus the hypotheses of Lemma 2.3 are verified. This implies in particular that ρ_b is the unique dominant singularity of the function $E(x)$, i.e. on the boundary of the disk of convergence, and that $E(x)$ admits the following analytic continuation in a domain of the form $\Delta(R_b, \phi_b)$ for some $R_b > \rho_b$ and $0 < \phi_b < \pi/2$:

$$E(x) = E_0 - E_1\sqrt{1 - \frac{x}{\rho_b}} + O\left(1 - \frac{x}{\rho_b}\right), \qquad (4.2)$$

for $x \sim \rho_b$ and $x \in \Delta(R_b, \phi_b)$, where $E_1 > 0$ is computed next. Since S is itself a function of x and F, (4.1) can be rewritten as $F = \Theta(x, F)$, where Θ is analytic at (ρ_b, F_0) with $F_0 = F(\rho_b)$. One checks that $\Theta_x(\rho_b, F_0) \neq 0$, $\Theta_F(\rho_b, F_0) = 0$ and $\Theta_{FF}(\rho_b, F_0) \neq 0$. And we can apply [7, Lemma VII.3] (see also [3, Remark 2.20]) to obtain

$$E_1 = F_1 = \sqrt{\frac{2\rho_b\Theta_x(\rho_b, F_0)}{\Theta_{FF}(\rho_b, F_0)}} \approx 0.092354. \qquad (4.3)$$

Furthermore, Lemma 2.3 implies a similar result for $S = S(x(1+F)^3)$. Note also that $\rho_b E_0^3 = 0.14665 < 4/27$, where $4/27$ is the dominant singularity of $S(x)$. This implies thar the composition scheme $S(xE(x)^3)$ is subcritical in the sense that S is not the source of the singularity.

With those results at hand, we can finally consider the generating function $B(x)$. Given the expression (3.5) and the fact that the scheme is subcritical, the dominant singularity of $B(x)$ is the same as that of $E(x)$ and it is furthermore unique. We show next that $B(x)$ admits a singular expansion at $x = \rho_b$ similar to $E(x)$. First we extend the system (4.1) so that it includes the variable y:

$$\begin{aligned} F &= y \exp\left(x(1+F)^2 + \frac{x(1+F)^2 S}{2}\right) - 1, \\ S &= x(1+F)^3(1+S)^3, \end{aligned} \qquad (4.4)$$

where now $F = F(x, y)$. By Lemma 2.3 (setting $u = y$) there exist three functions $\rho_b(y)$, $f_0(y)$ and $f_1(y)$ analytic in a neighbourhood W of 1 such that for $y \in W$ and $x \sim \rho_b(y)$ with $|\arg(x - \rho_b(y))| \neq 0$ the following singular expansion holds

$$\begin{aligned} E(x, y) &= 1 + F(x, y) \\ &= 1 + f_0(y) - f_1(y)\sqrt{1 - \frac{x}{\rho_b(y)}} + O\left(1 - \frac{x}{\rho_b(y)}\right), \end{aligned} \qquad (4.5)$$

where $\rho_b(1) = \rho_b$, $1 + f_0(1) = E_0$ and $f_1(1) = E_1$. From there, applying [3, Theorem 2.30] to (3.4) and setting $y = 1$, we obtain that $B(x) = B(x, 1)$ admits an analytic continuation of the form

$$B(x) = B_0 - B_2\left(1 - \frac{x}{\rho_b}\right) + B_3\left(1 - \frac{x}{\rho_b}\right)^{3/2} + O\left(1 - \frac{x}{\rho_b}\right)^2,$$

for $x \sim \rho_b$ and $x \in \Delta(R_b, \phi_b)$. The above coefficients can be computed by substituting into (3.5) the expansions of $E(x)$ and $S(xE(x)^3)$ when $x = \rho_b(1-X^2)$, with $X = \sqrt{1 - x/\rho_b}$. This gives

$$\begin{aligned} B_0 &\approx 0.0044796, \\ B_2 &\approx 0.0085328, \\ B_3 &\approx 0.00038321. \end{aligned}$$

The estimate on b_n follows from Lemma 2.2, with $b = 3B_3/(4\sqrt{\pi}) \approx 0.00016215$.

4.2 Connected and arbitrary graphs

The composition scheme (3.7) is subcritical because $B''(\rho_b) \to \infty$ (see [10]). This means in particular that the singularities of $C^\bullet(x)$ come from a branch point and not from those of $B(x)$ and are obtained by solving

$$\rho = \tau e^{-B'(\tau)} \quad \text{and} \quad \tau B''(\tau) = 1,$$

with $\tau = C^\bullet(\rho) < \rho_b$. To find such a solution, one must first compute $E'(x)$ and $E''(x)$ and then $B''(x)$. This is a routine but lengthy computation, best solved numerically together with equations (3.2) and (3.3), and which gives the following approximate solutions:

$$\tau \approx 0.092859 \quad \text{and} \quad E(\tau) \approx 1.16446. \tag{4.6}$$

From this we obtain that the singularity of $C^\bullet(x)$ at $x = \rho$ given by

$$\rho = \tau e^{-B'(\tau)} \approx 0.084088.$$

As before $C^\bullet(x)$ can be extended analytically to a domain of the form $\Delta(R, \phi)$ for some $R > \rho$ and $0 < \phi < \pi/2$. The same holds for $C(x)$ (see [10, Proposition 3.10.(1)]), which in fact verifies

$$C(x) = C_0 - C_2\left(1 - \frac{x}{\rho}\right) + C_3\left(1 - \frac{x}{\rho}\right)^{3/2} + O\left(1 - \frac{x}{\rho}\right)^2,$$

for $x \sim \rho$ and $x \in \Delta(R, \phi)$. The above coefficients are given by:

$$\begin{aligned} C_0 &= \tau(1 + \log\rho - \log\tau) + B(\tau) \approx 0.00037470, \\ C_2 &= \tau \approx 0.092859, \\ C_3 &= \frac{3}{2}\sqrt{\frac{2\rho\exp(B'(\tau))}{\tau B'''(\tau) - \tau B''(\tau)^2 + 2B''(\tau)}} \approx 0.00027194. \end{aligned}$$

The estimate for c_n is again a consequence of Lemma 2.2. The same goes for the series $G(x, y) = e^{C(x,y)}$ of arbitrary chordal planar graphs. Since

$G(x) = e^{C_0}(1 - C_2(1 - x/\rho) + C_3(1 - x/\rho)^{3/2} + O(1 - x/\rho)^2)$ for $x \sim \rho$ and $x \in \Delta(\phi, R)$, we have

$$\begin{aligned} G_0 &= e^{C_0} \approx 1.00037, \\ G_2 &= C_2 e^{C_0} \approx 0.092894, \\ G_3 &= C_3 e^{C_0} \approx 0.00027205, \end{aligned}$$

and the estimate for g_n follows. This concludes the proof of Theorem 1.

Chapter 5

Simple chordal planar maps

This chapter is the proof of Theorem 2. Section 5.1 is dedicated to the 2-connected simple chordal maps and Section 5.2 to all maps.

5.1 Decomposition of 2-connected simple chordal maps.

Let $D(x)$ be the generating function of simple 2-connected chordal maps, where x marks the number of edges minus 1, and let $S(x)$ be the generating function of ternary trees satisfying (3.2). Similarly to the case of graphs, a simple 2-connected chordal map can be decomposed into a sequence of smaller chordal maps. As opposed to the situation for graphs the planar embedding provides a linear ordering, which is why we use the sequence instead of the set construction. The maps in the sequence are either a triangle rooted at an edge, where each side of the two non-root edges (four sides in total) is replaced by a map, or 3-connected maps in which the two sides of every edge are replaced by a map. This gives

$$D(x) = \frac{1}{1 - x^2 D(x)^4 \left(1 + S\left(x^3 D(x)^6\right)\right)}. \tag{5.1}$$

Now, let $B(x)$ be the generating function counting simple 2-connected chordal maps, with x now marking the total number of edges, so that $B = B(x) = xD(x)$. Algebraic elimination between (3.2) and (5.1) gives the following irreducible polynomial equation satisfied by B:

$$B^9 - x^2 B^5 + x^3 B^4 + x^3 B^3 - 3x^4 B^2 + 3x^5 B - x^6 = 0. \tag{5.2}$$

Therefore, $B(x)$ is an algebraic function and its analysis in the rest of the proof will follow the approach detailed in [7, Chapter VII.7]. For instance, $B(x)$ can be represented at $x = 0$ as a Taylor series with non-negative coefficients and radius of convergence σ_b, for some $\sigma_b > 0$, corresponding to a branch of the curve (5.2) passing through the origin, as follows:

$$B(x) = x + x^3 + 5x^5 + x^6 + 35x^7 + 16x^8 + 288x^9 + O(x^{10}).$$

Next, we find the value of σ_b. By Pringsheim's theorem (see [7, Theorem IV.6]), it must be a singularity of $B(x)$. Since $B(x)$ is algebraic, its singularities must be among the roots of the discriminant of (5.2) with respect to B, which up to a trivially non-zero factor is equal to

$$387420489x^6 + 573956280x^5 + 184705272x^4 - 81168524x^3 \\ -15907392x^2 + 3326272x - 135424.$$

This polynomial admits $\sigma_b \approx 0.27370$ as unique positive real root and it can be readily checked that no other root ψ satisfies $|\psi| = \sigma_b$.

Finally, we determine the singular expansion of $B(x)$ locally around σ_b. As $B(x)$ is algebraic and has no other singularity on the circle of radius σ_b, there exists $R'_b > \sigma_b$ and $0 < \phi'_b < \pi/2$ for which its representation at $x = 0$ admits an analytic continuation to a domain at $x = \sigma_b$ of the form $\Delta(R'_b, \phi'_b)$. It can in fact be computed from (5.2) using Newton's *polygon algorithm*. This gives a singular expansion of the form:

$$B(x) = B(\sigma_b) + b_1\sqrt{1 - \frac{x}{\sigma_b}} + O\left(1 - \frac{x}{\sigma_b}\right), \tag{5.3}$$

for $x \sim \sigma_b$ and $x \in \Delta(R'_b, \phi'_b)$, where $B(\sigma_b) \approx 0.33301$ and $b_1 \approx 0.12704$. The estimate on B_n then follows from Lemma 2.2.

5.2 Decomposition of simple chordal maps

Let $M(x)$ be the generating function of all simple chordal maps, where x marks the total number of edges. The decomposition of a map into block is given by the equation

$$M(x) = B\left(x(1 + M(x))^2\right), \tag{5.4}$$

reflecting the fact that a map is obtained from its 2-connected core by attaching a (possibly empty) map at each corner [16]. Since being simple and chordal is preserved by taking 2-connected components, the same equation holds for simple chordal maps.

We proceed as in the previous section. First, by algebraic elimination between (5.2) and (5.4), we obtain an irreducible polynomial equation satisfied by $M = M(x)$:

$$
\begin{aligned}
0 = {} & x^6 M^{12} + 3x^5 (4x-1) M^{11} + x^3 \left(66x^3 - 30x^2 + 3x - 1\right) M^{10} \\
& + \left(220x^6 - 135x^5 + 24x^4 - 7x^3 + x^2 - 1\right) M^9 \\
& + x^2 \left(495x^4 - 360x^3 + 84x^2 - 21x + 4\right) M^8 \\
& + x^2 \left(792x^4 - 630x^3 + 168x^2 - 35x + 6\right) M^7 \\
& + x^2 \left(924x^4 - 756x^3 + 210x^2 - 35x + 4\right) M^6 \\
& + x^2 \left(792x^4 - 630x^3 + 168x^2 - 21x + 1\right) M^5 \\
& + x^3 \left(495x^3 - 360x^2 + 84x^1 - 7\right) M^4 \\
& + x^3 \left(220x^3 - 135x^2 + 24x - 1\right) M^3 \\
& + 3x^4 \left(22x^2 - 10x + 1\right) M^2 + 3x^5 (4x-1) M + x^6.
\end{aligned}
\tag{5.5}
$$

From the curve (5.5) we get that $M(x)$ can be represented at $x = 0$ as the following Taylor series with non-negative coefficients and radius of convergence $\sigma > 0$:

$$
\begin{aligned}
M(x) = {} & x + 2x^2 + 6x^3 + 22x^4 + 92x^5 + 419x^6 + 2025x^7 + 10214x^8 \\
& + 53192x^9 + O(x^{10}).
\end{aligned}
$$

The discriminant of (5.5) with respect to M is, up to a trivially non-zero factor, given by

$$
\begin{aligned}
& 2035256037376x^{12} - 2215690119168x^{11} + 6474387490048x^{10} \\
& + 1262789263168x^9 - 3620212090976x^8 + 1275725763644x^7 \\
& - 301902286683x^6 + 60575733276x^5 - 13112588384x^4 \\
& - 5212588972x^3 + 1812419712x^2 - 148471488x + 3656448.
\end{aligned}
\tag{5.6}
$$

It admits two positive real roots, given approximately by 0.15616 and 0.4951 2. However 0.49512 cannot be the radius of convergence of $M(x)$ since it is larger than σ_b. Therefore $\sigma \approx 0.15616$, and it can be readily checked that (5.6) admits no other zero of modulus σ. By a standard compactness argument, this means that there exists $R' > \sigma$ and $0 < \phi' < \pi/2$ for which the representation of $M(x)$ at $x = 0$ admits an analytic continuation to a domain at $x = \sigma$ of the form $\Delta(R', \phi')$. It is given by

$$
M(x) = M(\sigma) + m_1 \sqrt{1 - \frac{x}{\sigma}} + O\left(1 - \frac{x}{\sigma}\right), \tag{5.7}
$$

for $x \sim \sigma$ and $x \in \Delta(R', \phi')$, where $M(\sigma) \approx 0.31055$ and $m_1 \approx 0.22326$. Note that the class of simple chordal maps is subcritical in the sense, similar to (3.6), that the composition scheme in (5.4) is subcritical, that is, $\sigma(1 + M(\sigma)^2) \approx 0.26821 < \sigma_b$. The estimate on M_n is obtained from Lemma 2.2 as before, and this concludes the proof of Theorem 2.

Chapter 6

Concluding remarks

From the system (4.4) and [3, Theorem 2.35] we could obtain, applying the so-called quasi-powers theorem [7], a central limit theorem for the number of edges in a uniform random 2-connected chordal planar graph with n vertices as $n \to \infty$. This result is to be expectd and fits into a general scheme of similar Gaussian parameters in subcritical graph classes (see for instance [6], and [5] and [13] for some generalisations). It would be of interest to study in the context of chordal planar graphs other parameters, particularly extremal parameters [10].

Furthermore, by sligthly adapting the scheme developed in this paper, one could in principle enumerate several related families of chordal graphs, such as outerplanar graphs, series-parallel graphs, planar multigraphs and also non-planar graphs, such as forbidding $K_{3,3}$ or K_5 as a minor. For chordal graphs, forbidding K_5 as a minor is equivalent to the property of having tree-width at most three.

A future line of research is to enumerate chordal graphs with bounded tree-width. An interesting aspect of this class of graphs is that the composition scheme (3.7) naturally generalizes to any connectivity. In other words, since separating sets form cliques, k-connected chordal graphs can be obtained by gluing $(k+1)$-connected chordal graphs through k-cliques.

Another possible continuation of this work is to enumerate unlabelled chordal planar graphs. This should be fesible with the help of Polya theory by taking into account the symmetries of the 3-connected graphs.

To conclude, we display the first numbers of labelled chordal planar graphs (resp., chordal maps) counted by vertices (resp., edges) in Table 6.1 (resp., Table 6.2).

n	g_n	c_n	b_n
1	1	1	0
2	2	1	1
3	8	4	1
4	61	35	7
5	821	540	110
6	17962	13116	2880
7	589912	462868	108486
8	26990539	22189056	5376448
9	1611421595	1364476032	330554736
10	119106036226	102768330140	24223100940
11	10475032926304	9150009283316	2056900853260
12	1064759262580675	937871756182824	198279609266376
13	122455558249650523	108501459033647056	21365210239261824
14	15683814373288014514	13957140054455406368	2542622031178234096
15	2210104382919809469776	1973316500054545453200	331005569819483825280
16	339419270505312015418873	303844760227083629476736	46769563108388612386560

Table 6.1: Numbers of arbitrary, connected and 2-connected labelled chordal planar graphs with n vertices.

n	M_n	B_n
1	1	1
2	2	0
3	6	1
4	22	0
5	92	5
6	419	1
7	2025	35
8	10214	16
9	53192	288
10	283921	210
11	1545326	2607
12	8544766	2612
13	47867107	25155
14	271091848	31885
15	1549624321	254255
16	8929009486	386672

Table 6.2: Numbers of arbitrary and 2-connected simple chordal maps with n edges.

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