Blossoming bijection for constellations of higher genus

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with Marie Albenque and Éric Fusy

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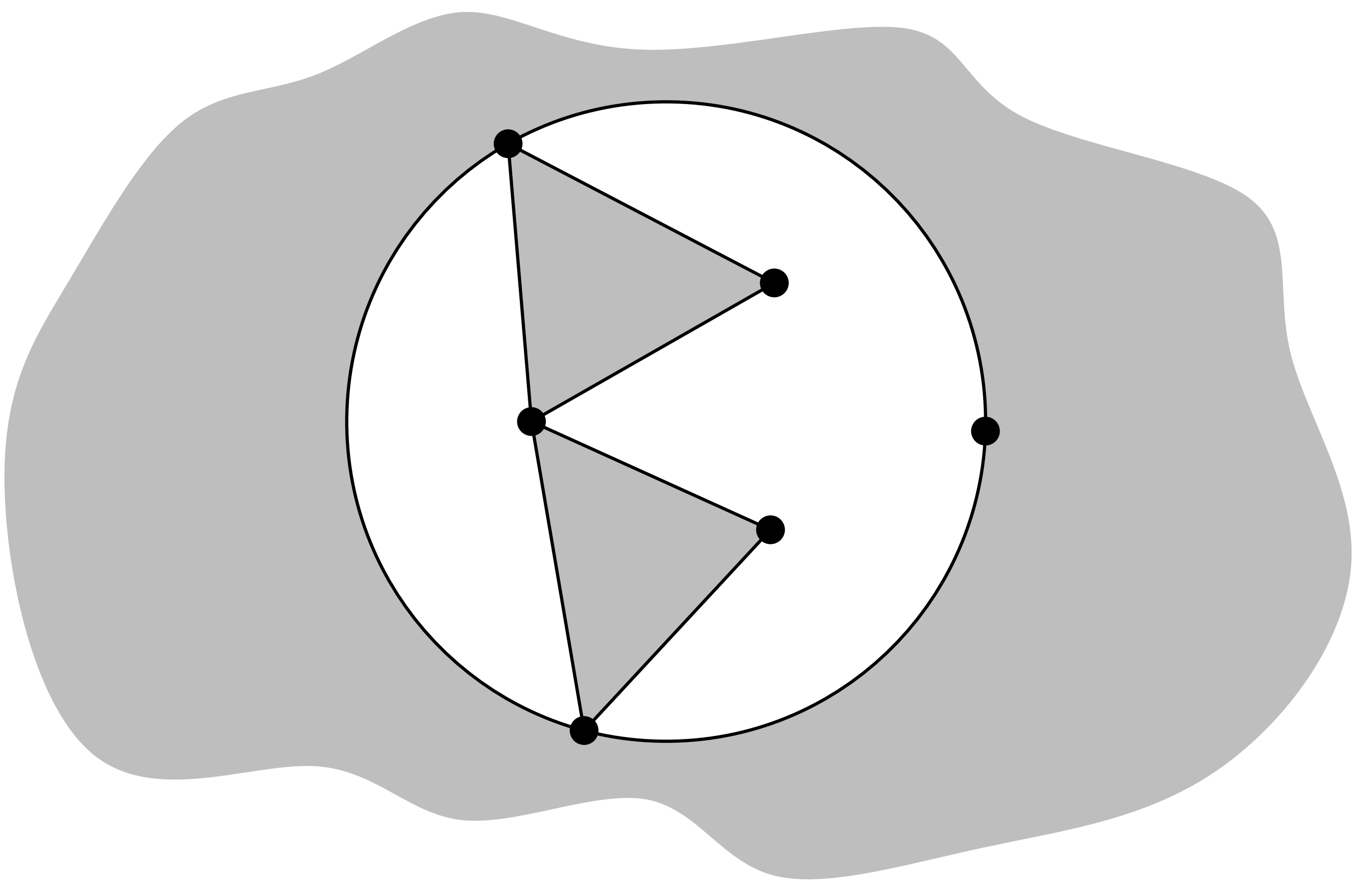
m-constellations and m-Eulerian maps

Let $m \geq 2$. We say that a planar map whose faces are bicolored (black and white) is a planar **m-constellation** if

(i) adjacent faces have different colors,

(ii) black faces have degree m and white faces have degree mi for some integer $i \geq 1$ (which can be different among white faces).



Example of a planar 3-constellation

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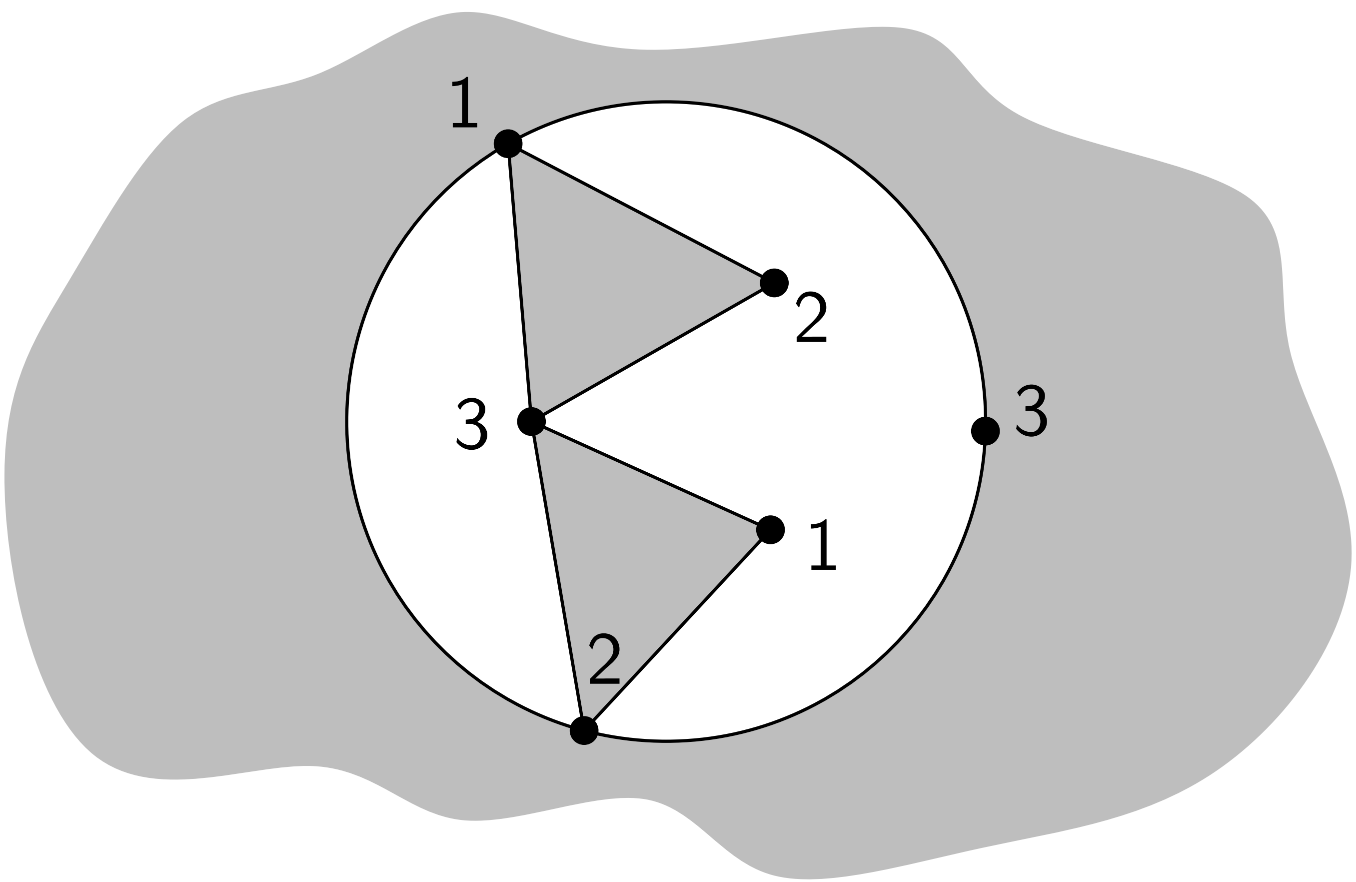
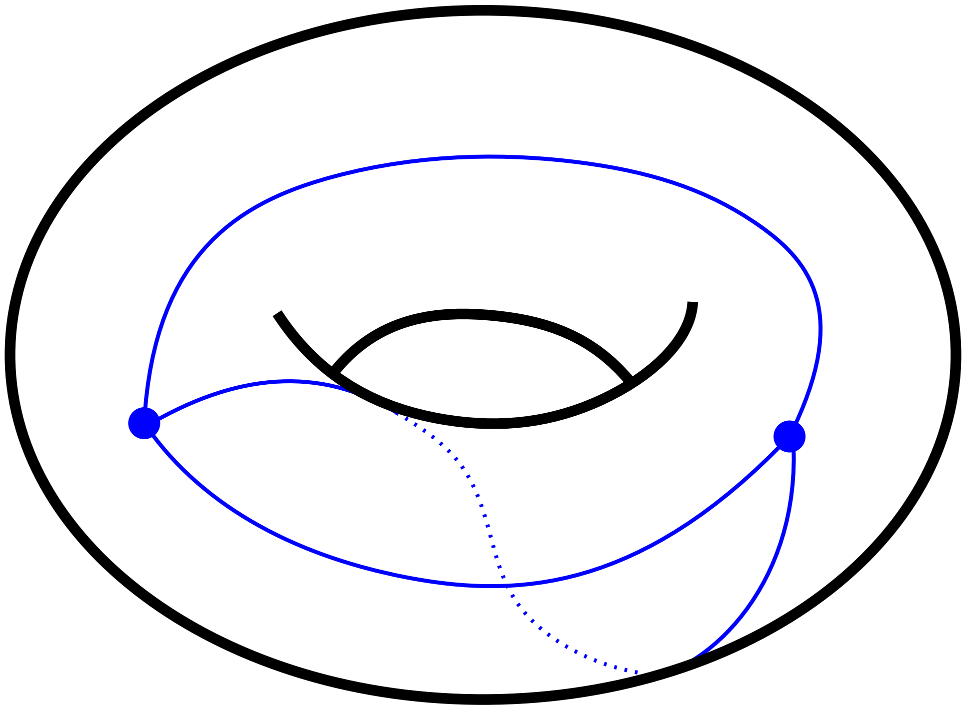
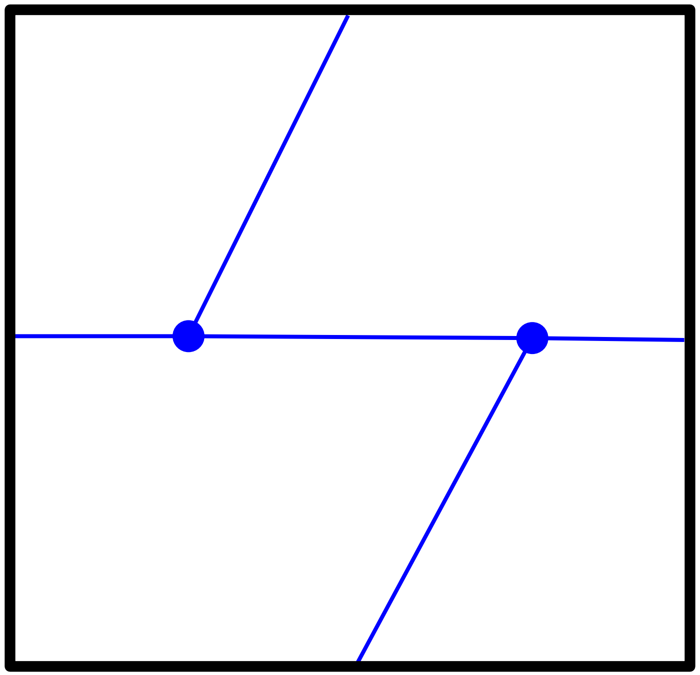


Example of a planar 3-constellation

Maps of higher genus

A map of genus g is a graph embedded in the torus with g holes such that its faces are contractible. Maps are considered up to orientation preserving homeomorphisms.





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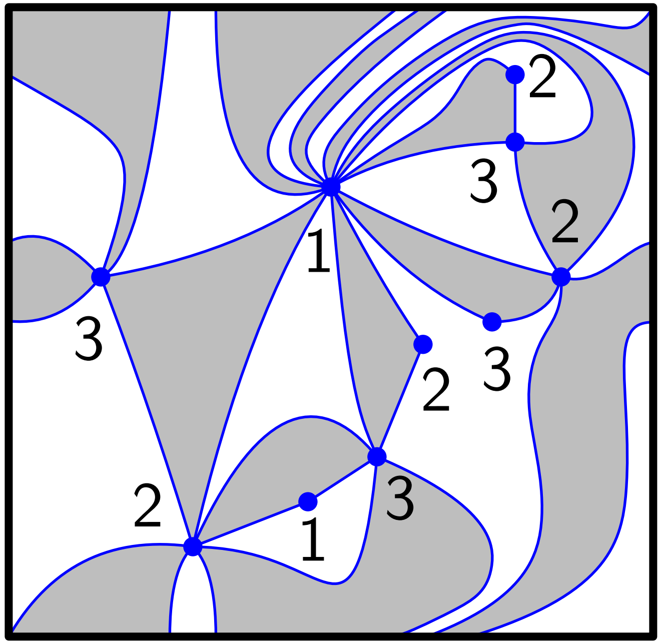

Example of a 3-constellation of genus 1

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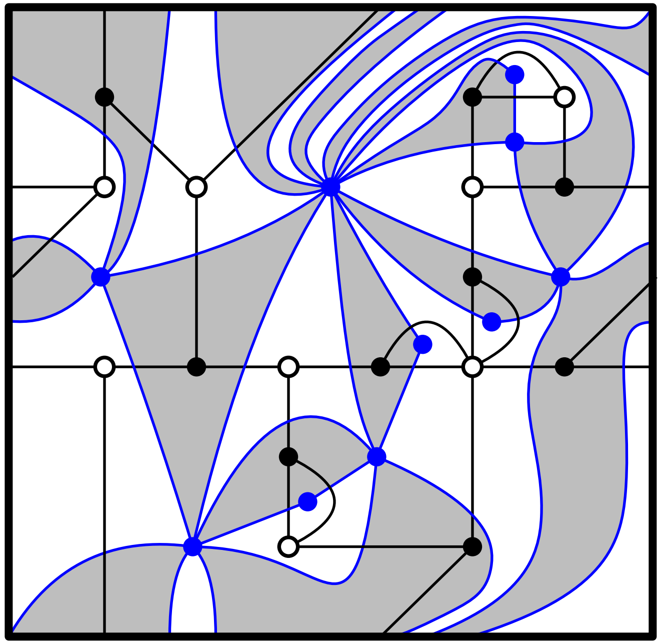
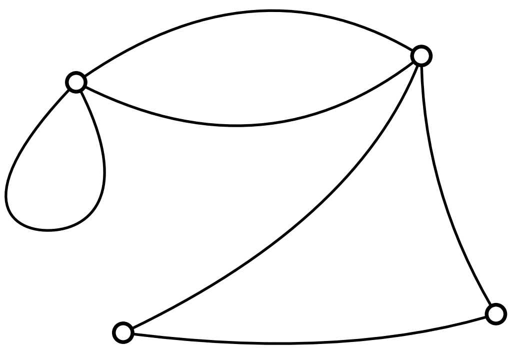
m-Eulerian maps are the dual maps of m-constellations



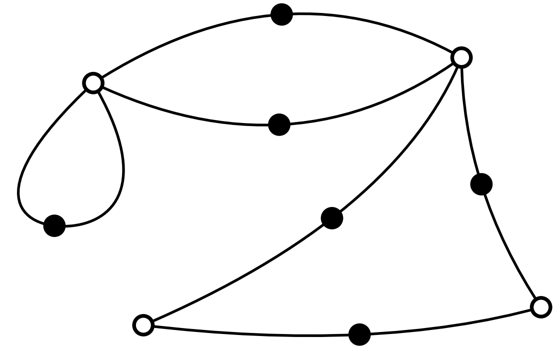
Example of a 3-constellation of genus 1

m-constellations and m-Eulerian maps

• In the planar case, 2-Eulerian maps are essentially the same as the well-known Eulerian maps.



Planar Eulerian map.



Planar 2-Eulerian map.

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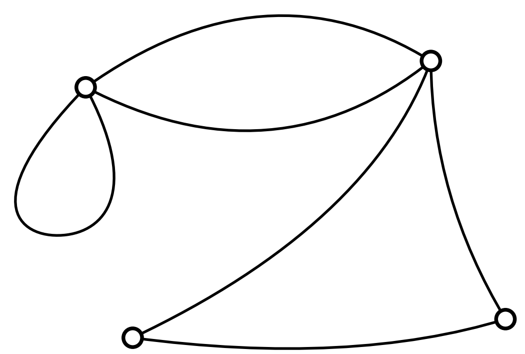

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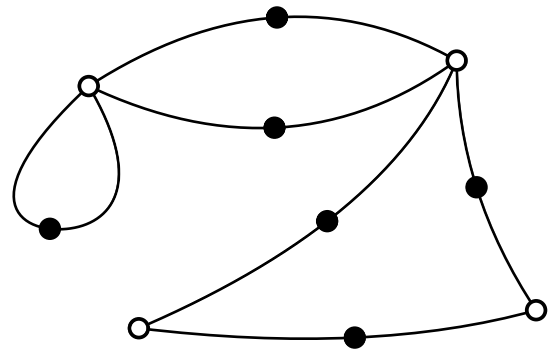

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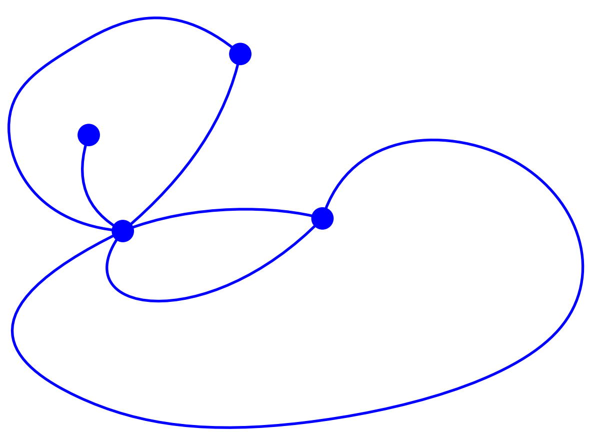
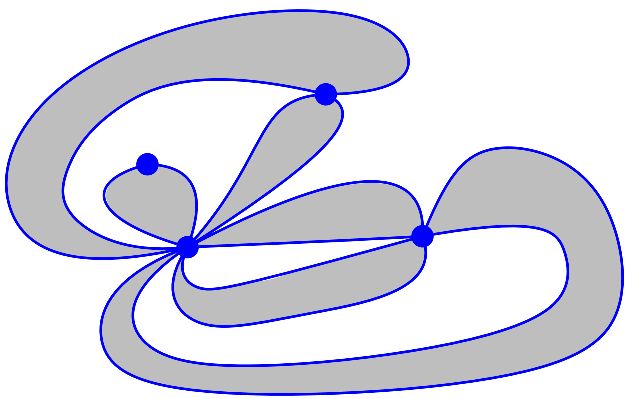


Its dual map.



Its dual planar 2-constellation.

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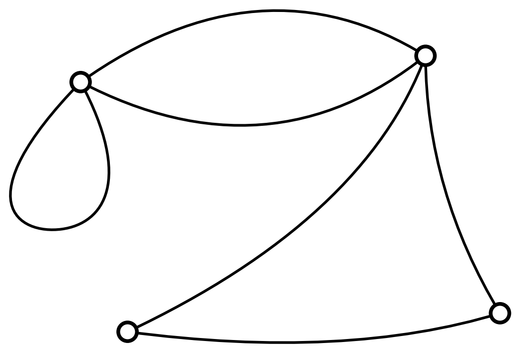

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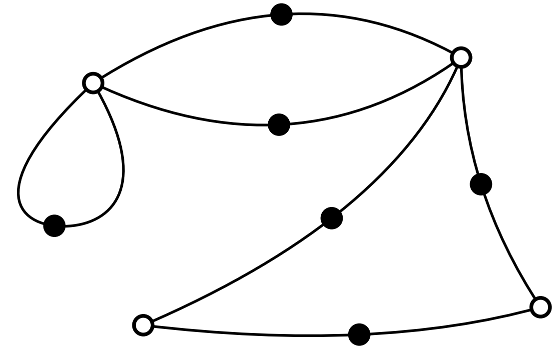
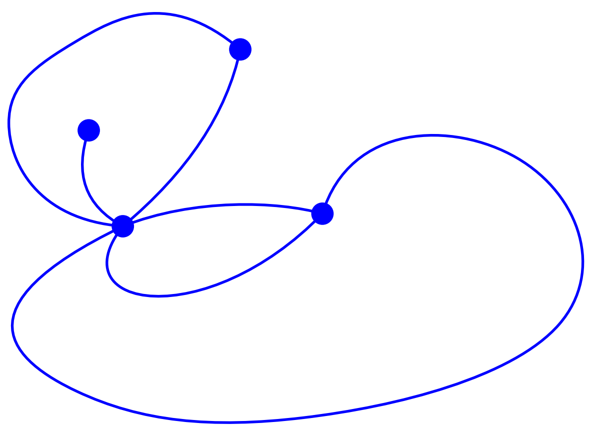


Planar 2-Eulerian map.



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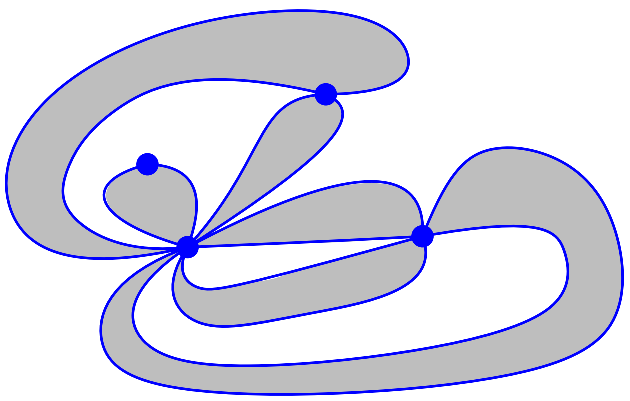


Its dual planar 2-constellation.

• In higher genus, 2-Eulerian maps have the additional condition (iii). They are essentially face-bicolorable maps.

m-constellations and m-Eulerian maps

The method introduced by Tutte for the enumeration of planar maps based on catalytic variables has not been effective to deal with constellations.

The successful attempts of [Bousquet-Mélou,Schaeffer'00] and [Bouttier,Di Francesco,Guitter'04] use bijections. They give explicit formulas for the enumeration of constellations:

Theorem. The number of rooted planar m-constellations with d_i white faces of degree mi is

$$m\,(m-1)^{f-1}\,\frac{[(m-1)n]!}{[(m-1)n-f+2]!}\prod_{i\geq 1}\frac{1}{d_i!}\binom{mi-1}{i-1}^{d_i},$$

where $n=\sum id_i$ is the number of black faces and $f=\sum d_i$ is the number of white faces.

m-constellations and m-Eulerian maps

The work of [Chapuy'09] extends the bijection of [Bouttier,Di Francesco,Guitter'04] to higher genus.

Our goal is to extend the bijection of [Bousquet-Mélou,Schaeffer'00] to higher genus and obtain similar explicit formulas.

Since their formulation of the bijection is hard to extend to higher genus, we reformulate it in the planar case in such a way that it is easy to generalize.

This reformulation is inspired by the work of [Lepoutre'19], on face-bicolorable maps of higher genus.

Our results

Theorem. Rooted m-constellations of genus g are in bijection with well-rooted m-bipartite unicellular maps of genus g.

This theorem extends the bijections of [Bousquet-Mélou,Schaeffer'00] and [Lepoutre'19] at the same time.

Using this bijection, we give the enumeration of a very particular case.

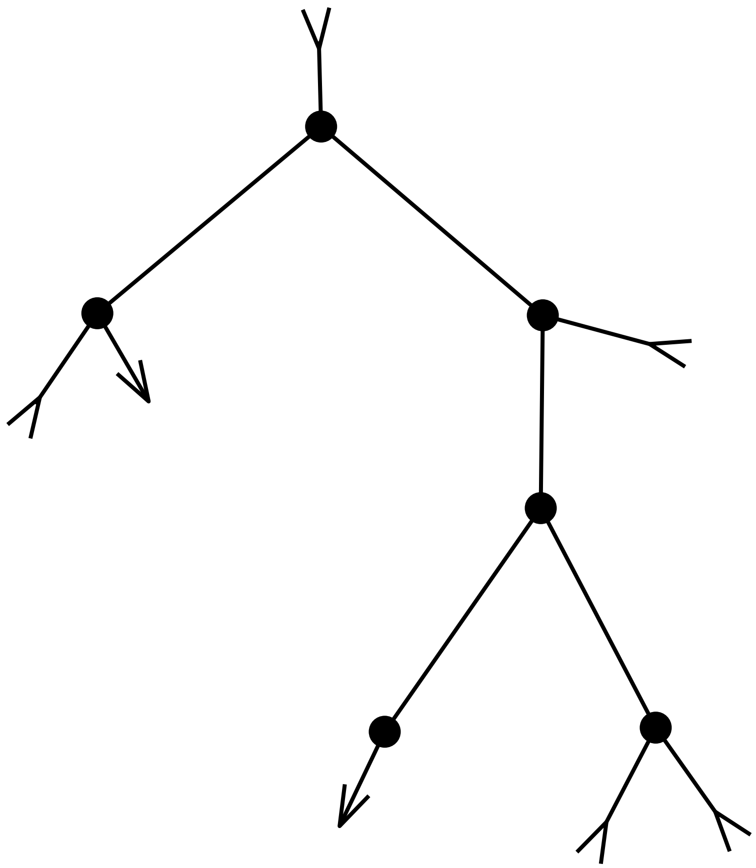
Corollary. Rooted 3-constellations of genus 1 whose white faces are triangles (counted by their number of white faces) are enumerated by

$$C(z) = \frac{T(z)^3}{(1-T(z))(1-4T(z))^2}$$

where $T(z)$ is the unique generating function satisfying $T(z) = z + 2T(z)^2$.

Blossoming trees

Blossoming trees are plane trees decorated with half-edges or stems.
There are two types of stems: instems and outstems.



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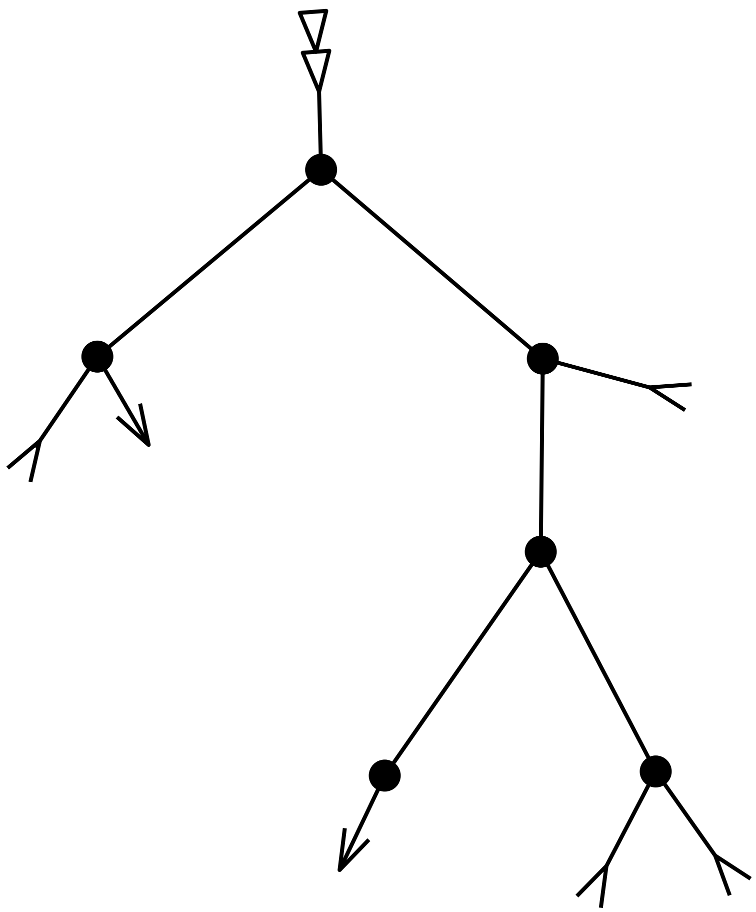

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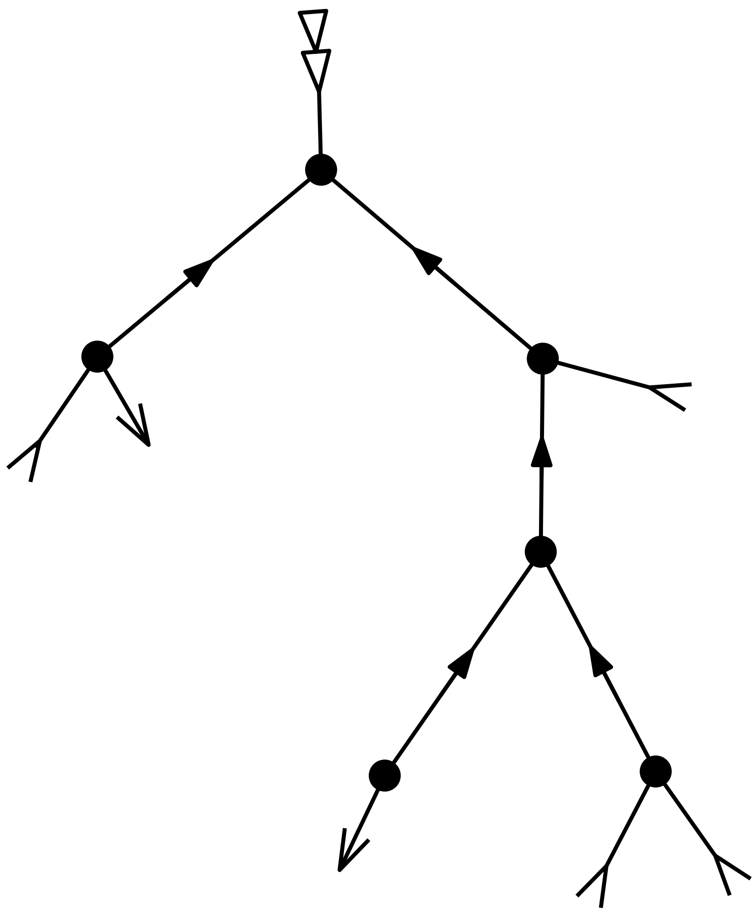

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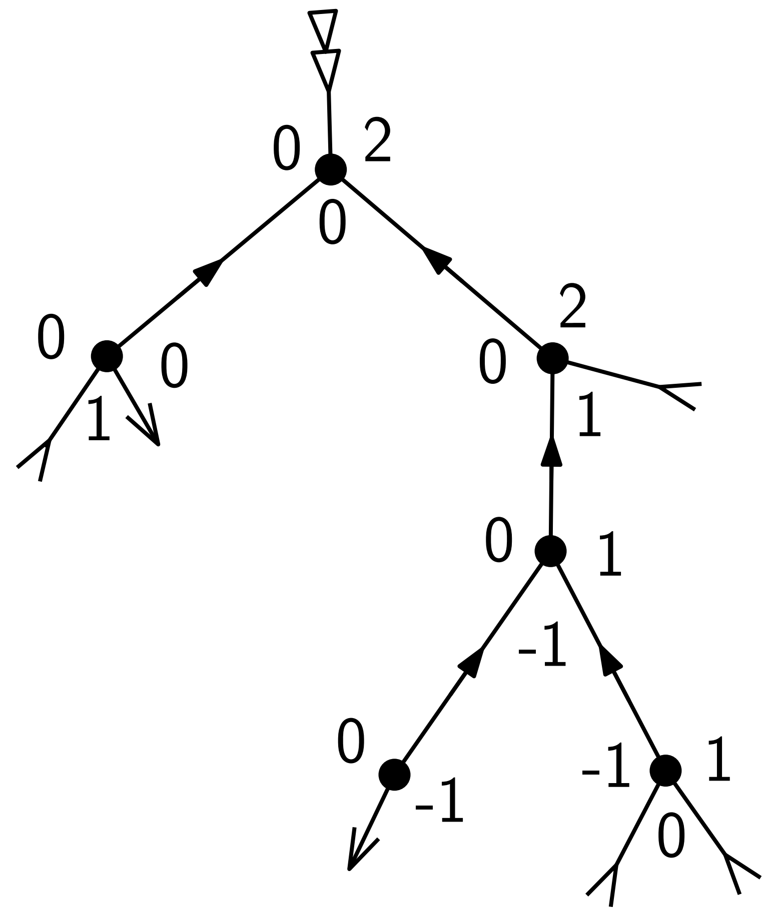

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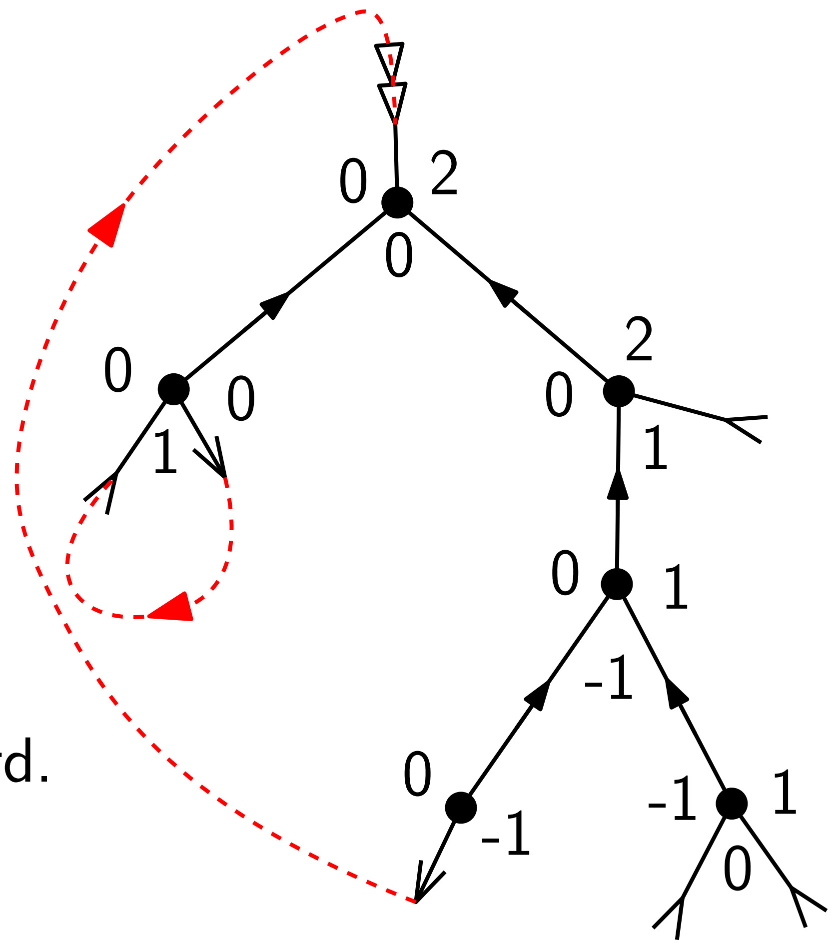
- The stems form a cyclic parentheses word. We can define their **matching**.



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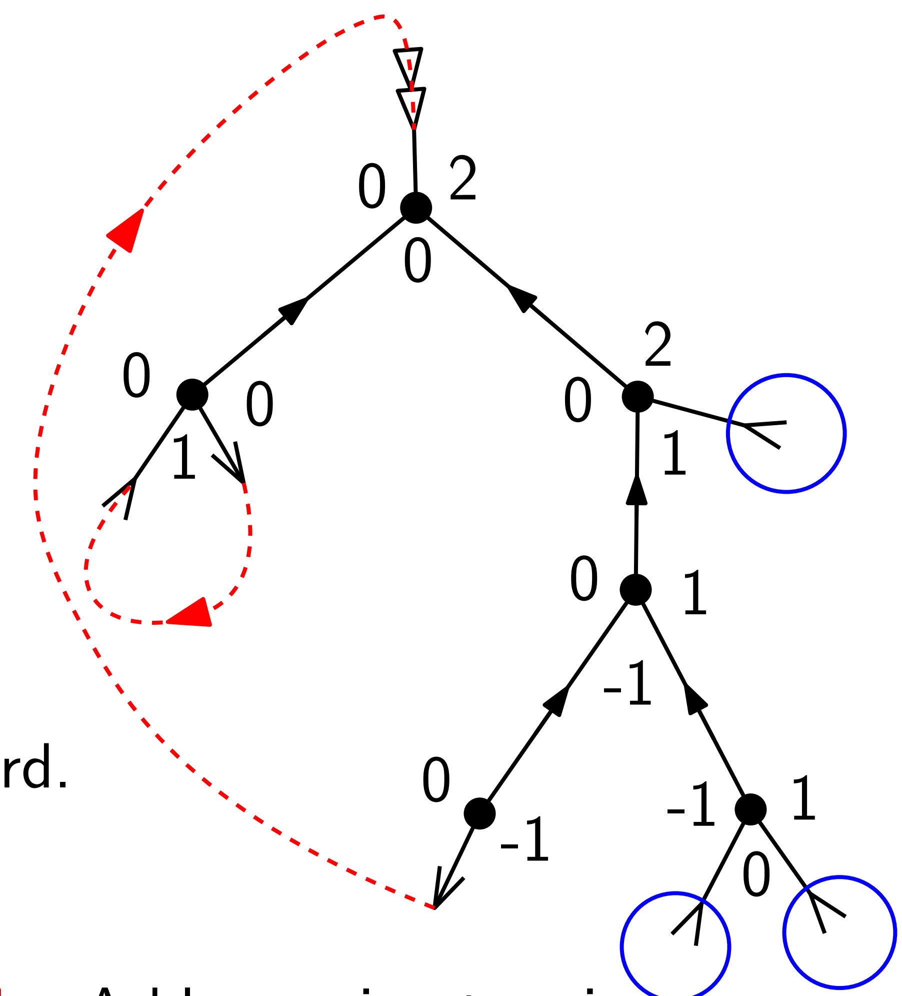

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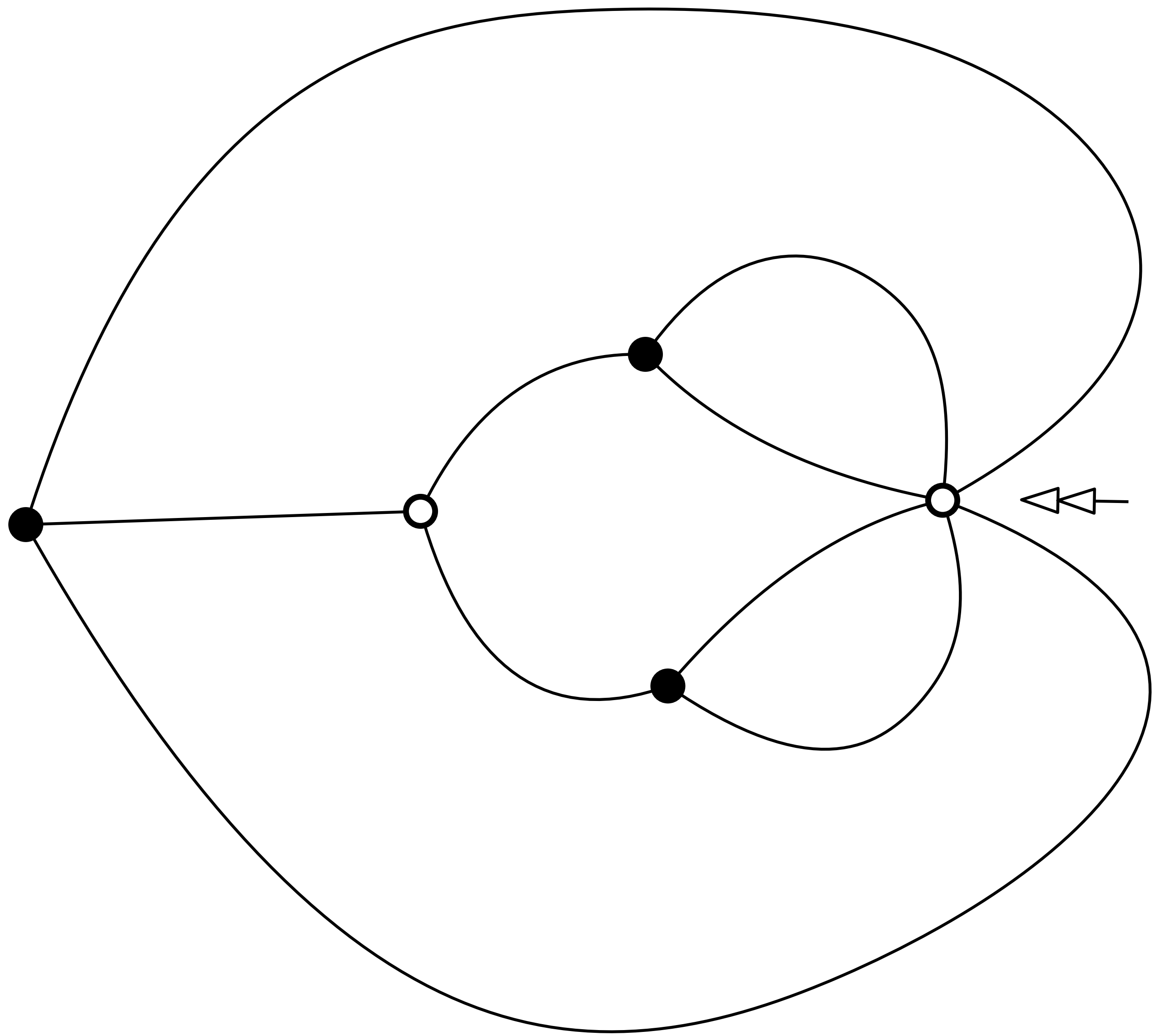
- We can define a **good labelling** of their corners.
- The stems form a cyclic parentheses word. We can define their **matching**.
- The unmatched instems are called **single**. A blossoming tree is **well-rooted** if its root instem is single.

The opening of a rooted m-Eulerian map



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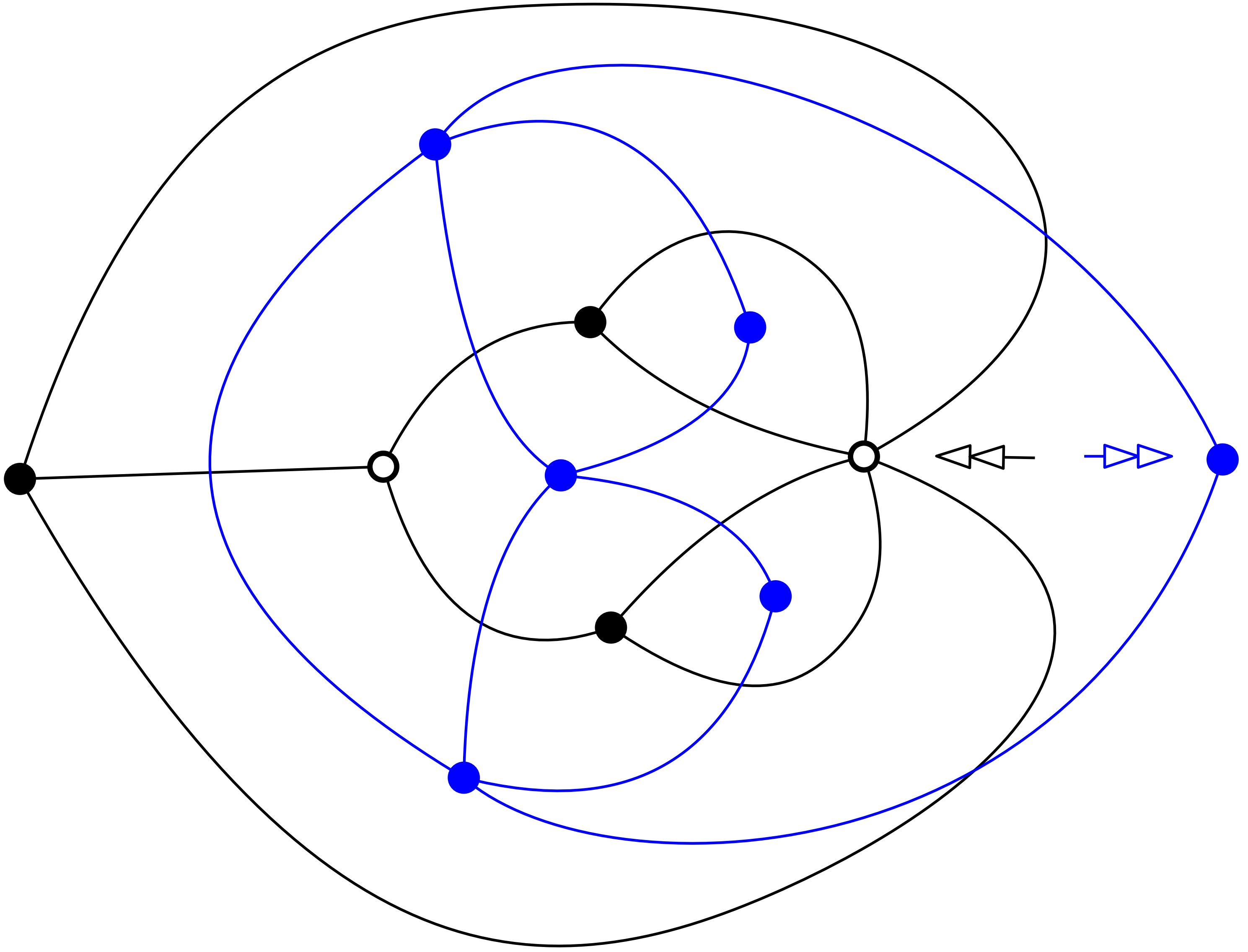

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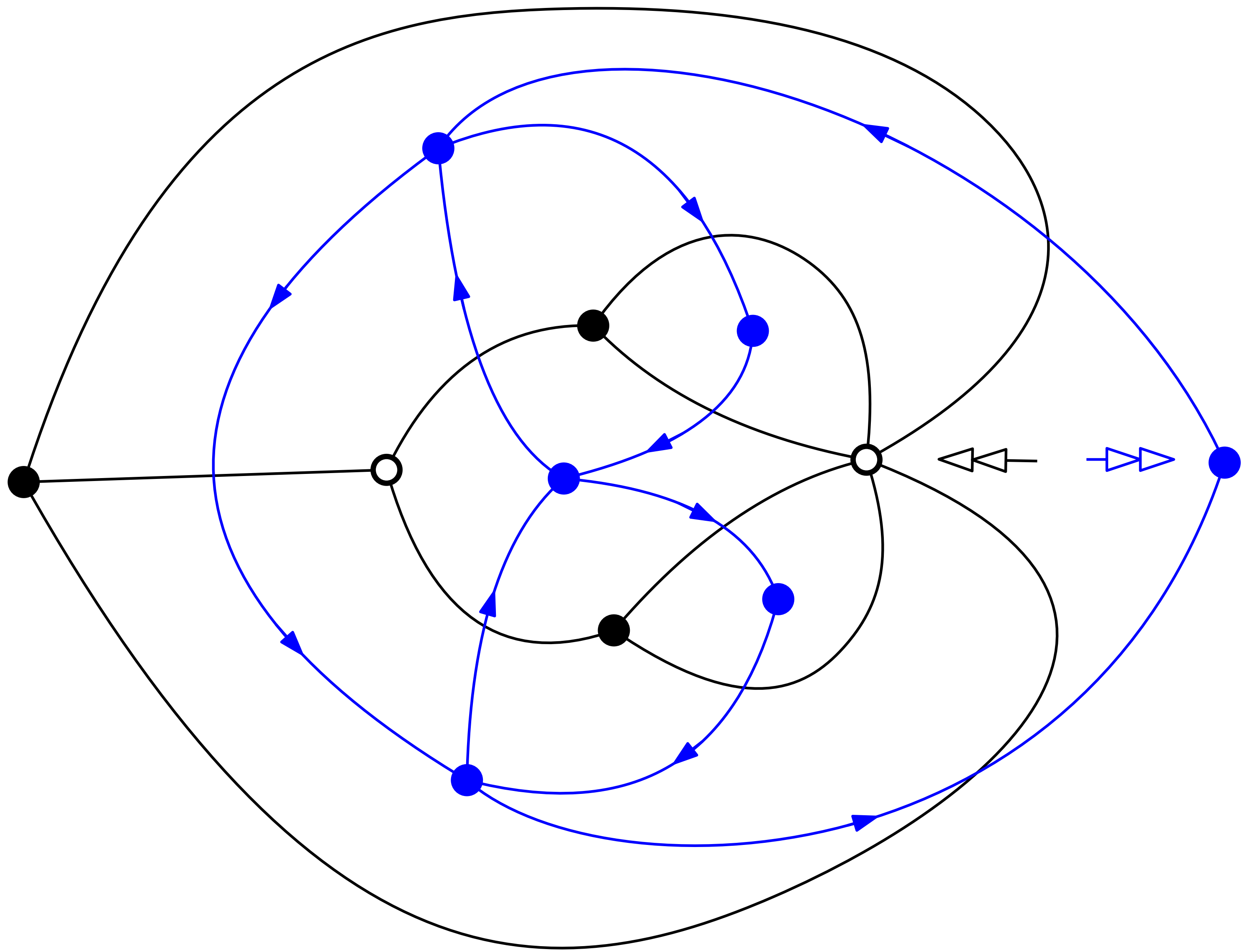

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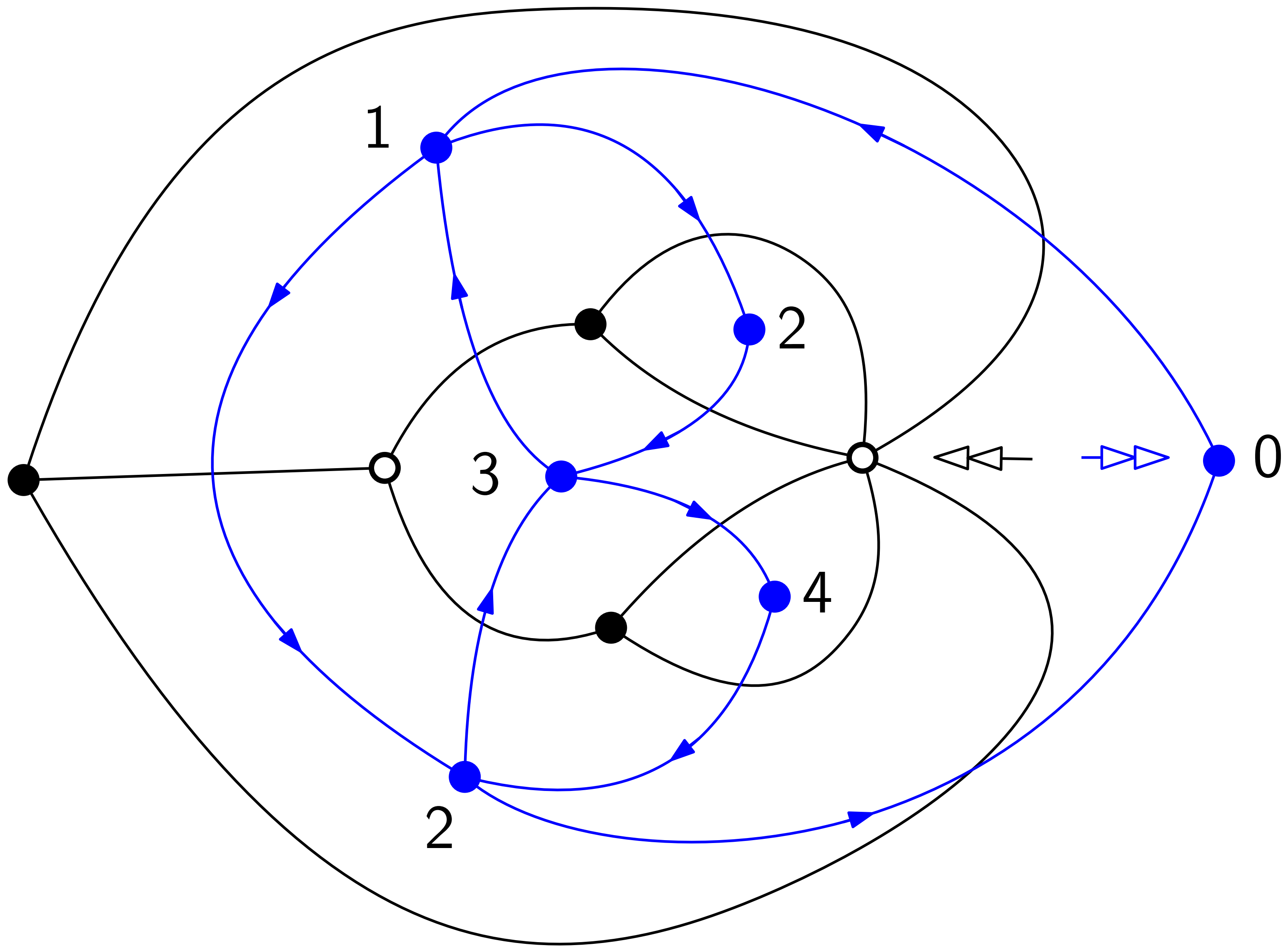

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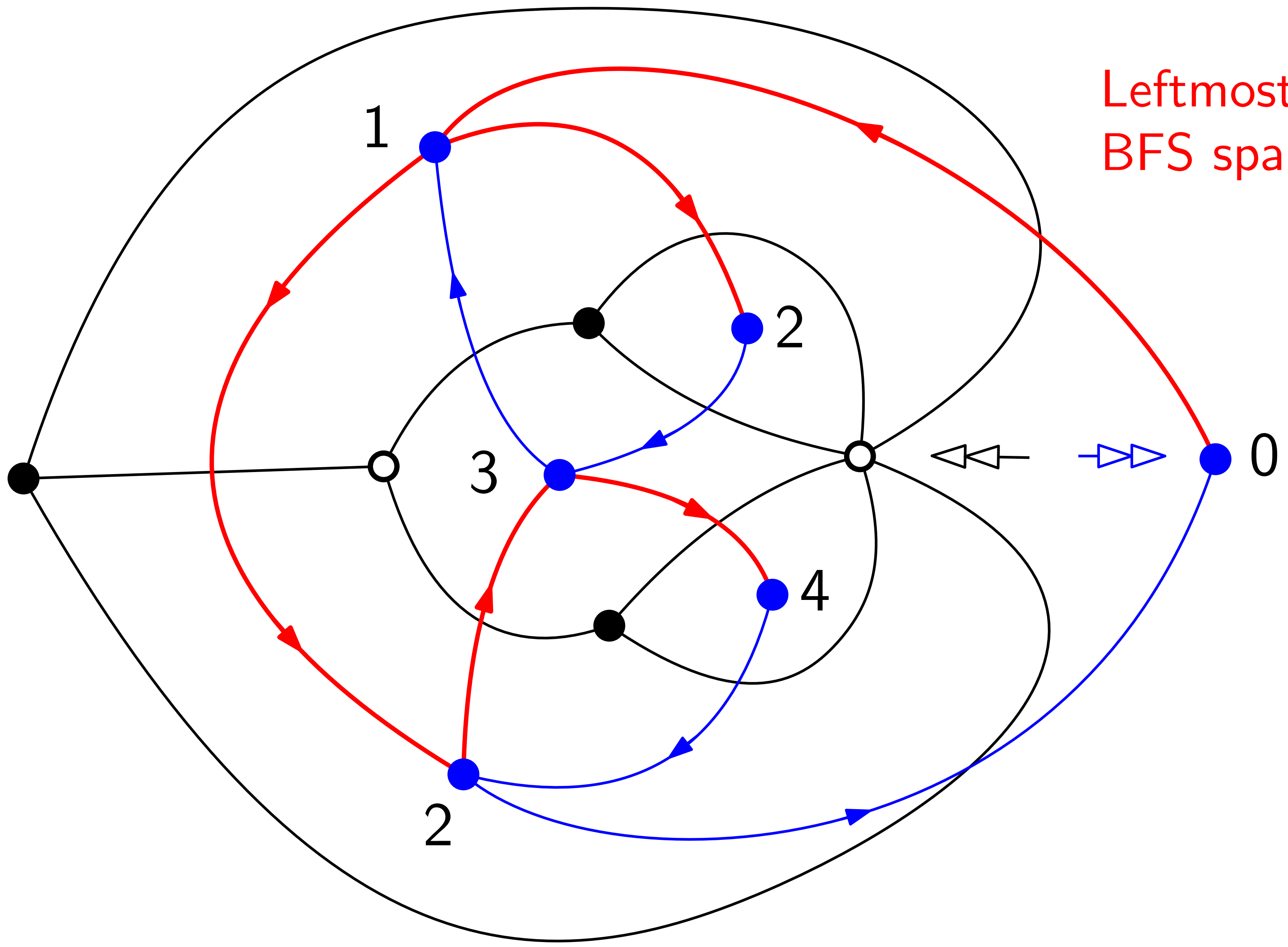

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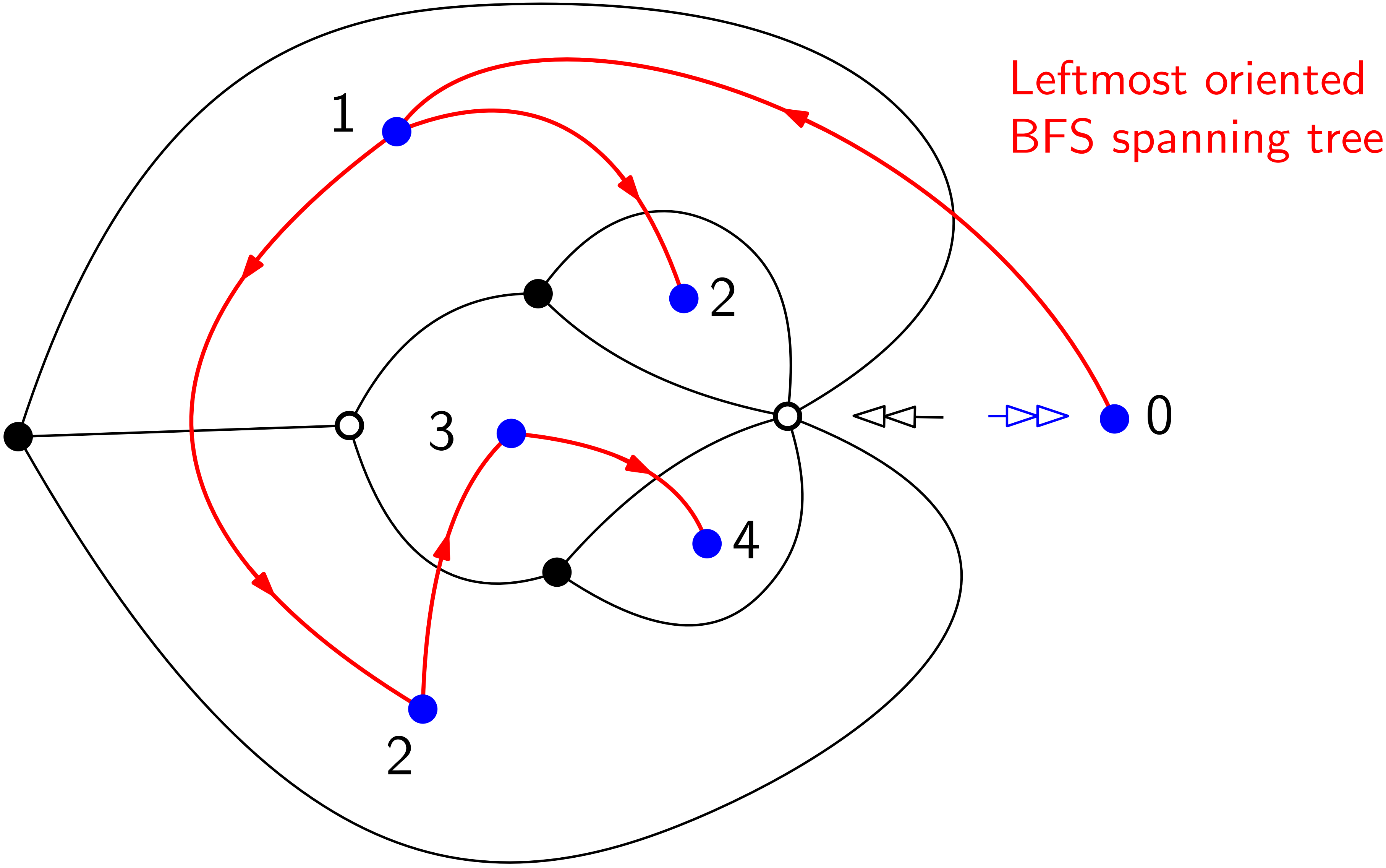

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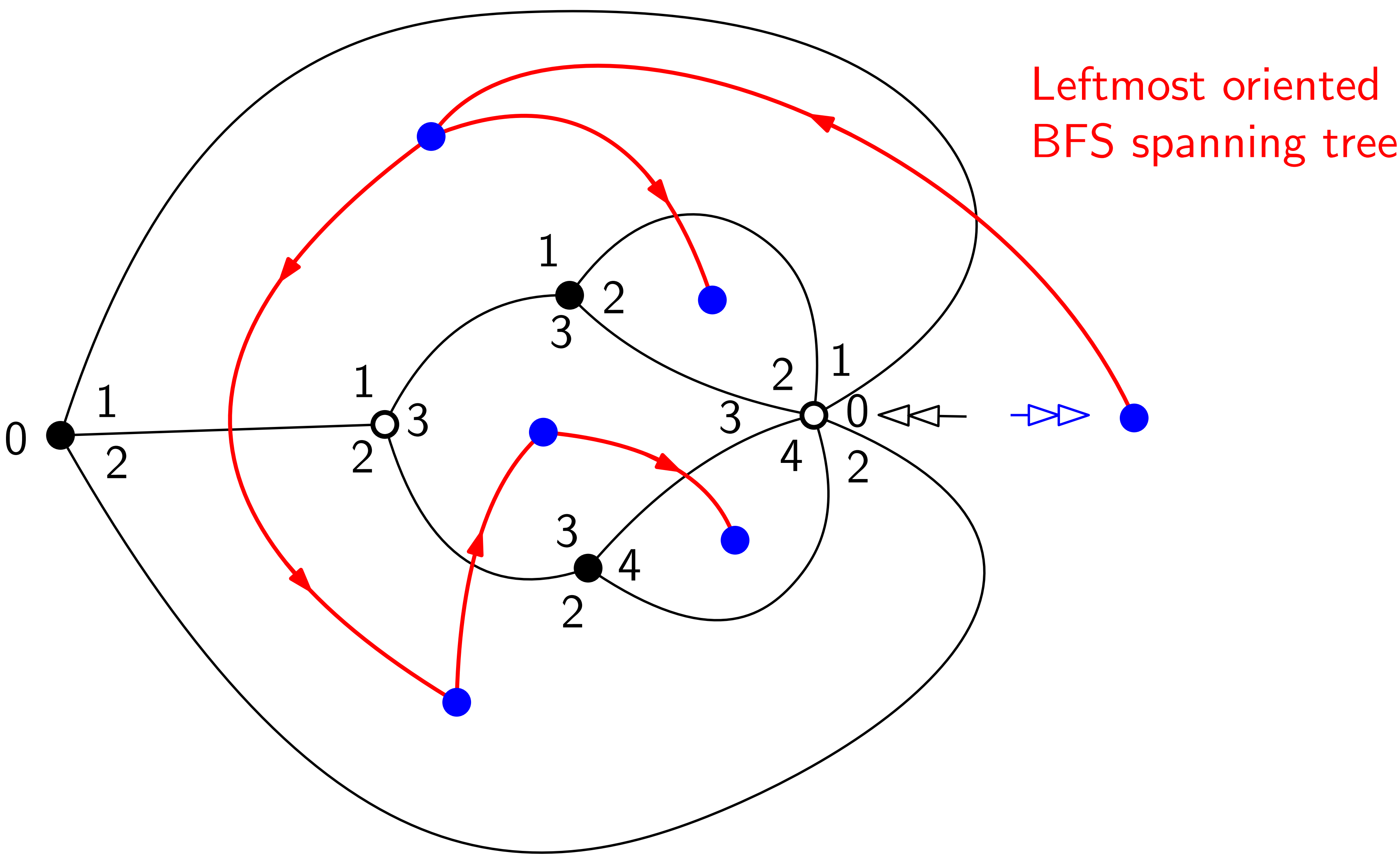

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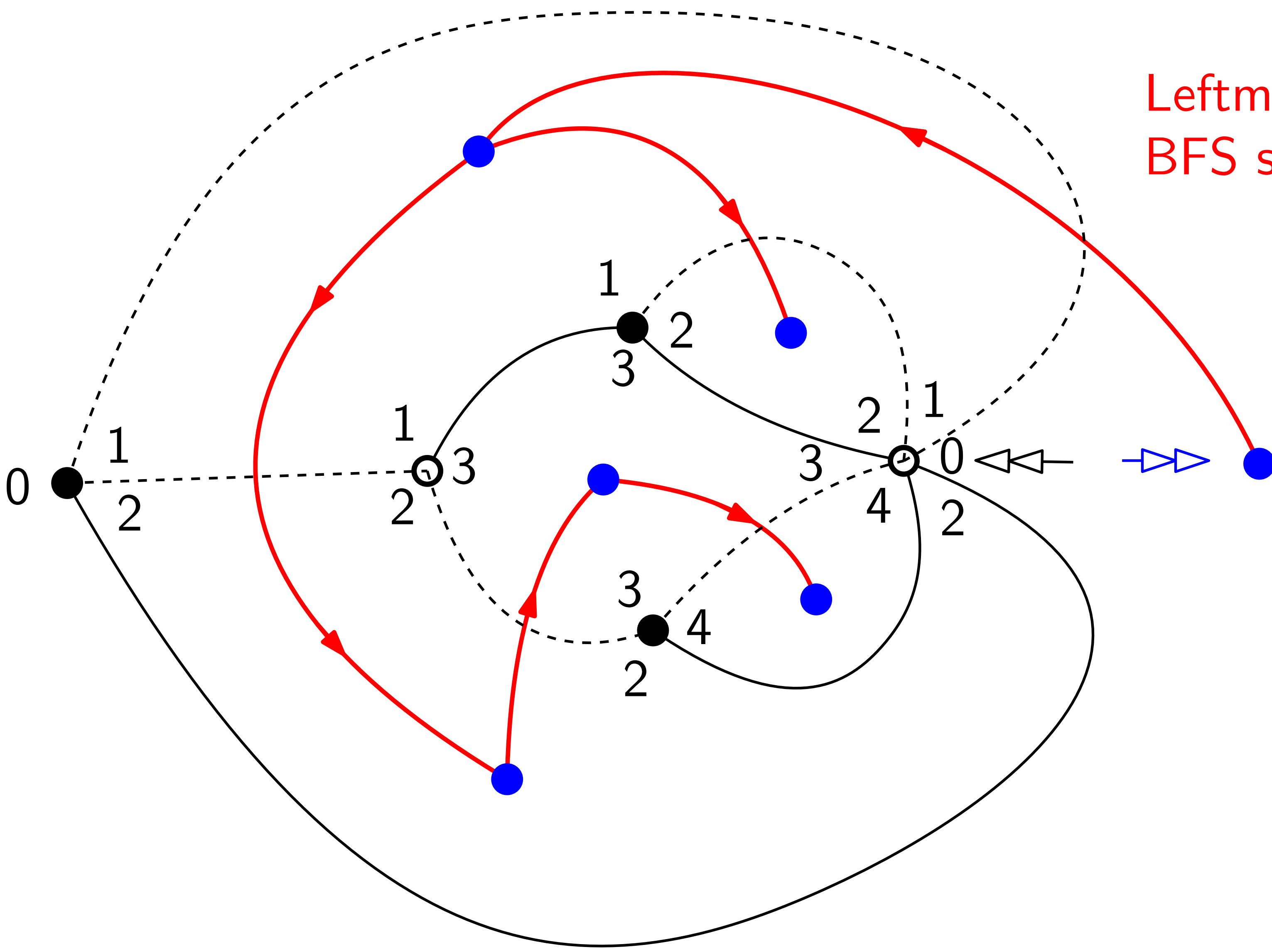

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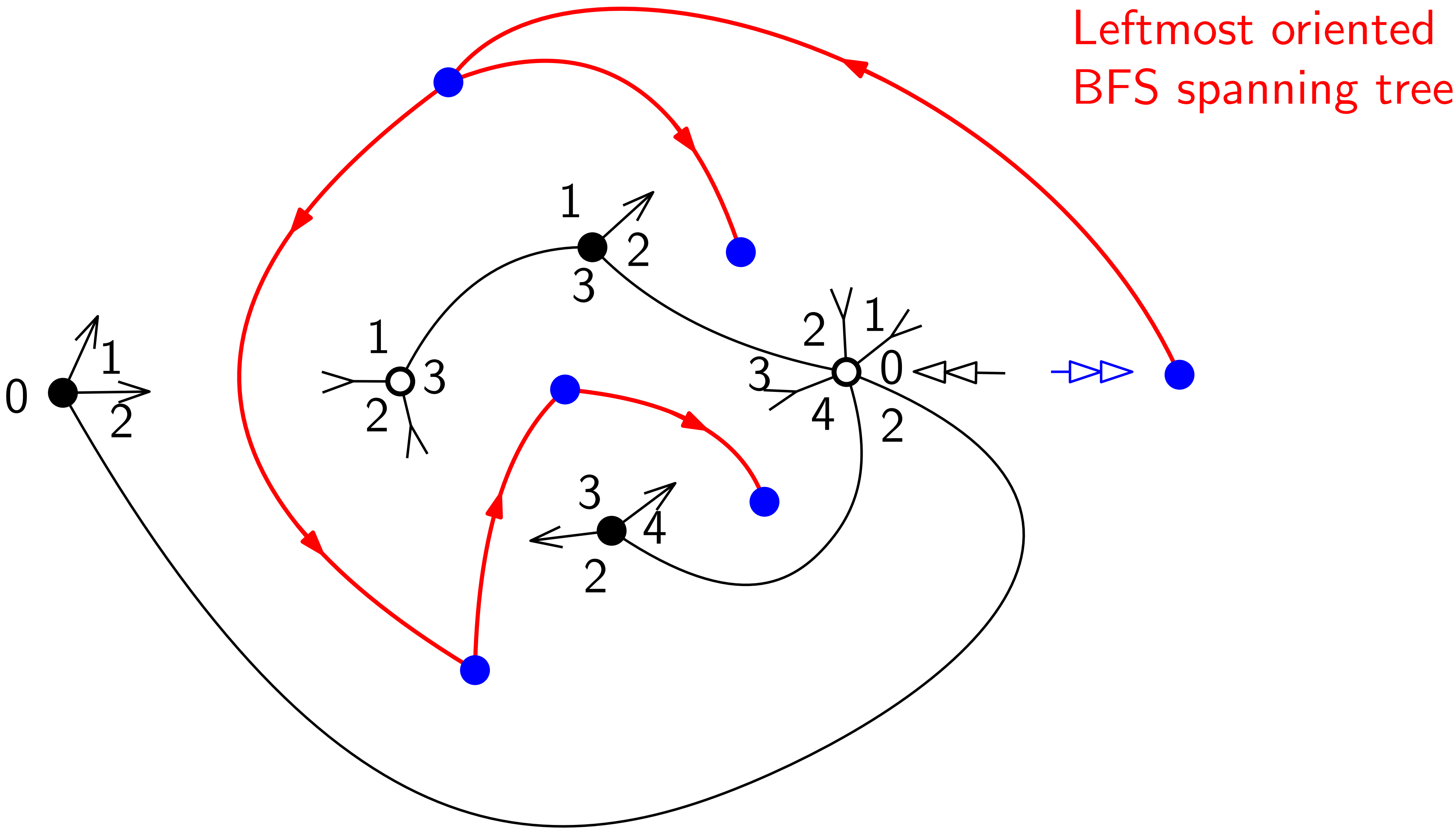

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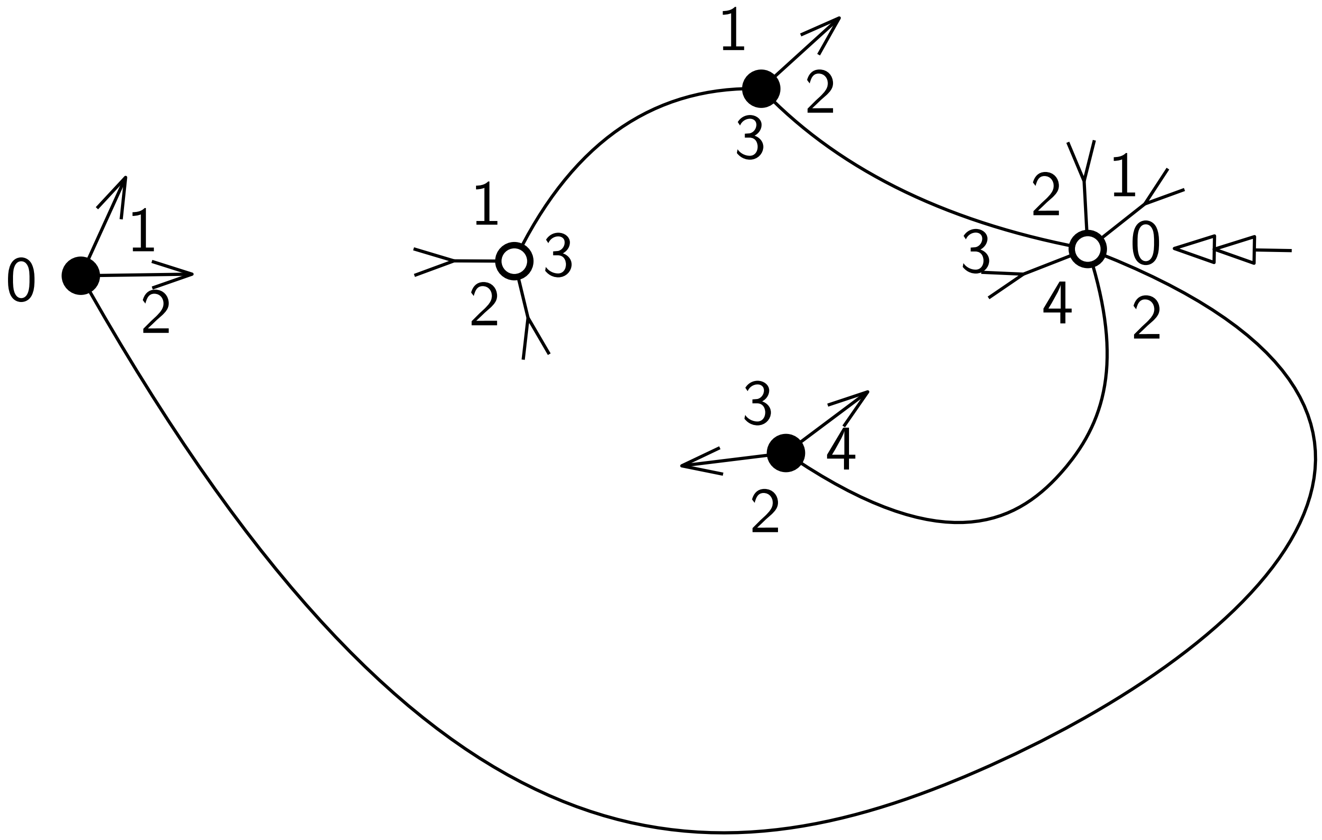

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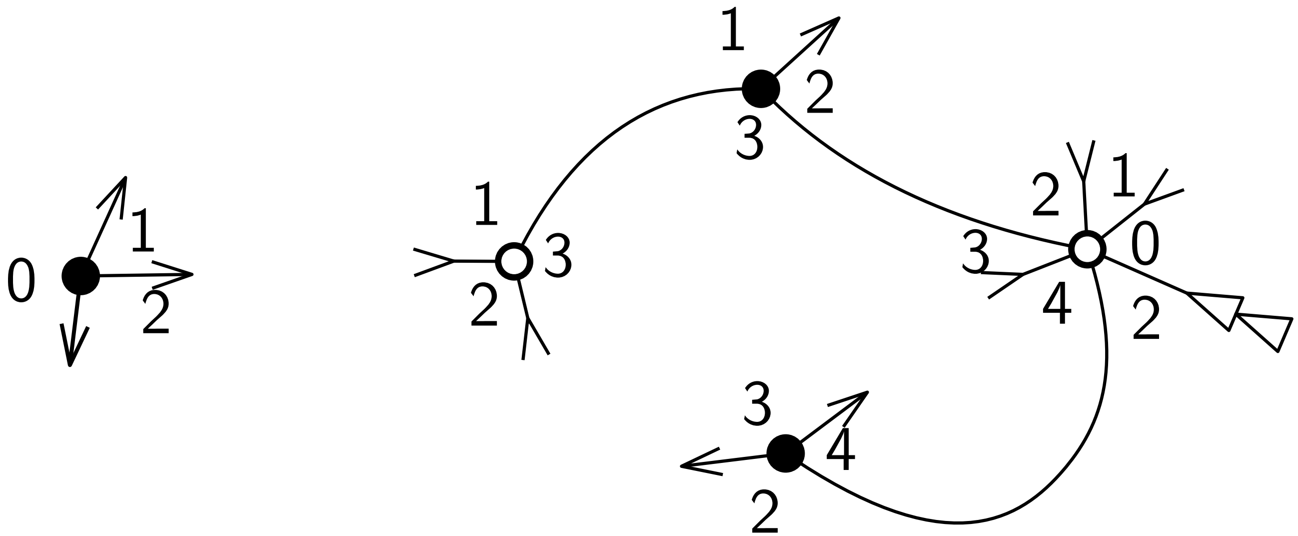

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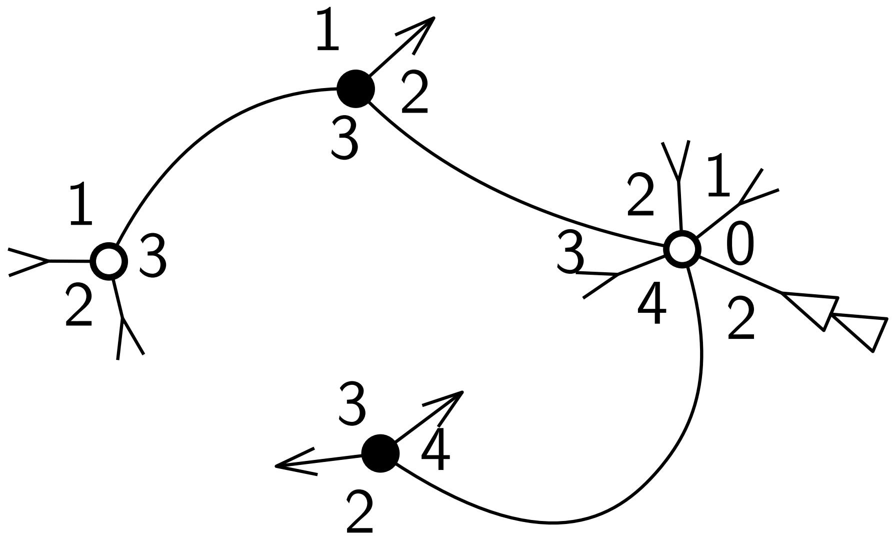
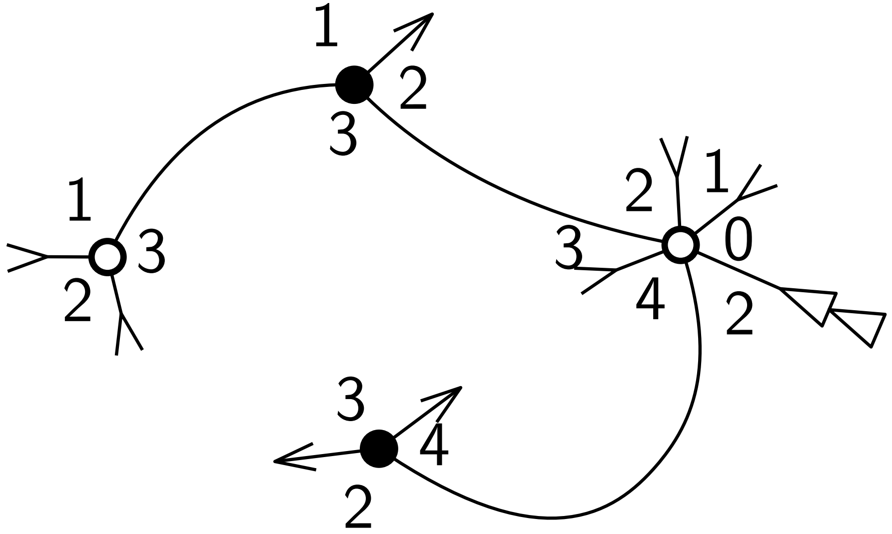


The opening of a rooted m-Eulerian map

What does it look like?



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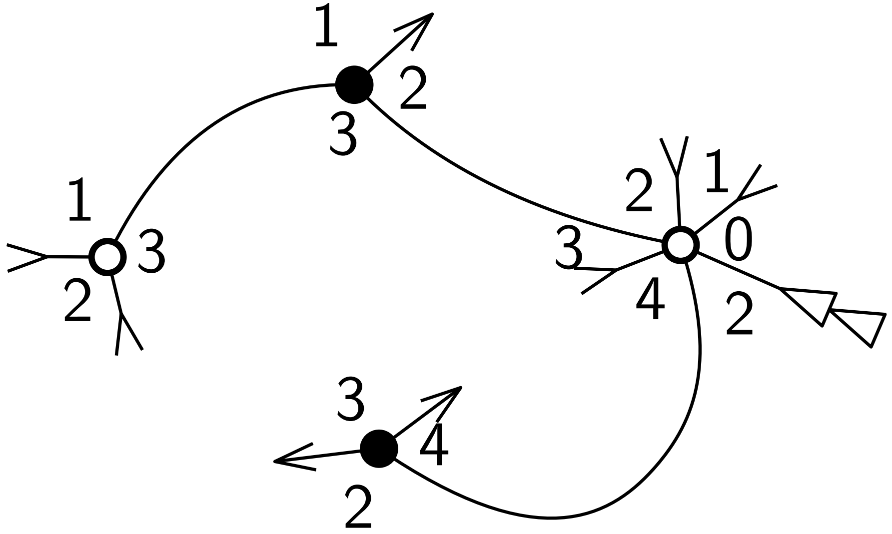

• It is a rooted blossoming tree.

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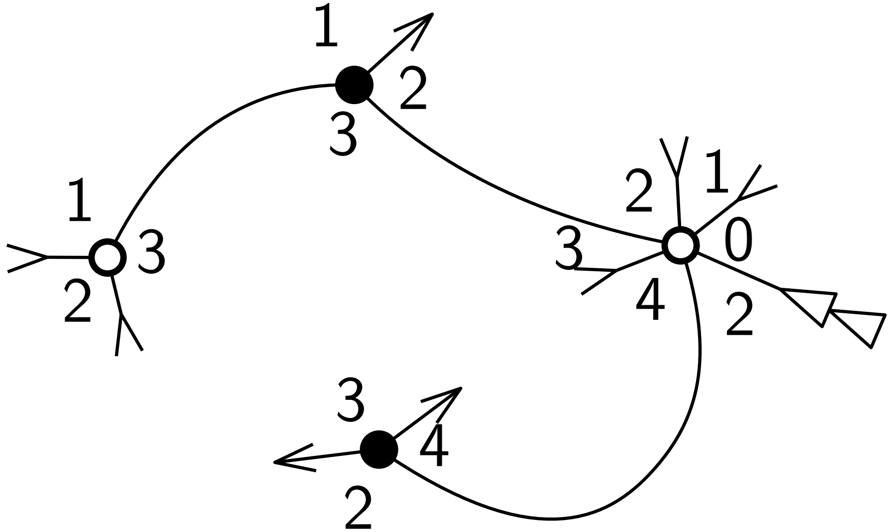

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• The degrees are preserved and it is bicolored.

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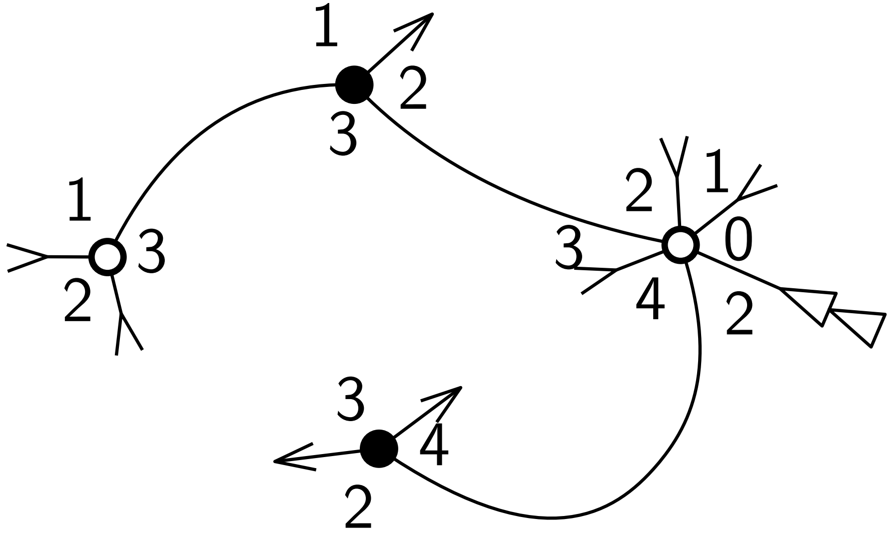

- It is a rooted blossoming tree.
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- It has a good labelling.

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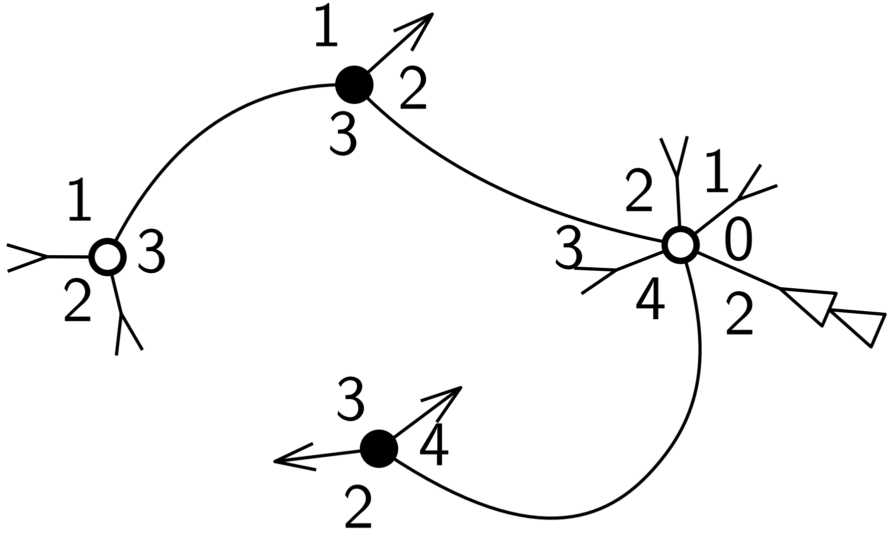

• It is a rooted blossoming tree.

• The degrees are preserved and it is bicolored.

• It has a good labelling.

• It is well-rooted.

m-bipartite trees

Let $m \geq 2$. We say that a rooted blossoming tree with m more instems than outstems and whose vertices are bicolored (black and white) is an m-**bipartite tree** if

(i) neighbouring vertices have different colors, instems are attached to white vertices and outstems are attached to black vertices,

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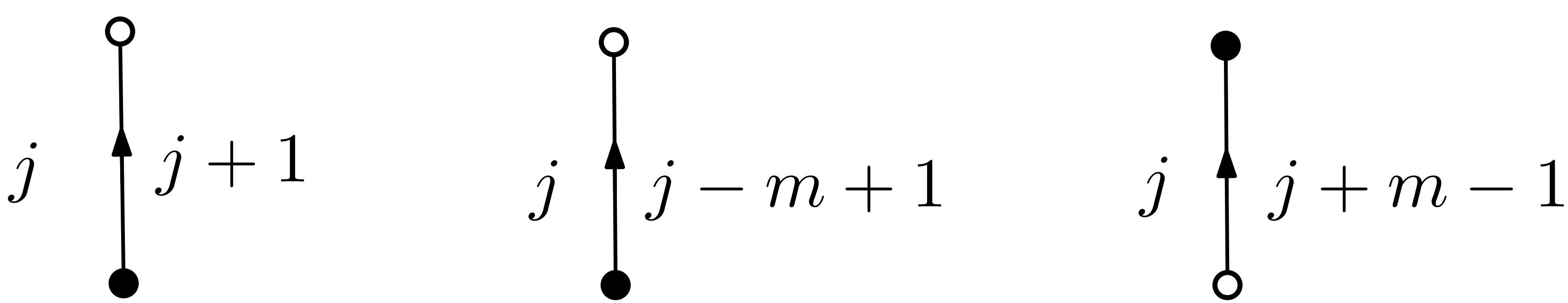
and, when endowed with its good labelling and good orientation,

(iv) the edges whose origin is a black vertex either decrease by 1 or increase by $m-1$,

(v) the edges whose origin is a white vertex decrease by $m-1$.



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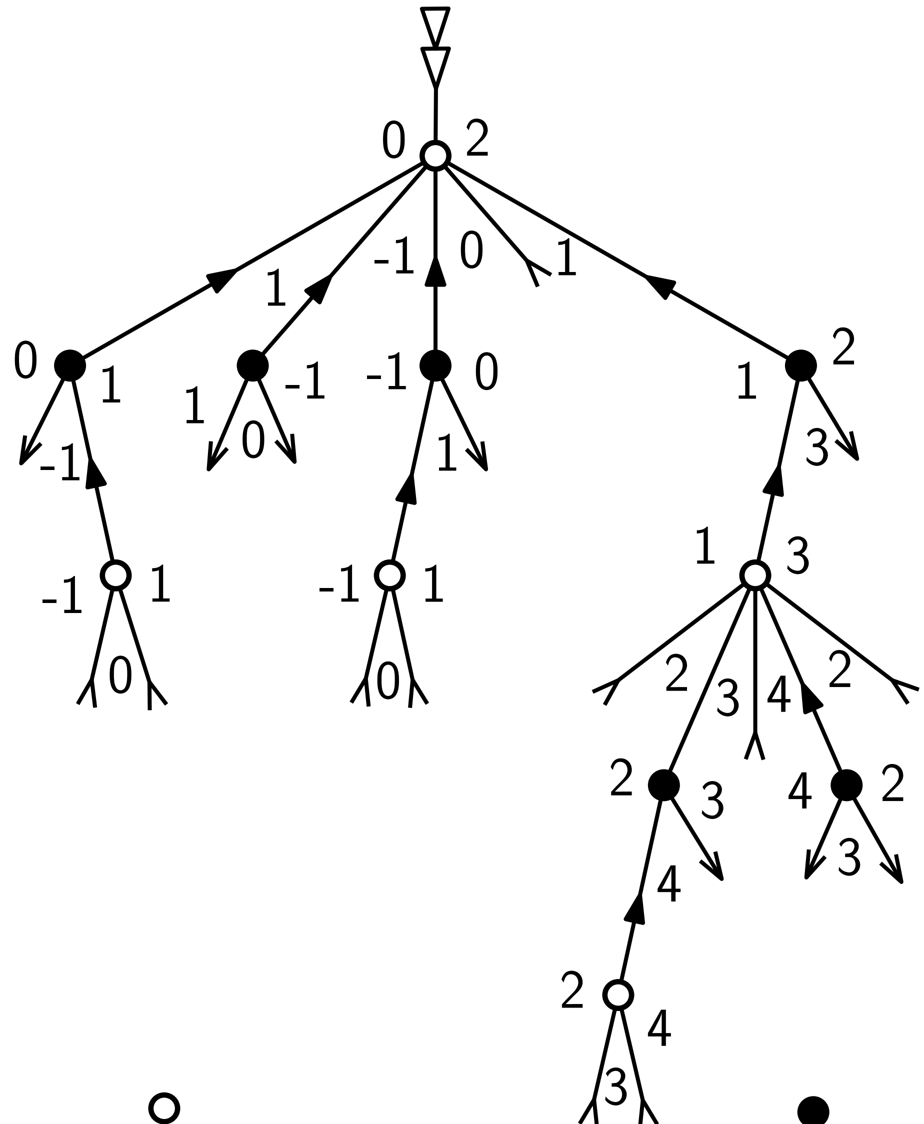
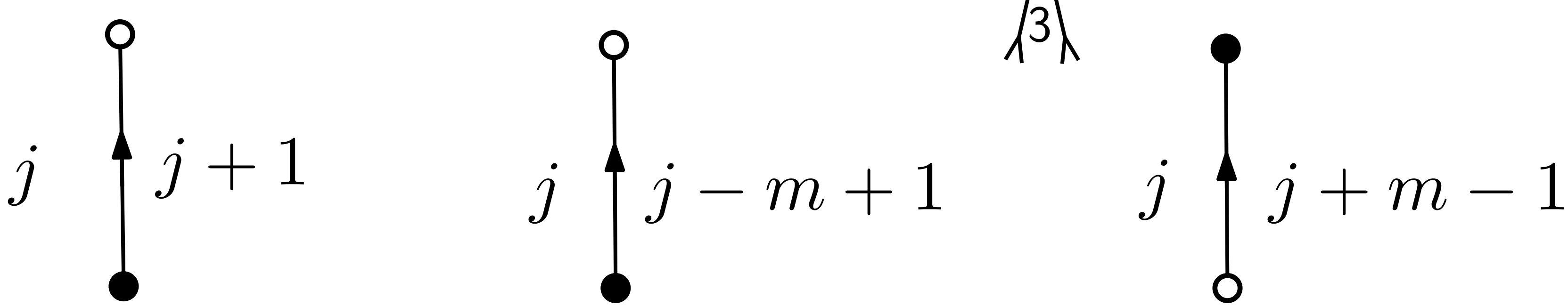
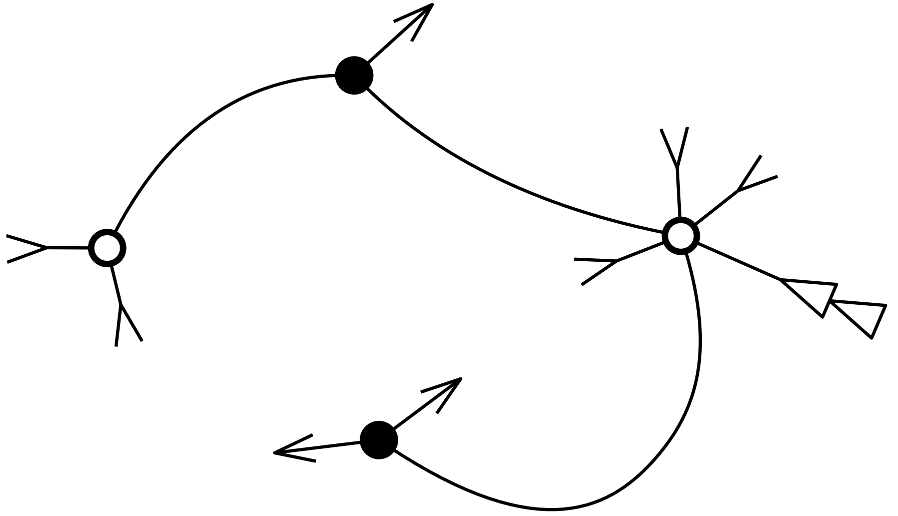




The closure of a well-rooted m-bipartite tree



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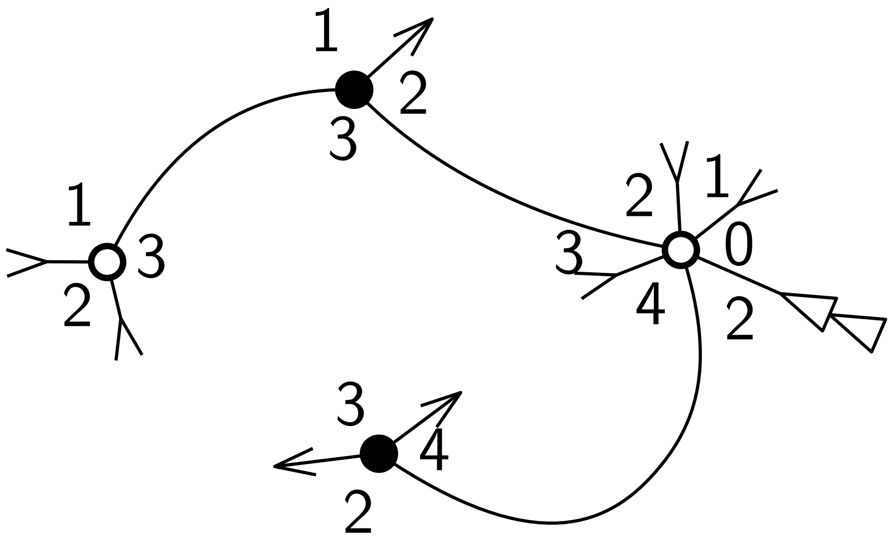

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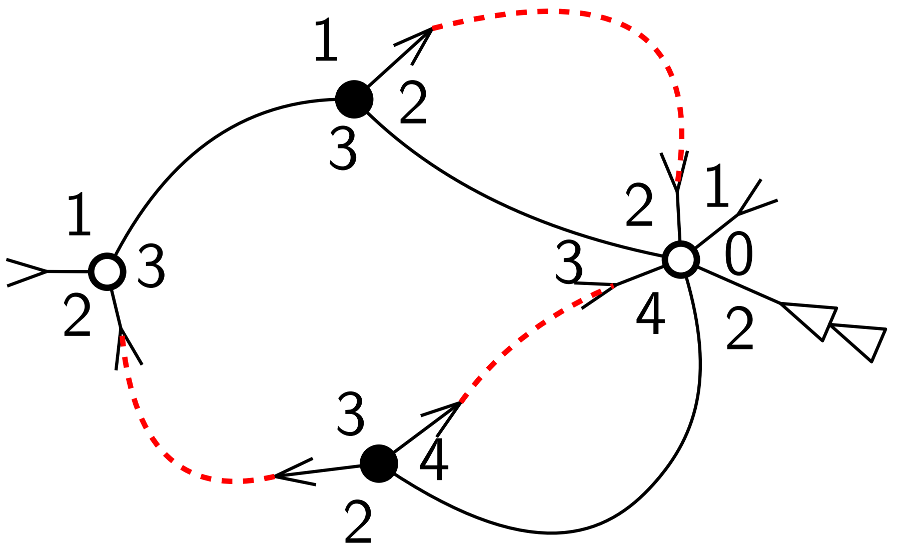

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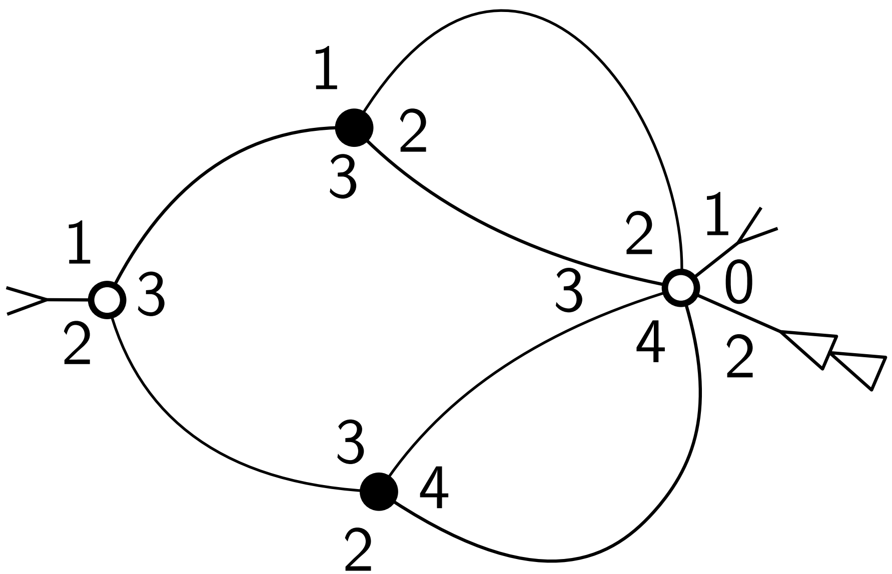

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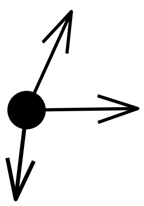

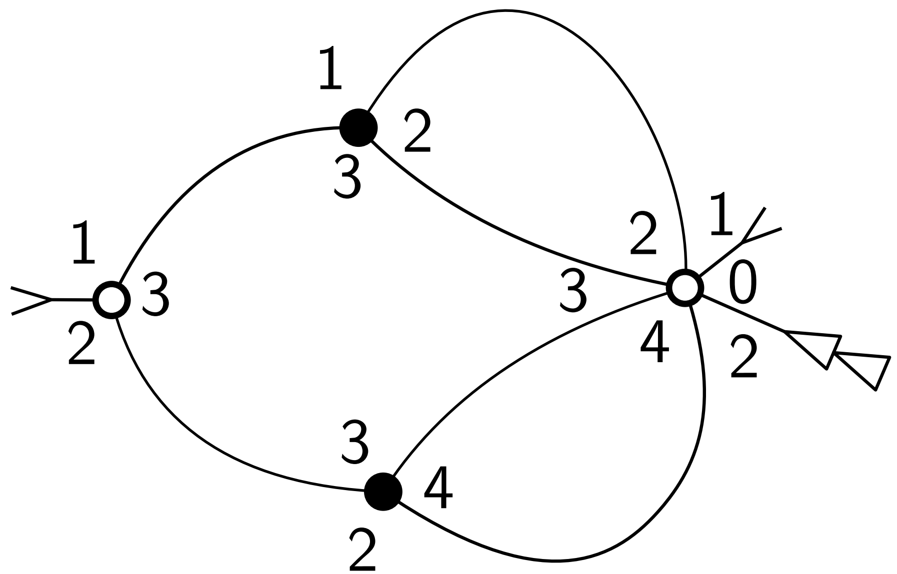

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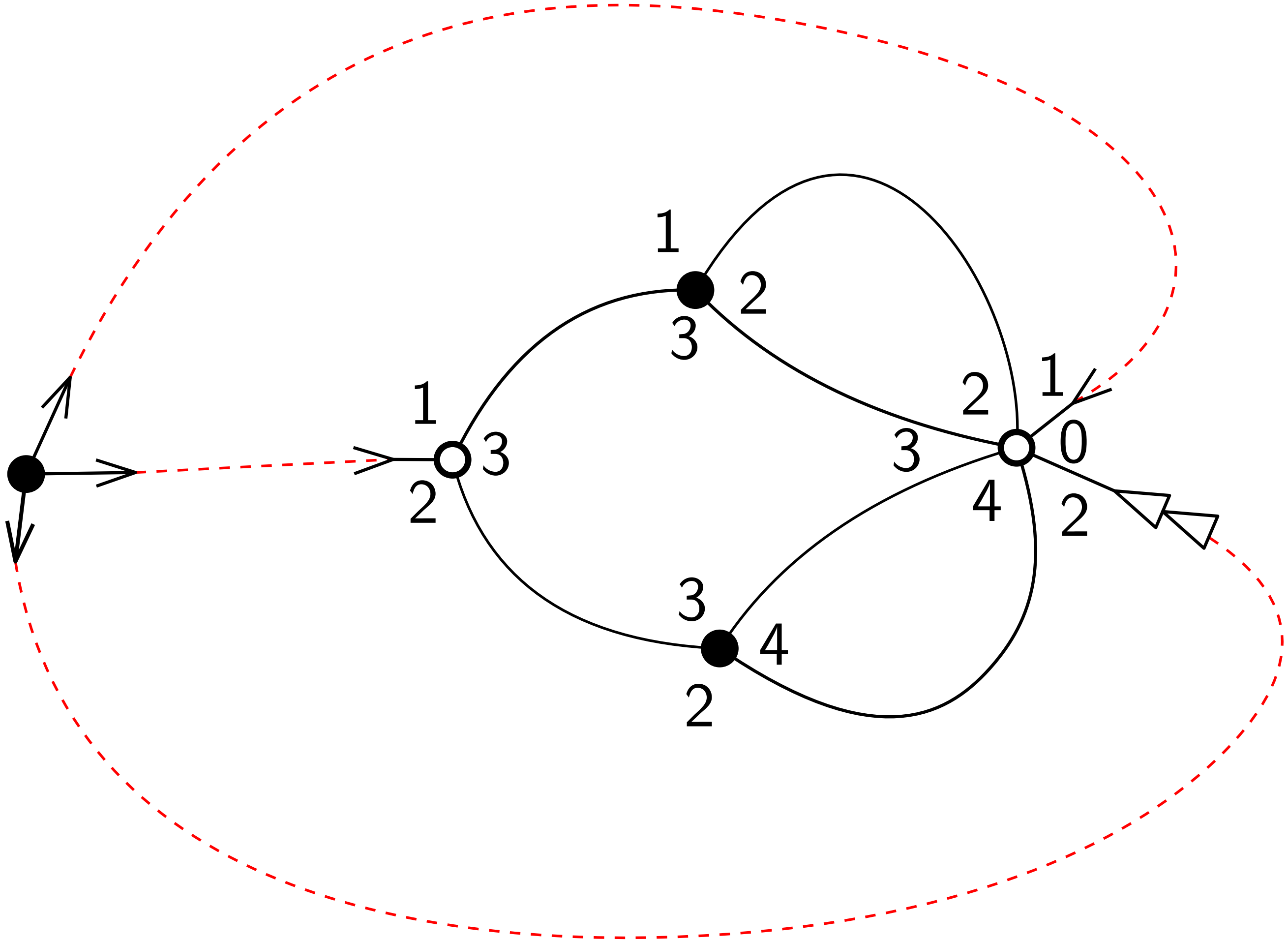

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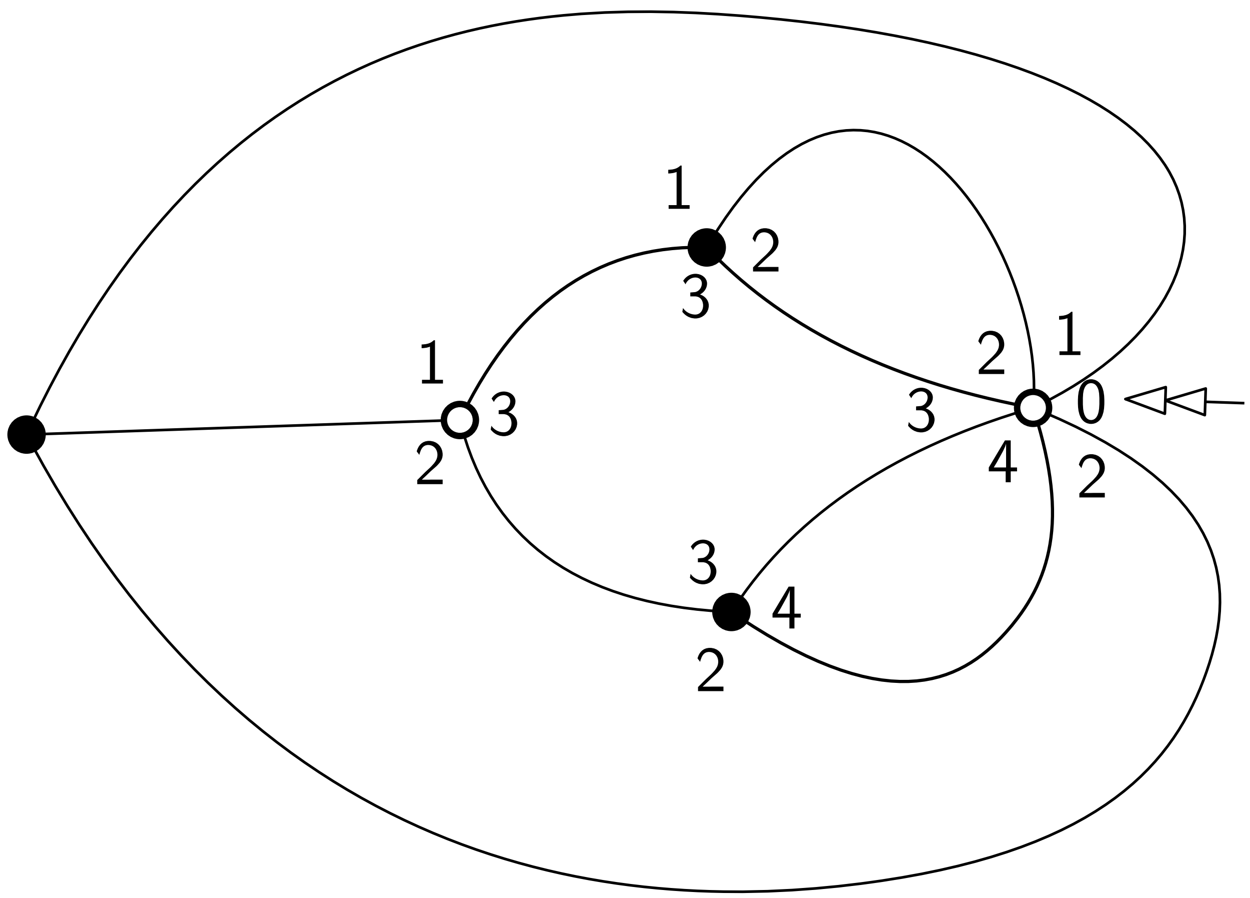

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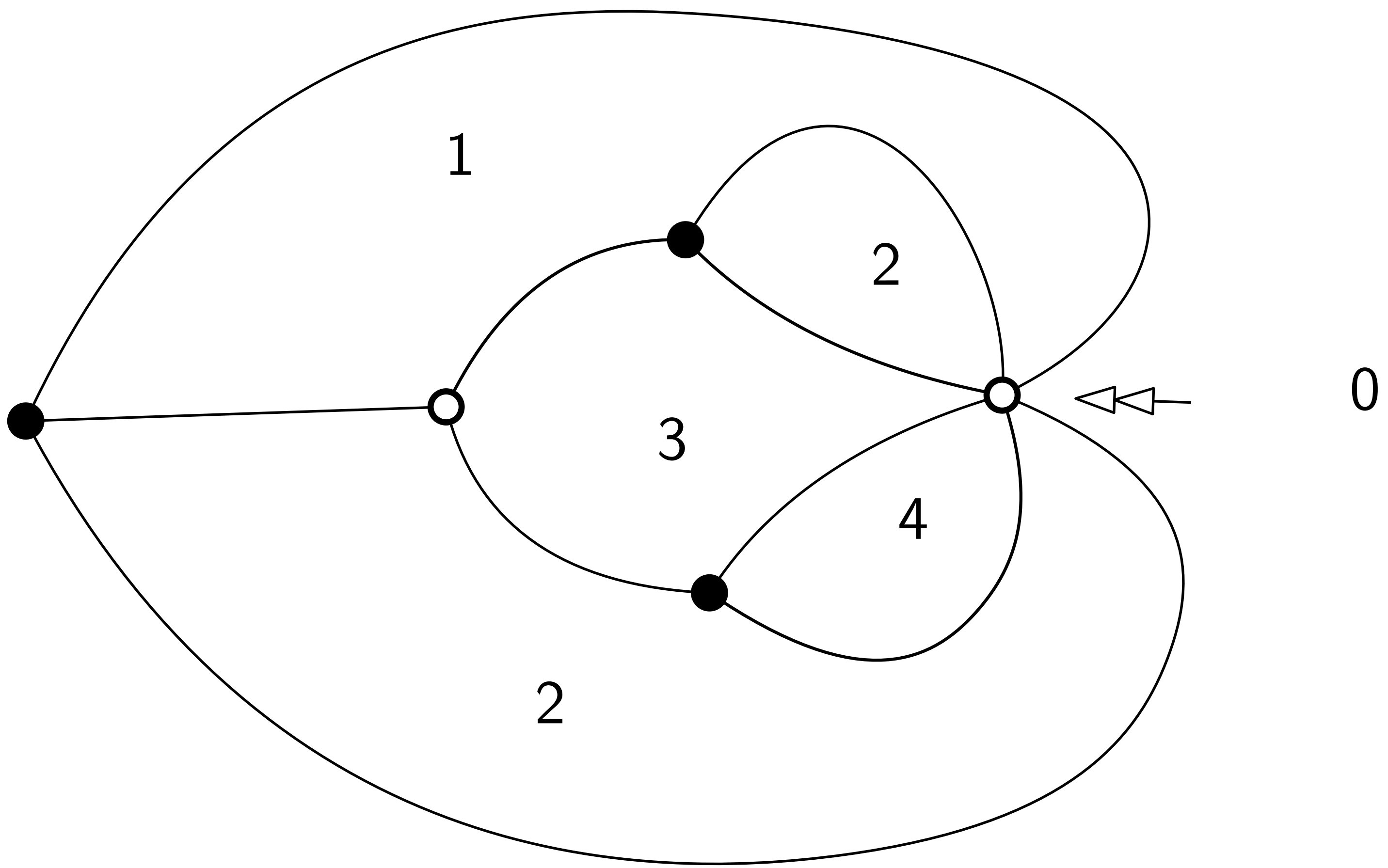

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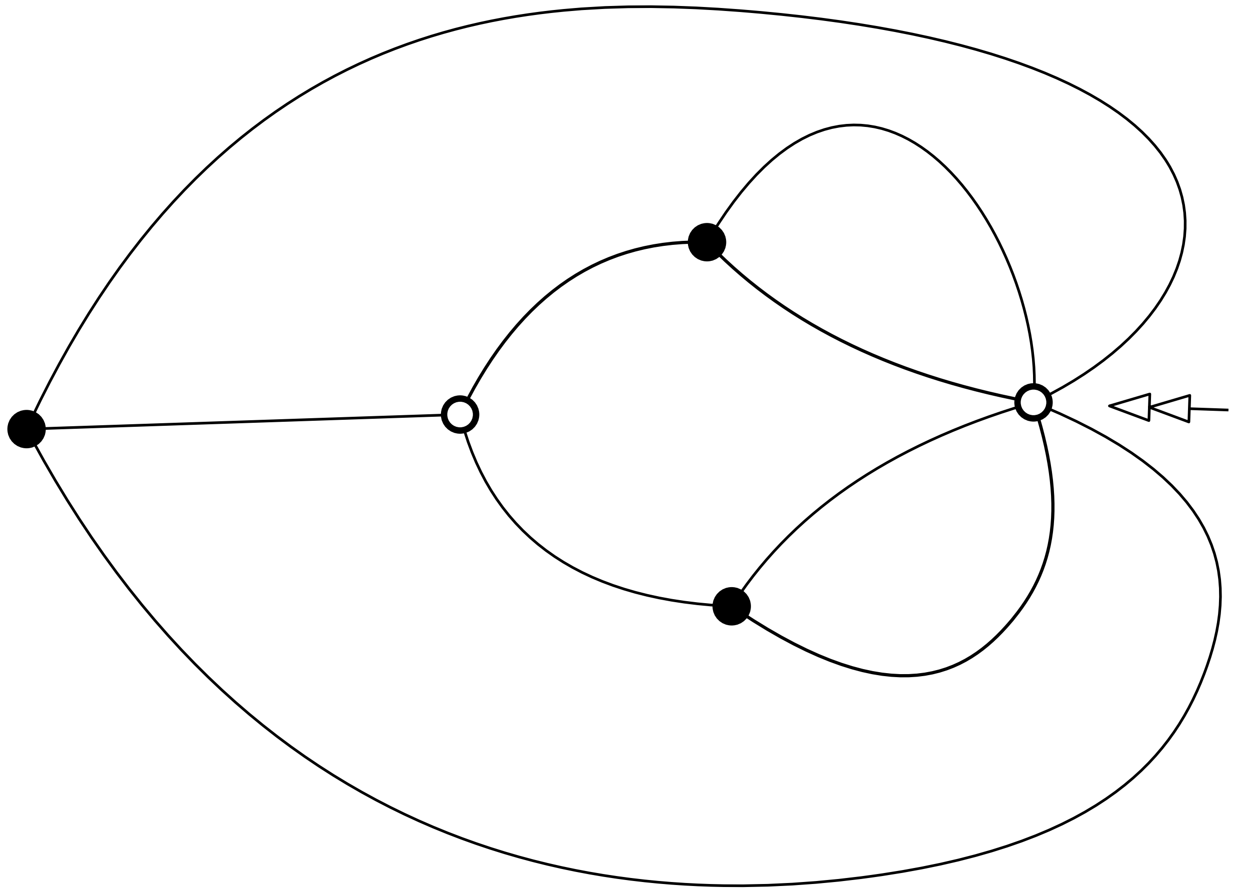

The bijection in higher genus

The procedure is
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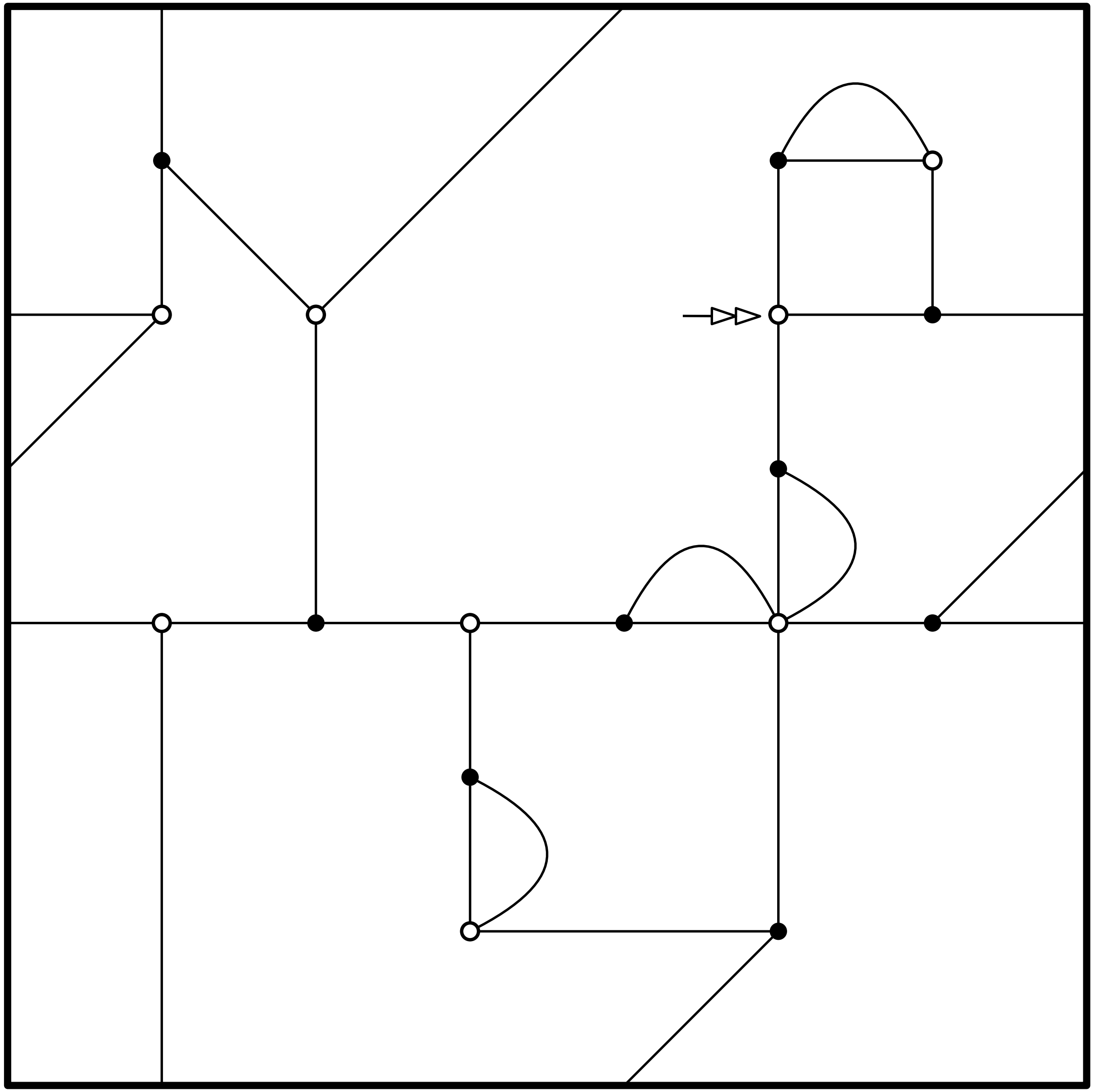

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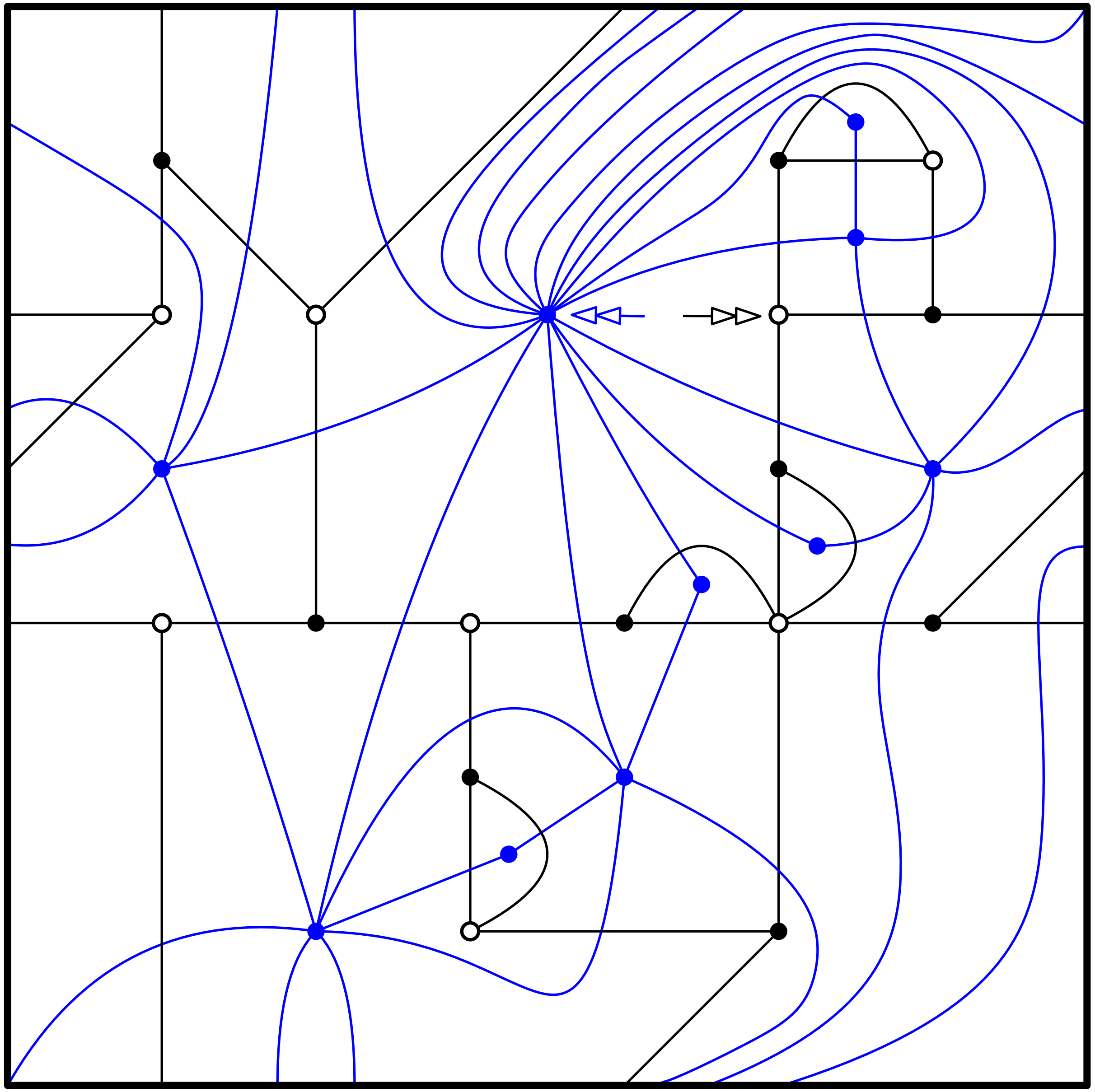

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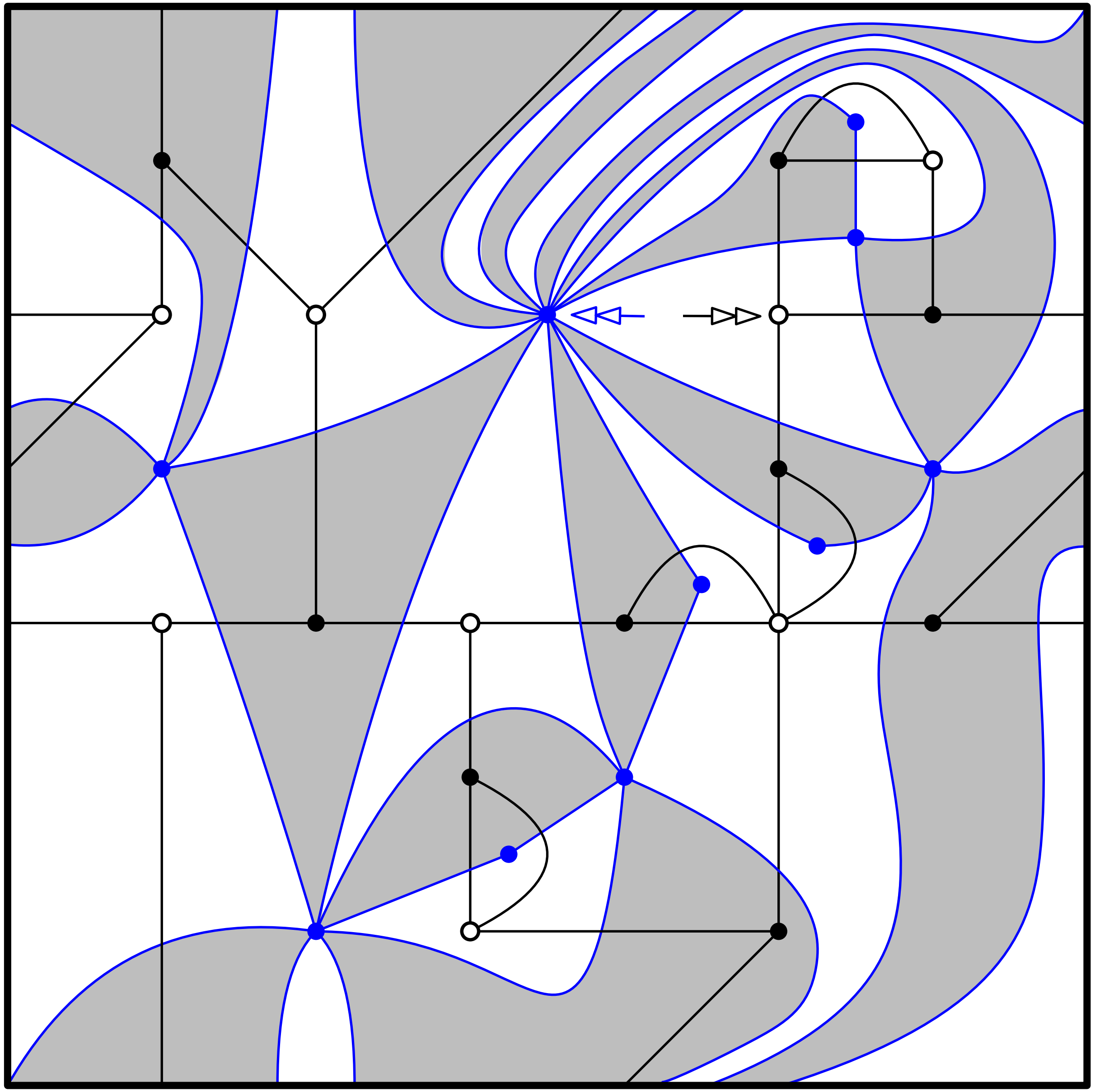

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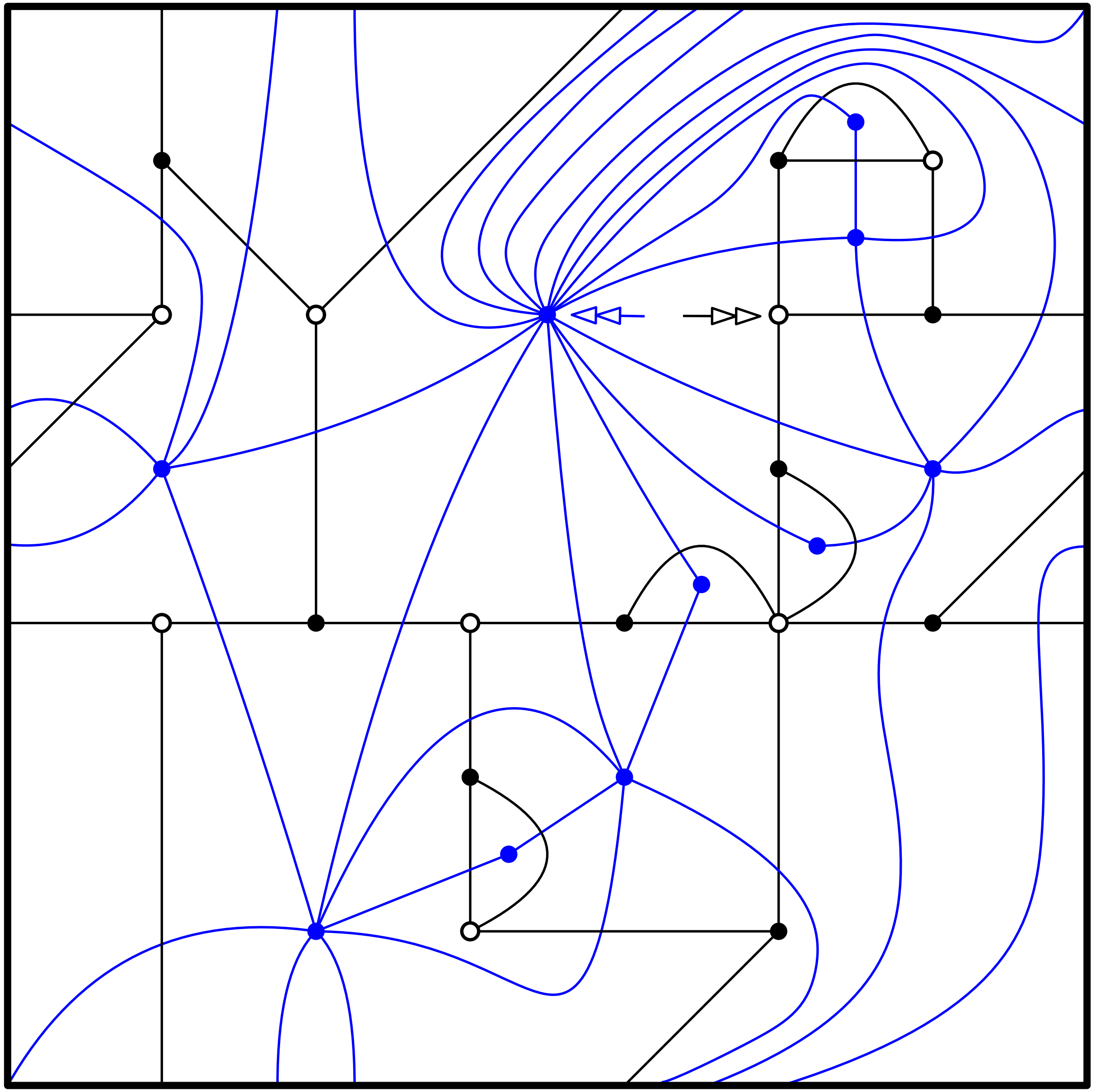

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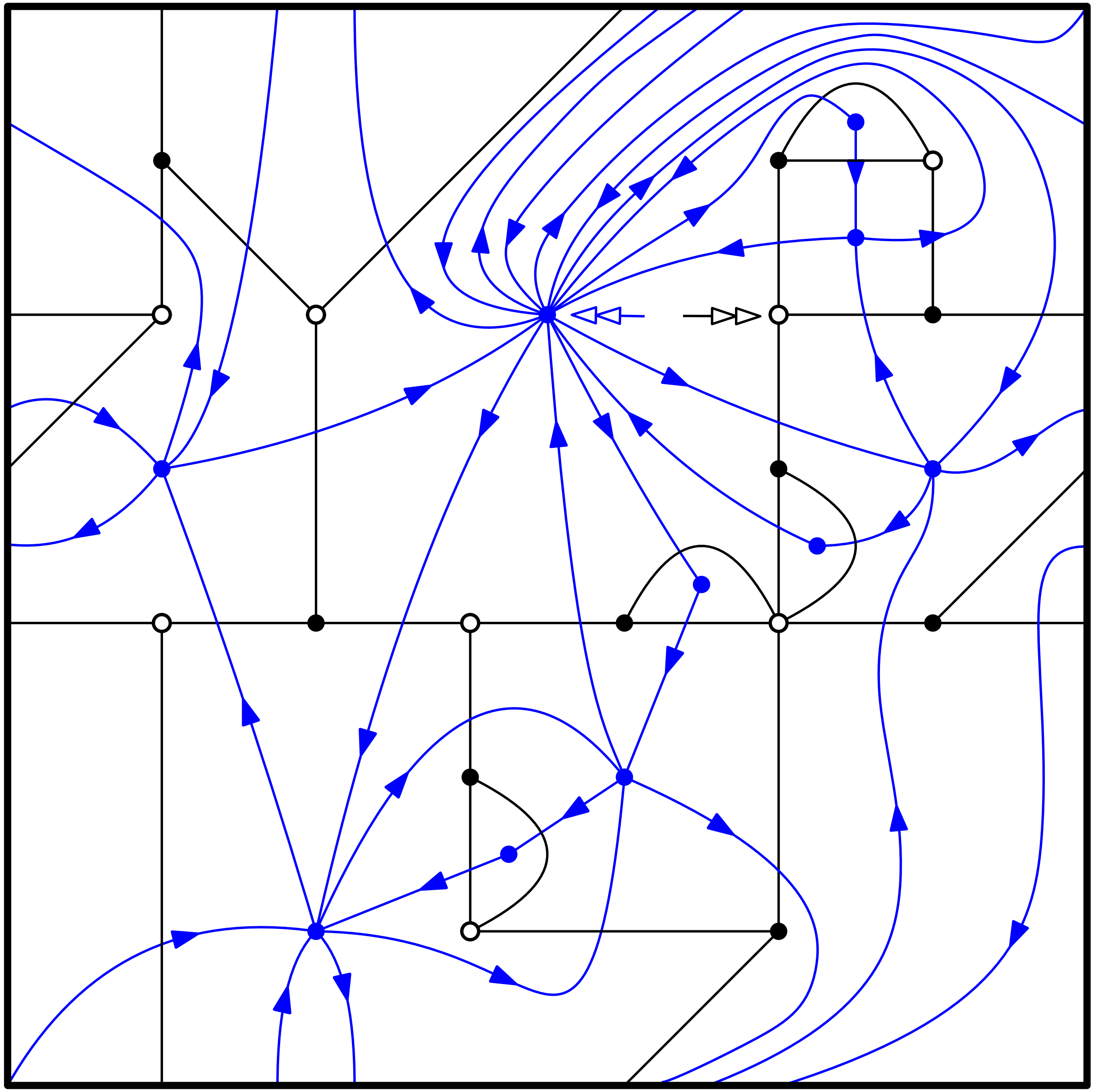

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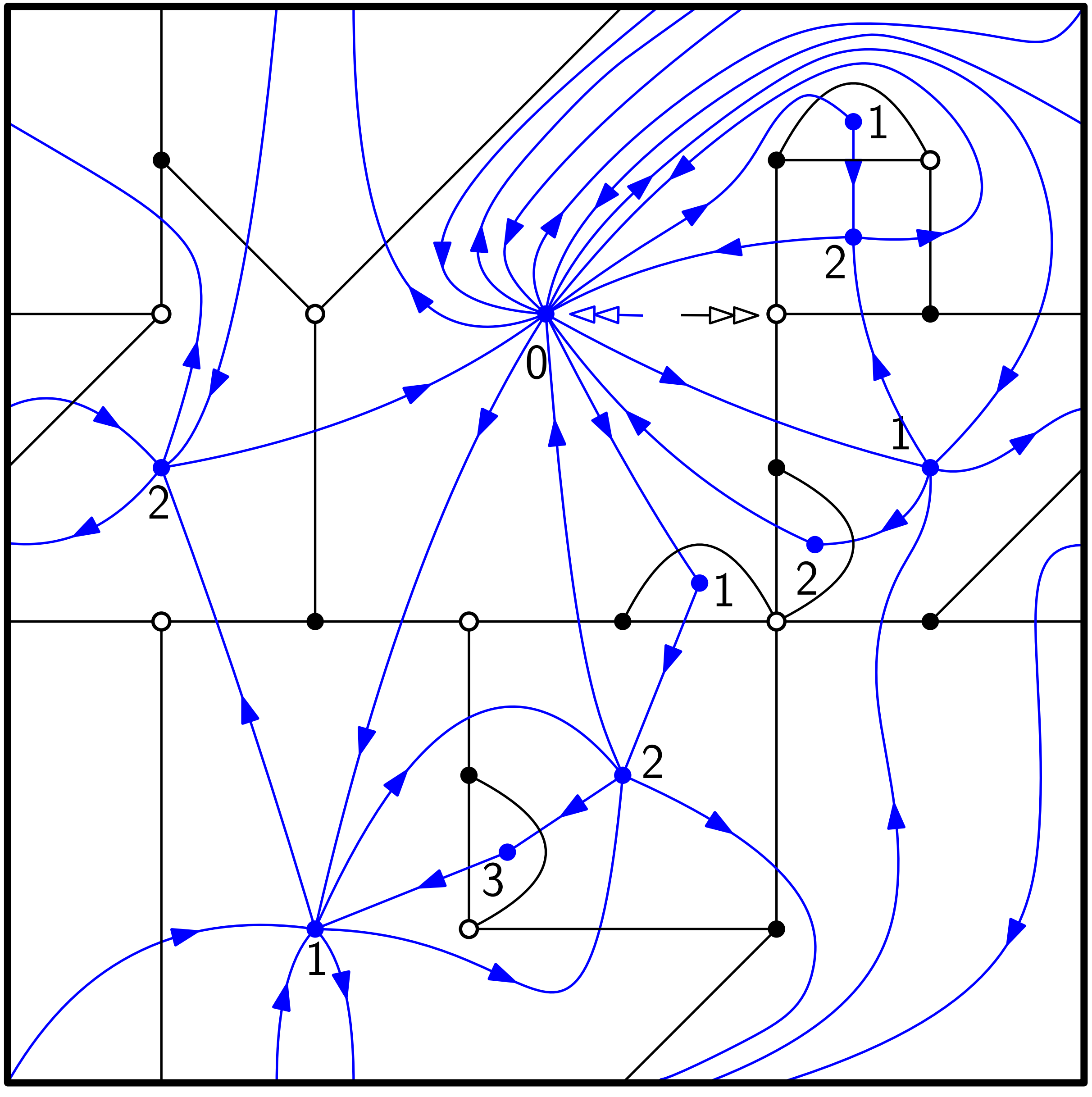

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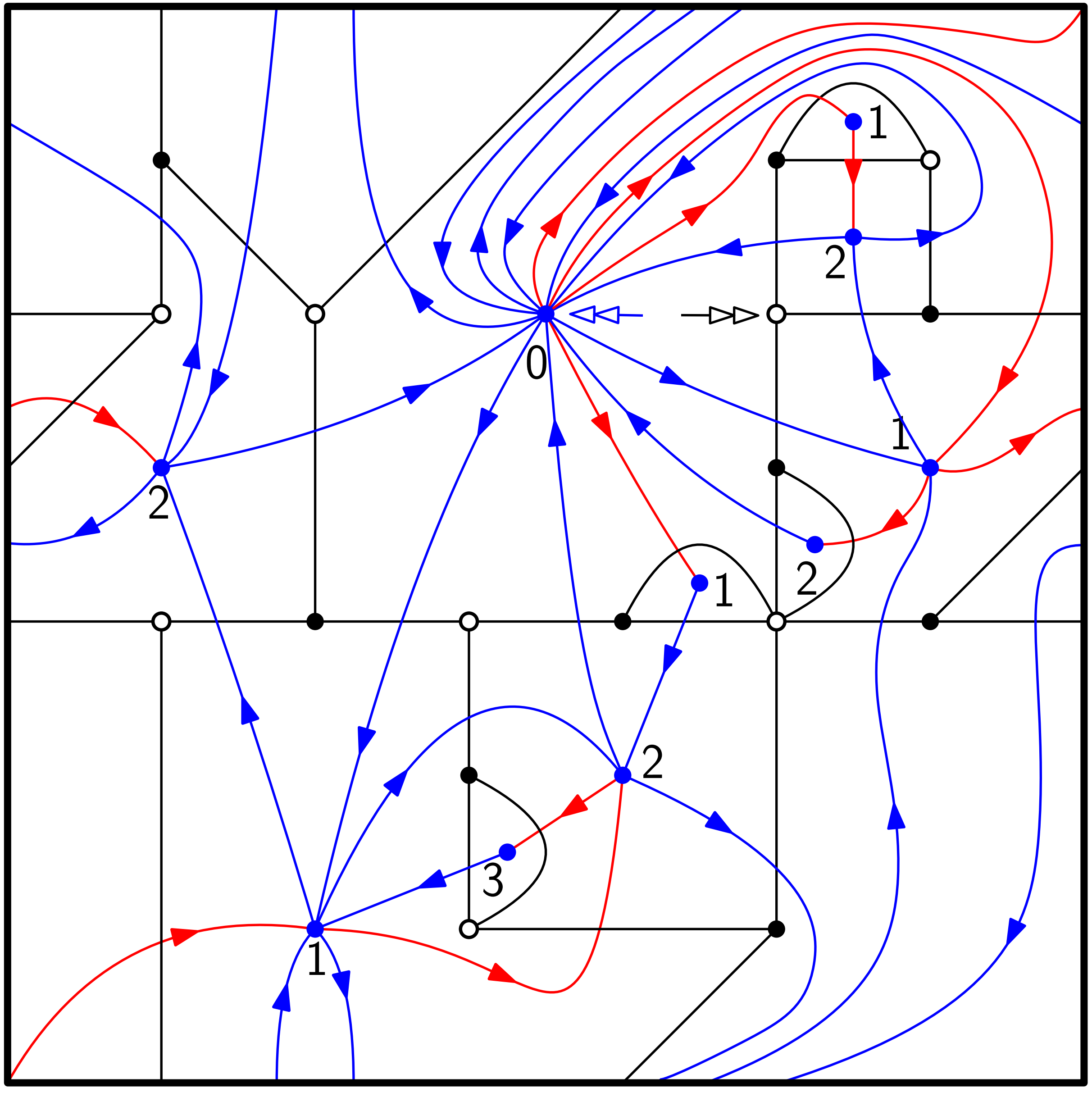

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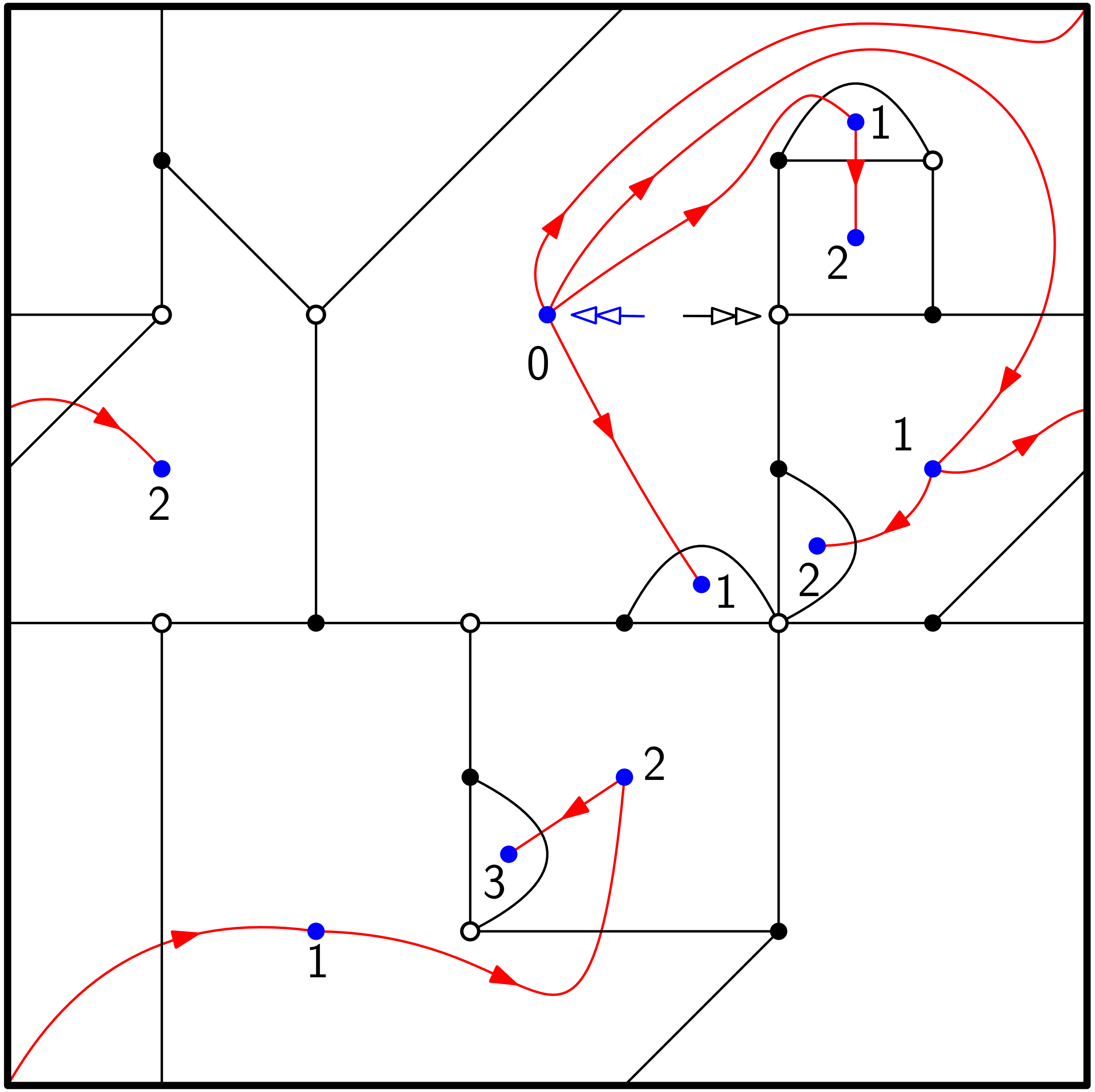

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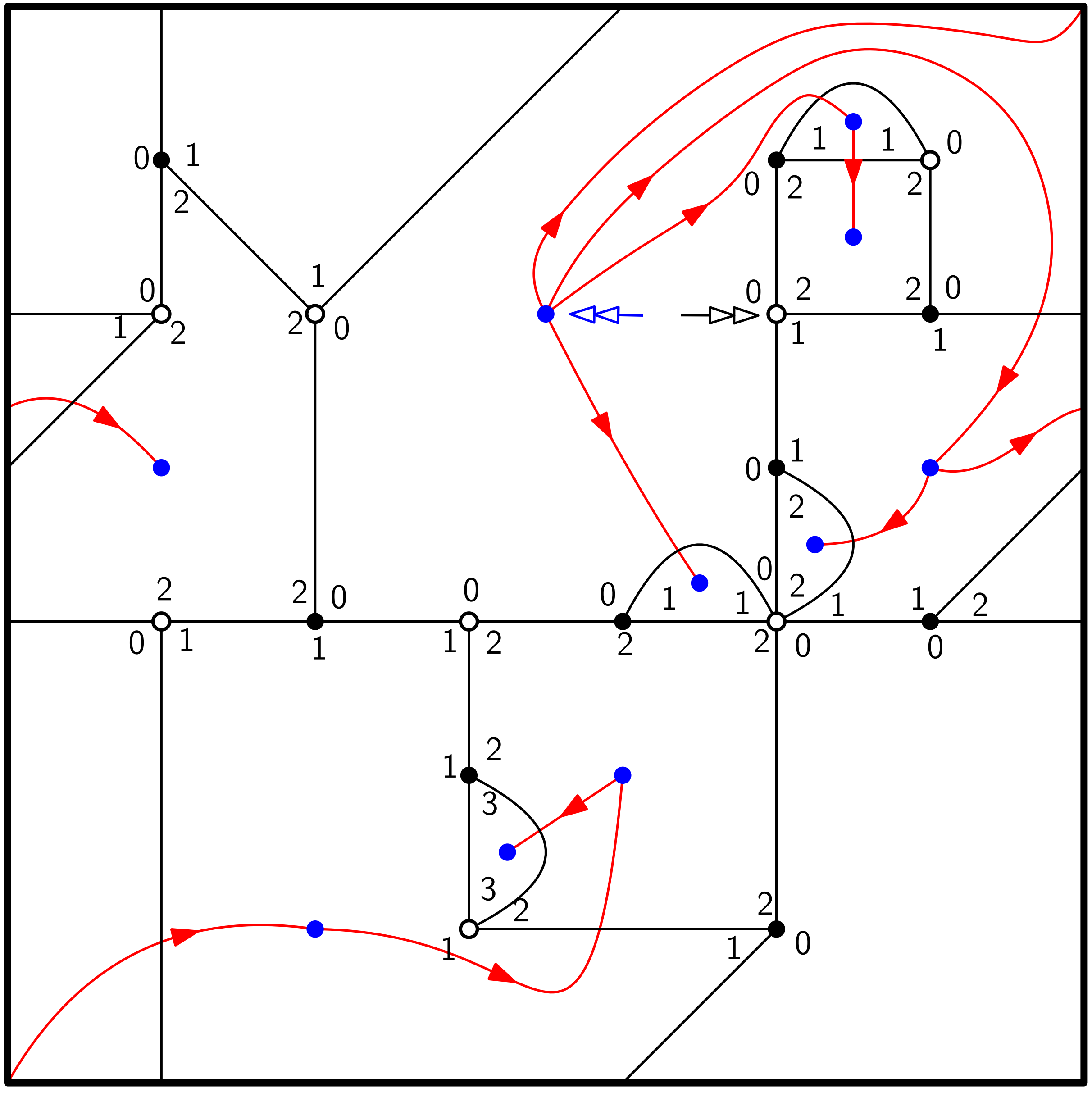

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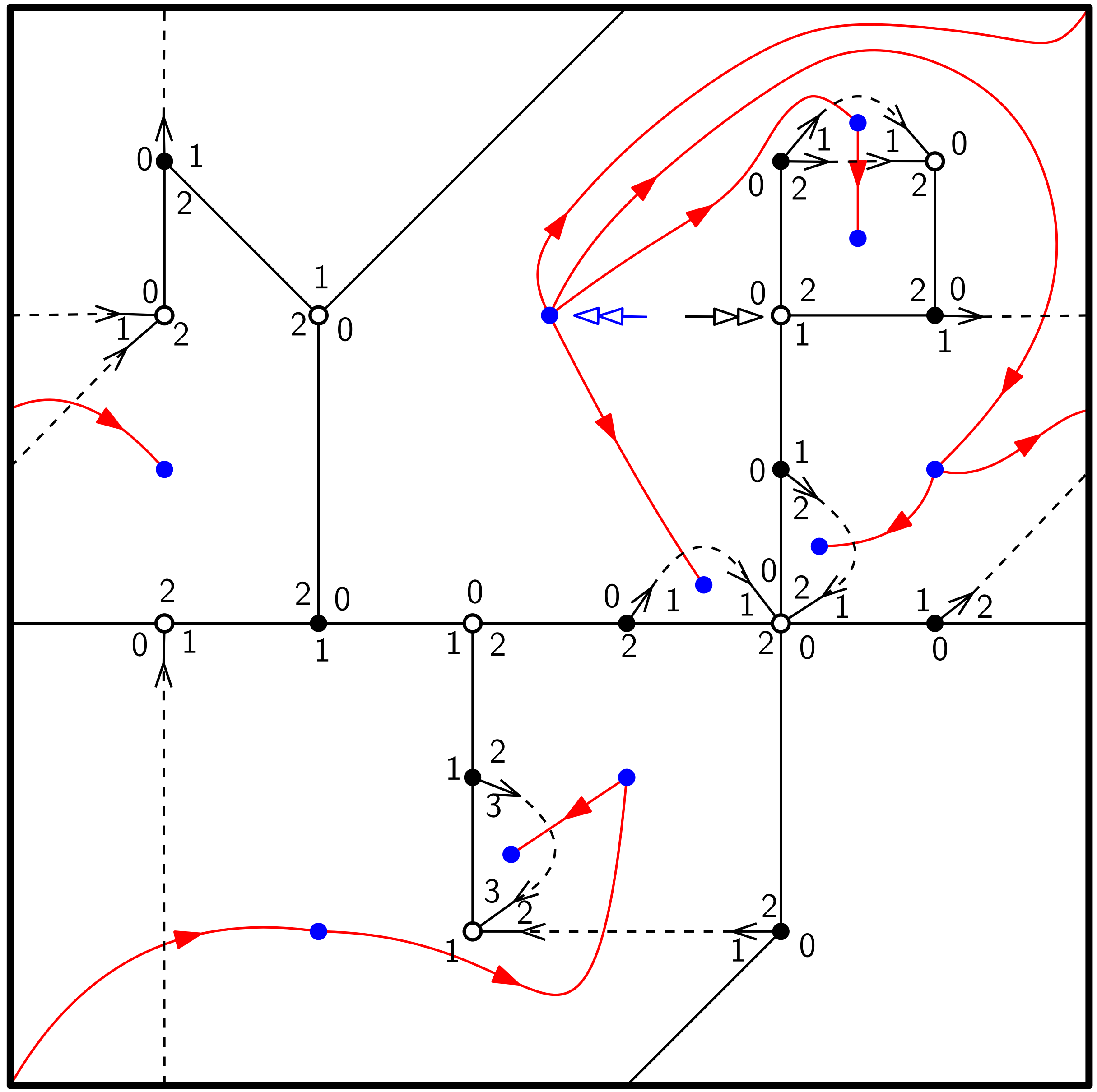

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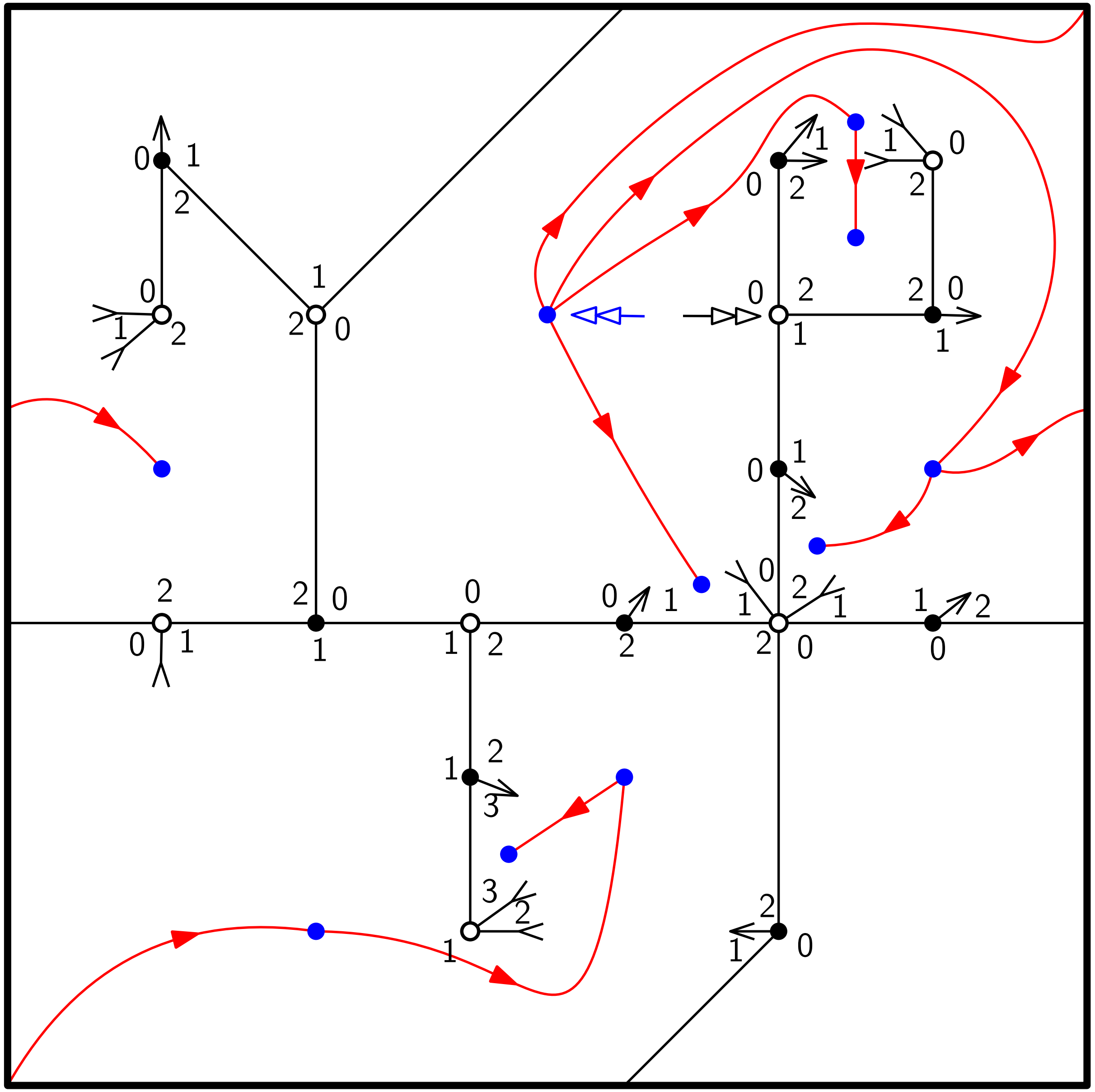

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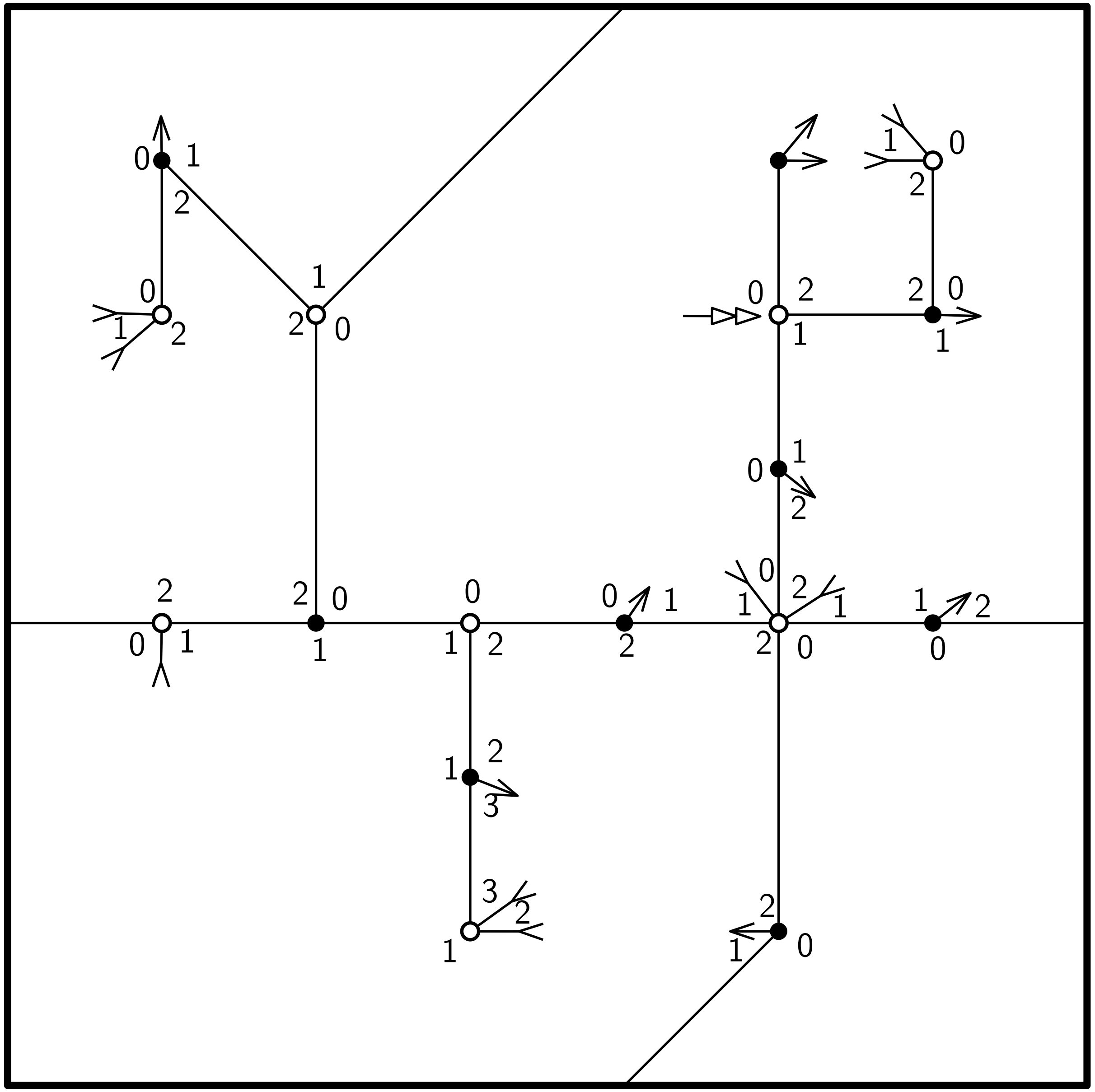

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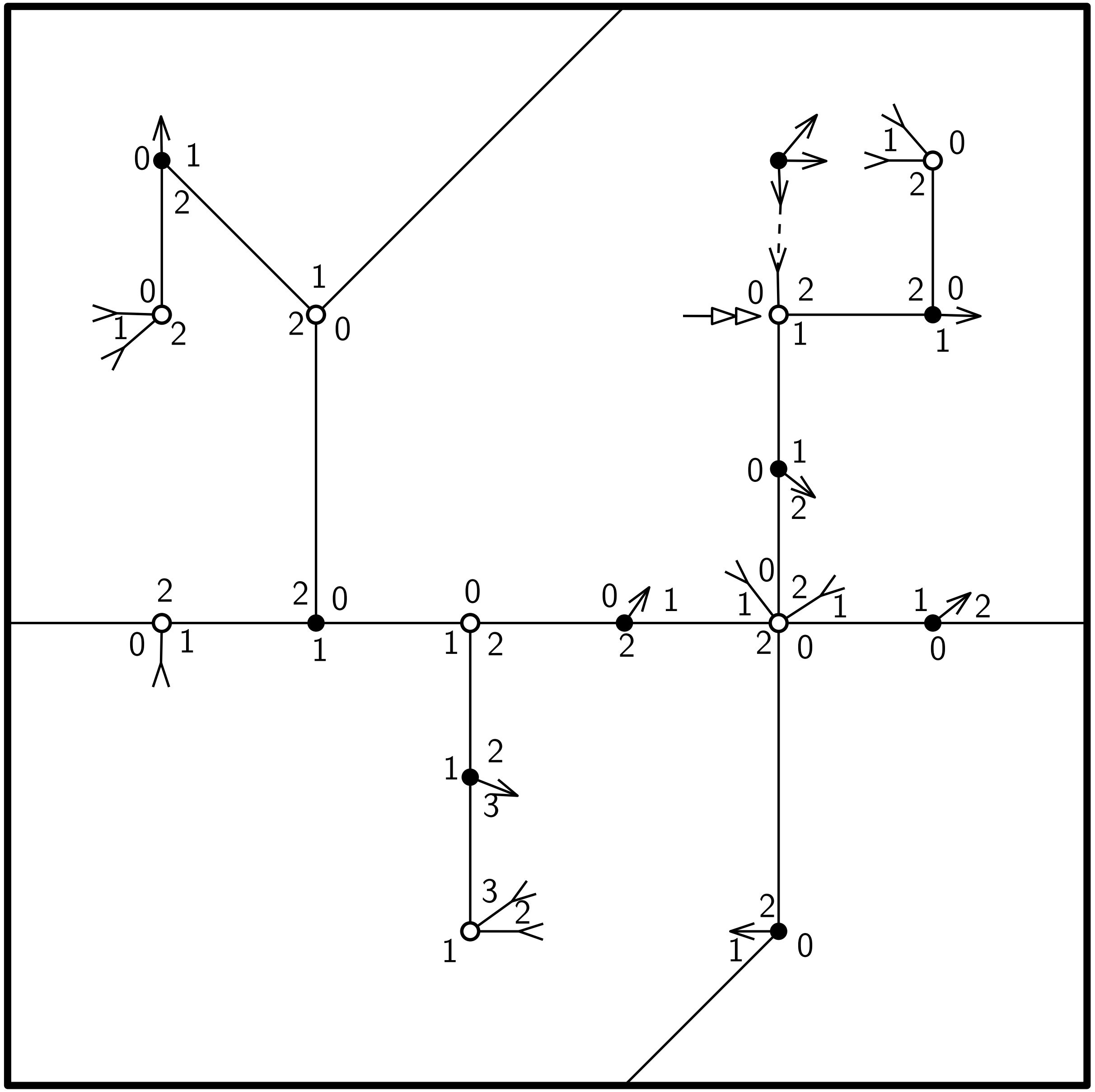

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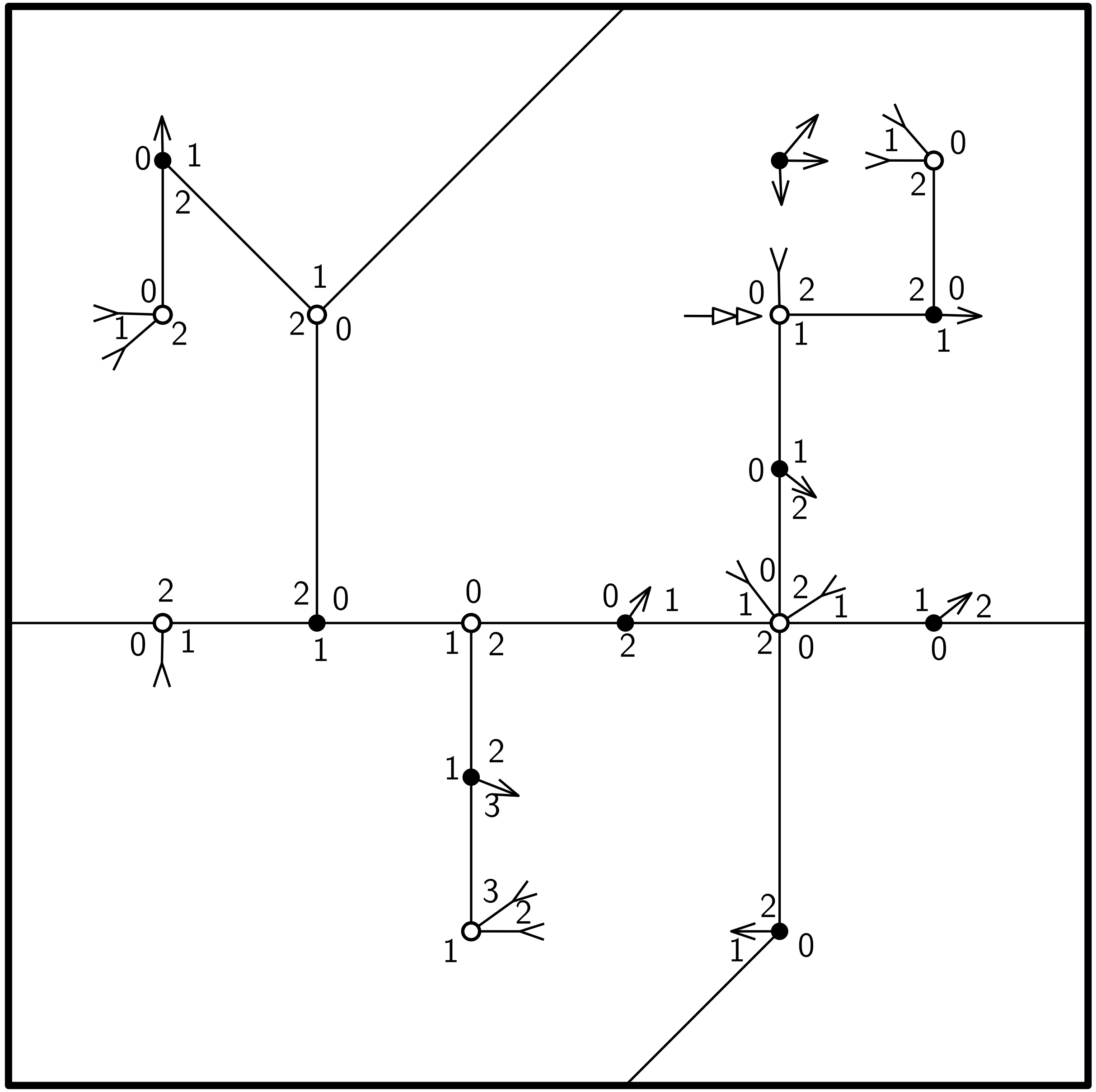

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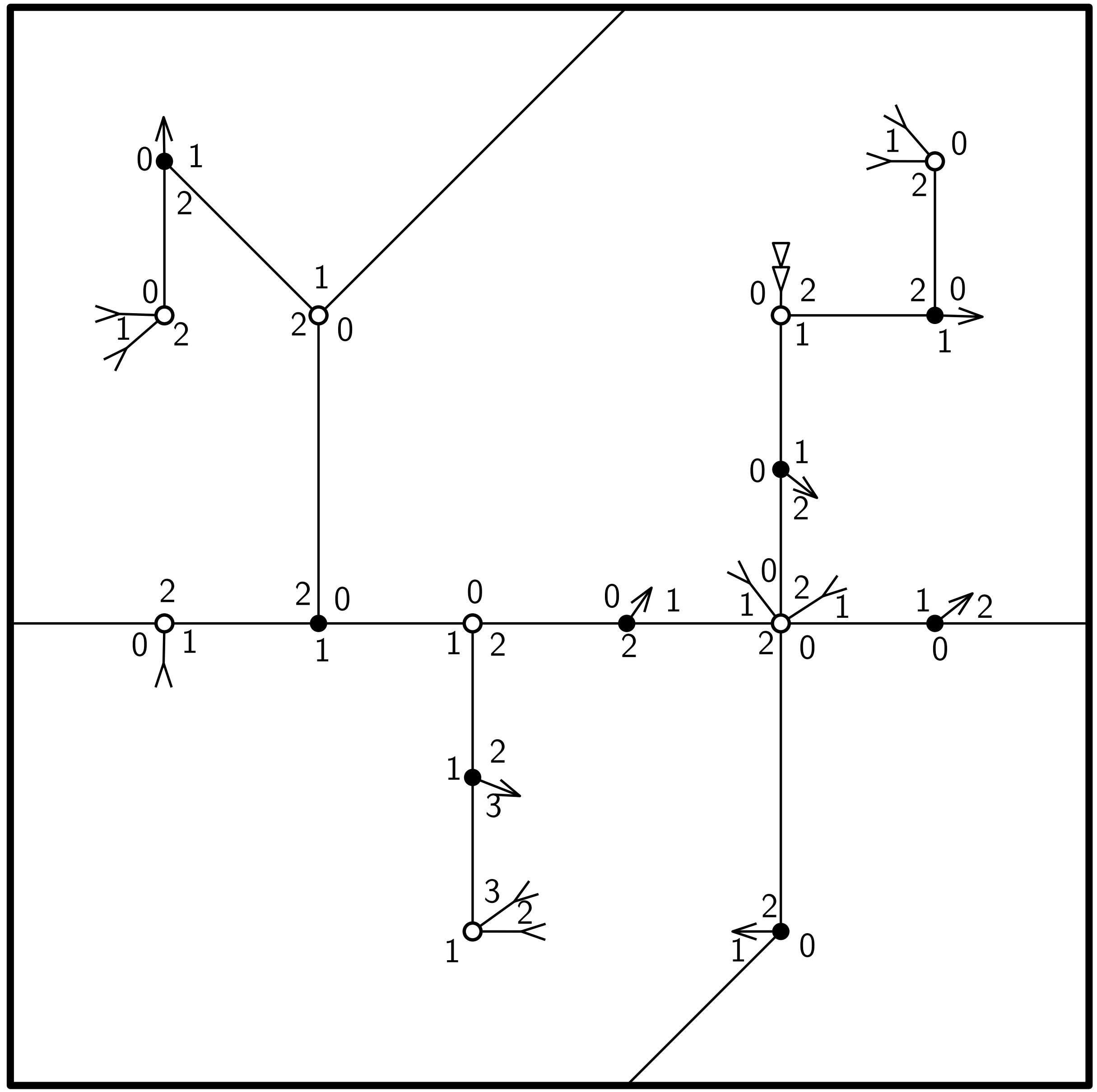
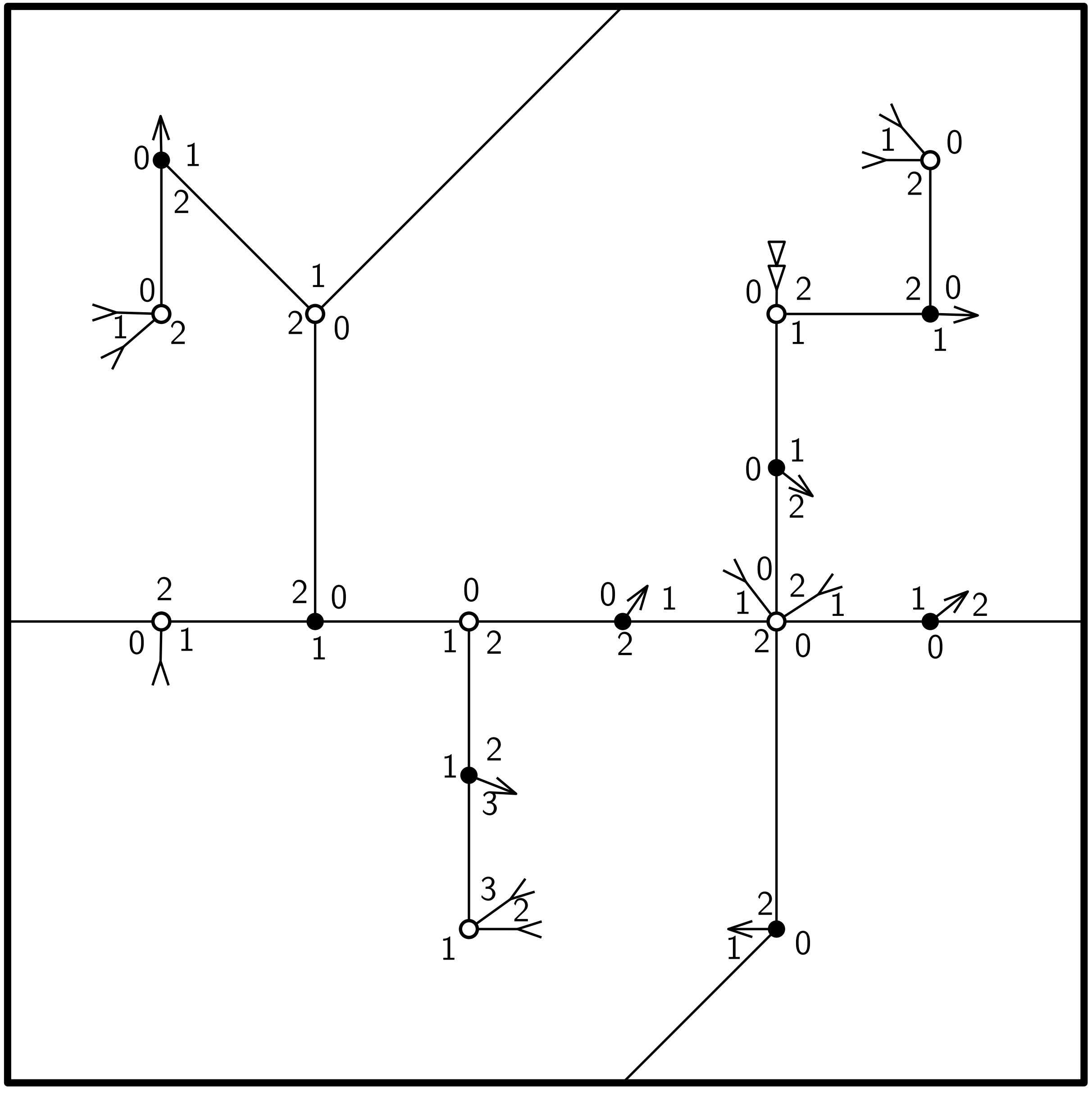


The bijection in higher genus

The procedure is **exactly** the same.

We have obtained a **unicellular blossoming map**.

We want to characterize it.



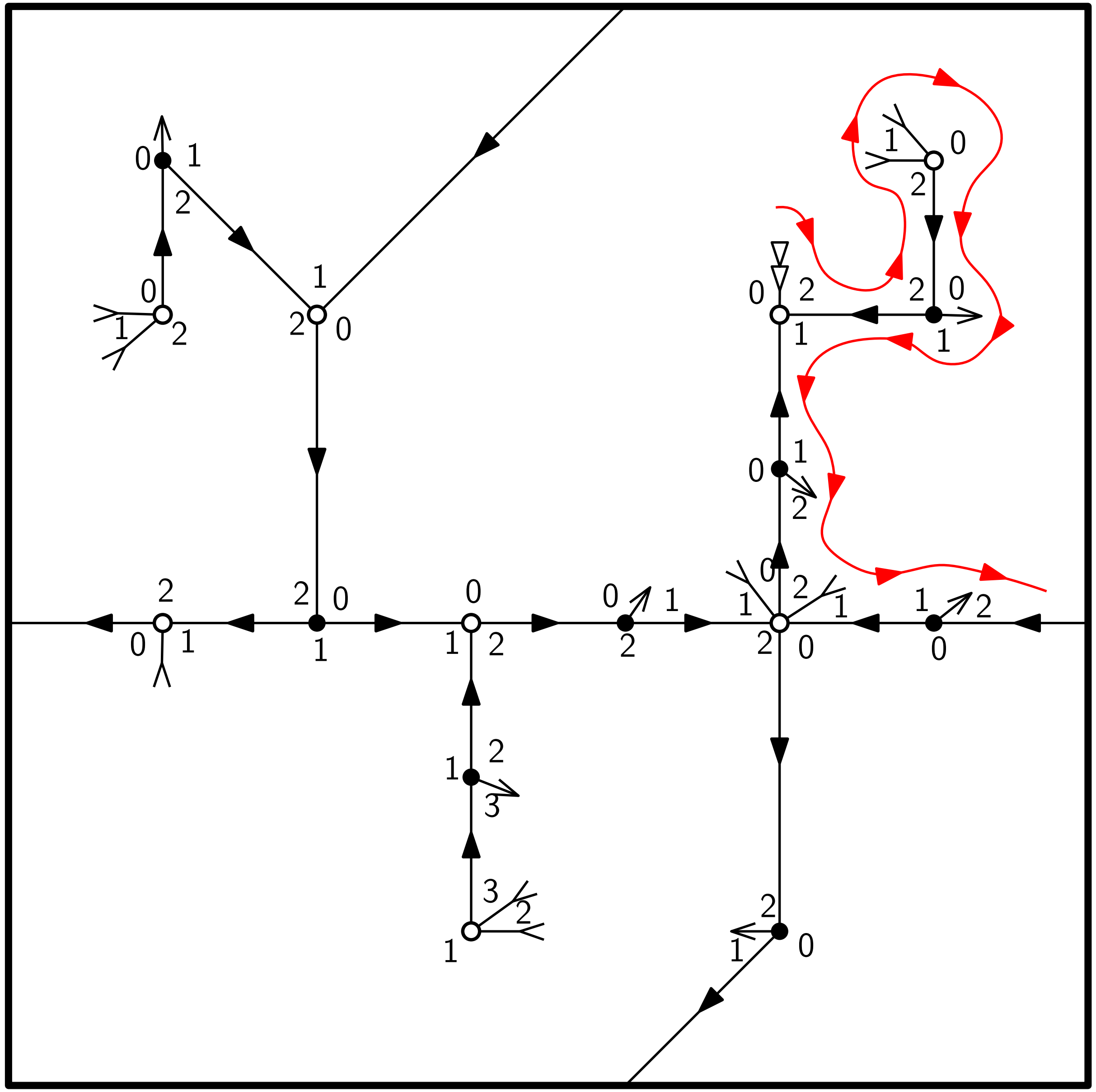
The bijection in higher genus

The procedure is **exactly** the same.

We have obtained a **unicellular blossoming map**.

We want to characterize it.

The map is endowed with a **good orientation**.



m-bipartite unicellular maps

Let $m \geq 2$. We say that a rooted blossoming unicellular map with m more instems than outstems and whose vertices are bicolored (black and white) is an **m-bipartite unicellular map** if

(i) neighbouring vertices have different colors, instems are attached to white vertices and outstems are attached to black vertices,

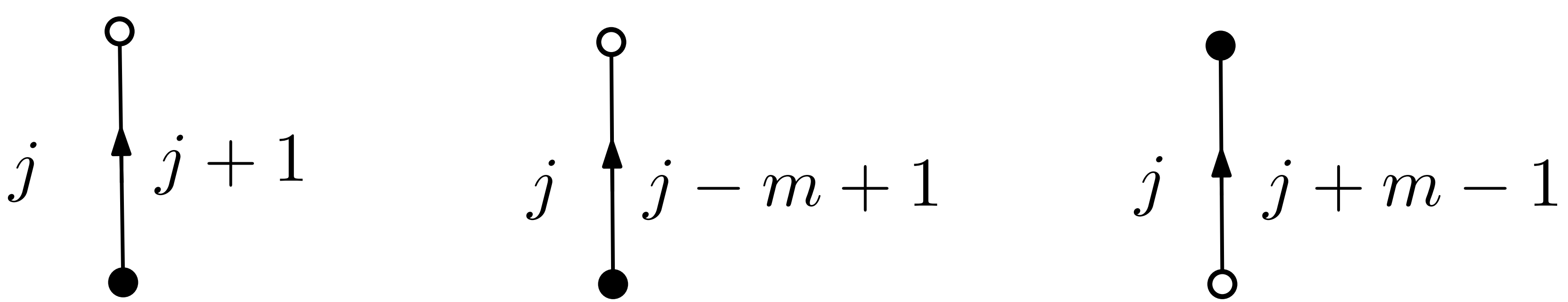
(ii) black vertices have degree m,

(iii) white vertices have degree mi for some integer $i \geq 1$ (which can be different among white vertices),

and, when endowed with its good labelling and good orientation,

(iv) the edges whose origin is a black vertex either decrease by 1 or increase by $m-1$,

(v) the edges whose origin is a white vertex decrease by $m-1$.



The bijection theorem and its application

Theorem. Rooted m-constellations of genus g are in bijection with well-rooted m-bipartite unicellular maps of genus g.

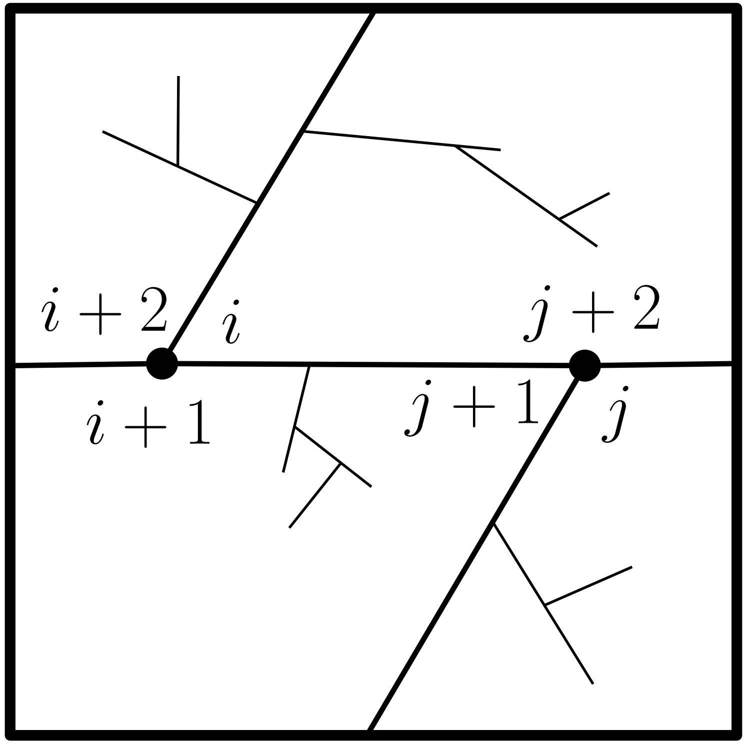
The bijection theorem and its application

Theorem. Rooted m-constellations of genus g are in bijection with well-rooted m-bipartite unicellular maps of genus g.

We now restrict ourselves to 3-constellations of genus 1 whose white faces are triangles. Their dual maps are **bipartite 3-face-colorable cubic maps of genus 1**.

We follow the framework introduced by [Chapuy,Marcus,Schaeffer'09] to study unicellular maps.

The bijection theorem and its application



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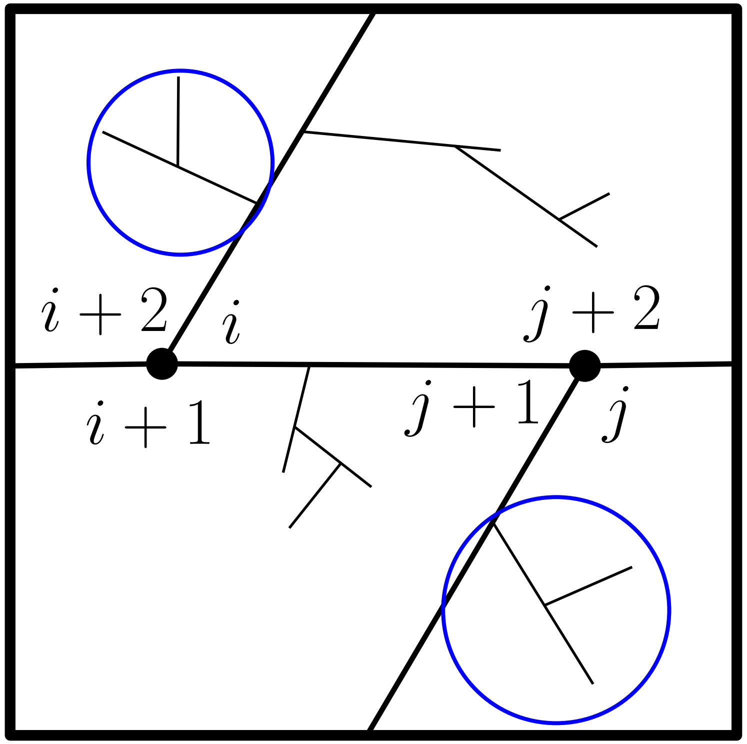
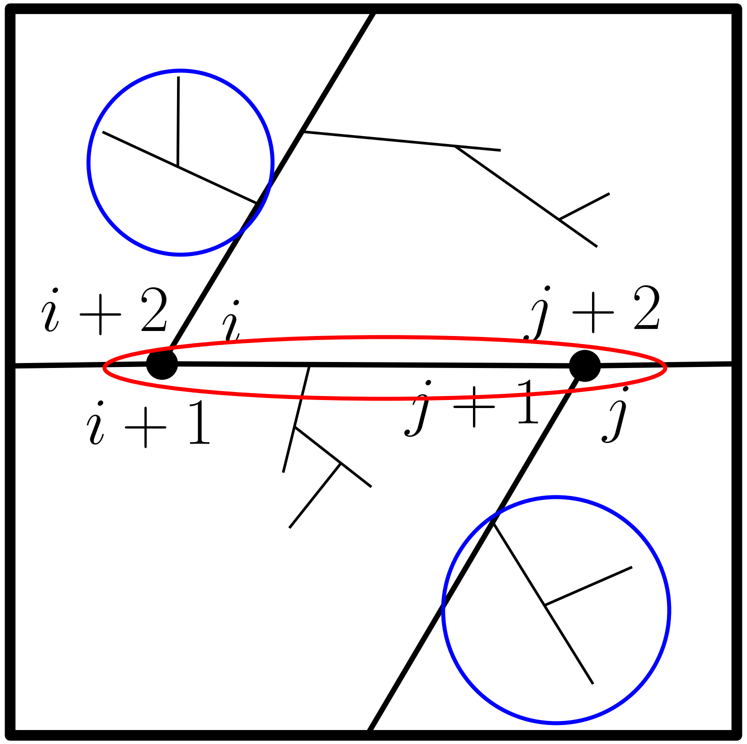


- The treelike parts are enumerated by

$$T(z) = z + 2T(z)^2.$$

The bijection theorem and its application



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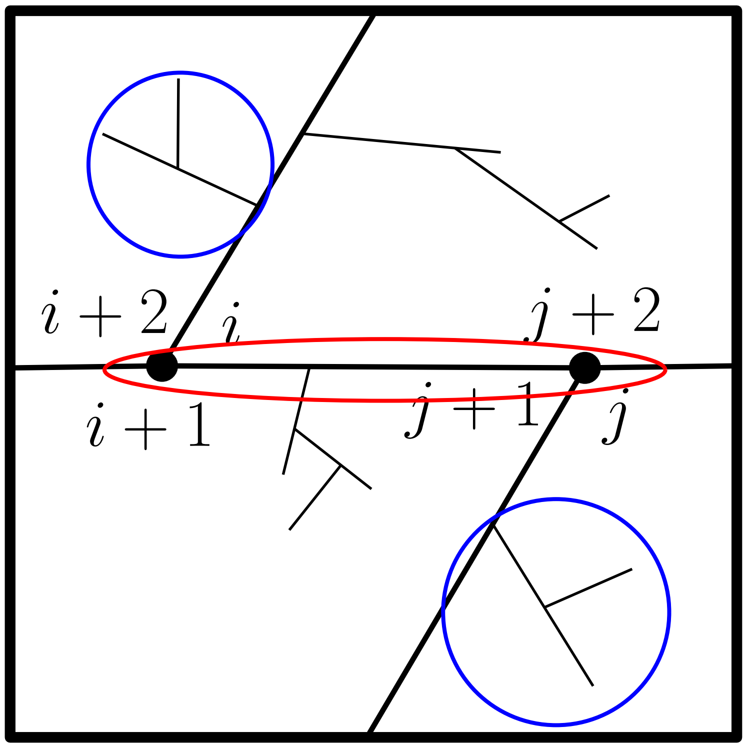
The bijection theorem and its application



- The treelike parts are enumerated by

$$T(z) = z + 2T(z)^2.$$

- The branches are **weighted Motzkin paths** that have to agree with the labels.

Theorem. Rooted 3-constellations of genus 1 whose white faces are triangles (counted by their number of white faces) are enumerated by

$$C(z) = \frac{T(z)^3}{(1-T(z))(1-4T(z))^2}.$$

Further work

Conjecture. Rooted 3-constellations of **arbitrary genus** whose white faces are triangles are enumerated by a rational function of $T(z)$.

Then try to extend the results to $m > 3$.