Enumeration of chordal planar graphs and maps

Master thesis

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under the direction of Marc Noy

16th May 2022

Introduction

The work can be divided into two main parts.

1. Work out the combinatorics and find the equations that define our generating functions using the symbolic method. We also make use of the dissymmetry theorem.
2. Do the singularity analysis of our equations to obtain the asymptotic behaviour. We use theorems from *Analytic Combinatorics* [Flajolet, Sedgewick '09] and *Random Trees* [Drmota '09].

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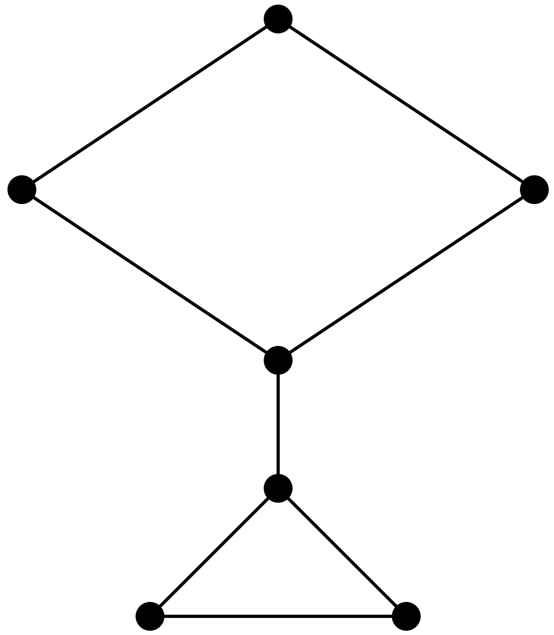
Our graphs are labelled, counted by their number of vertices and we use exponential generating functions $\sum_{n\geq 0} g_n \frac{x^n}{n!}$, where g_n is the number of graphs in the class with n vertices.

Maps are counted by edges and we use regular generating functions $\sum_{n\geq 0} M_n x^n$, where M_n is the number of maps in the class with n edges.

Chordal graphs

Definition. A graph is **chordal** if it has no induced cycle of lengh greater than 3.



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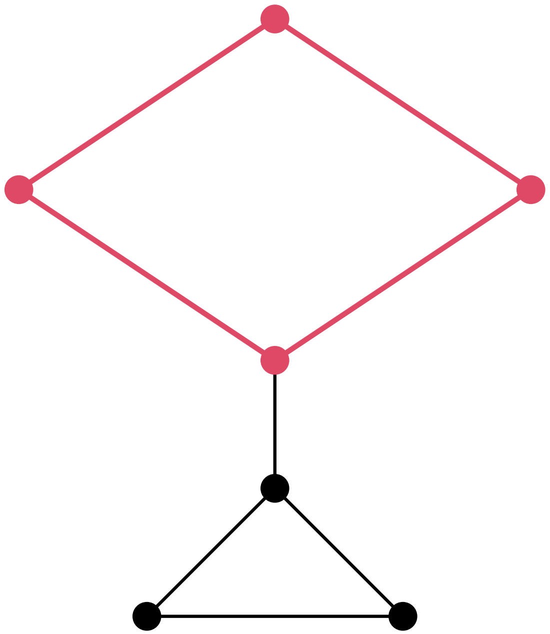

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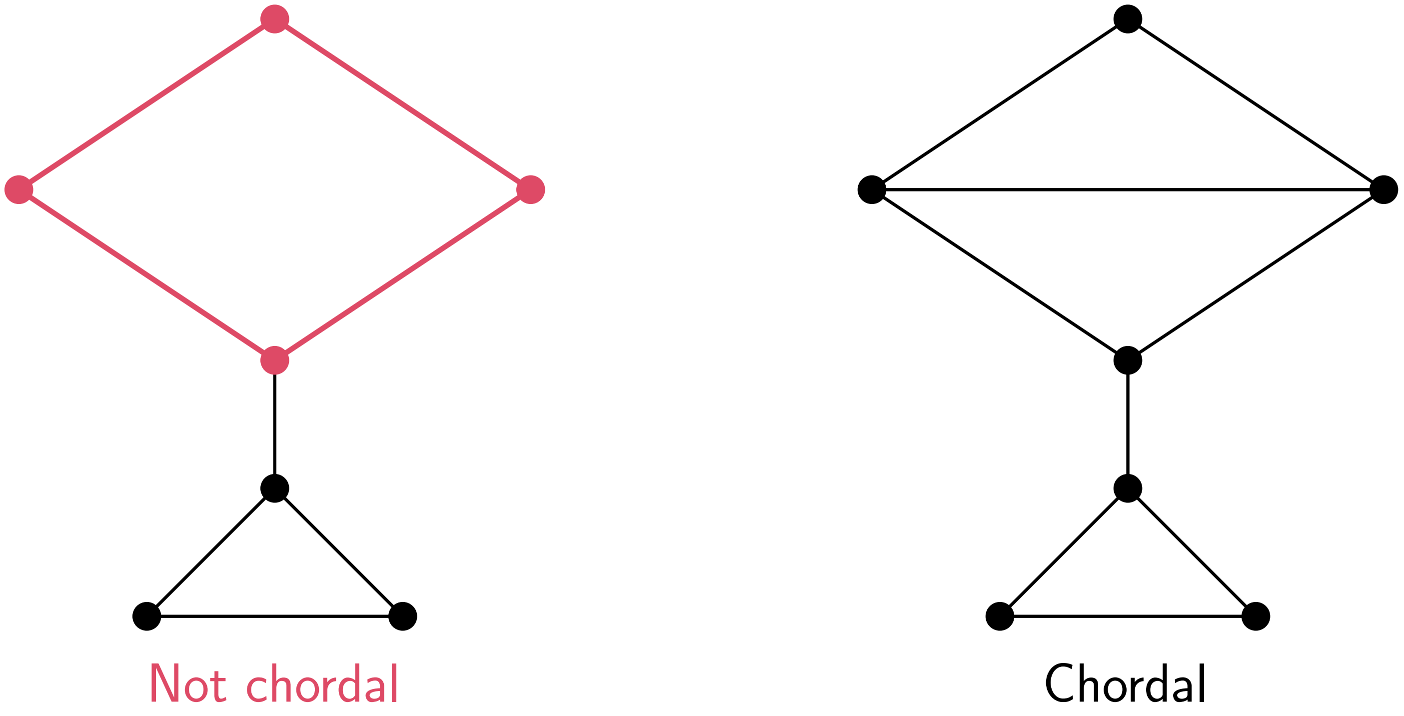

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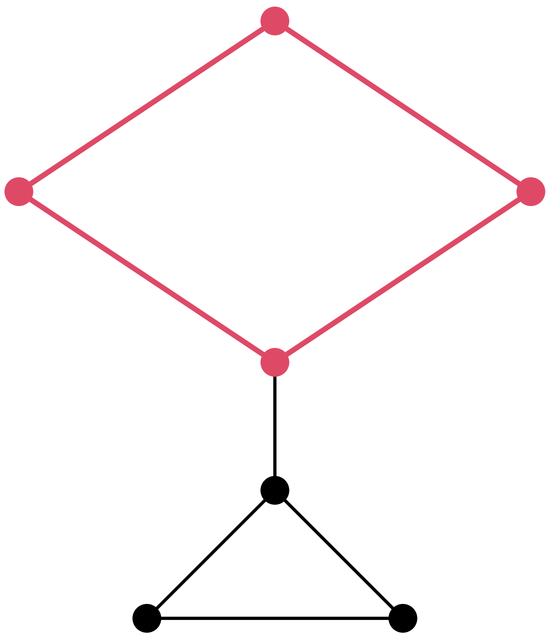

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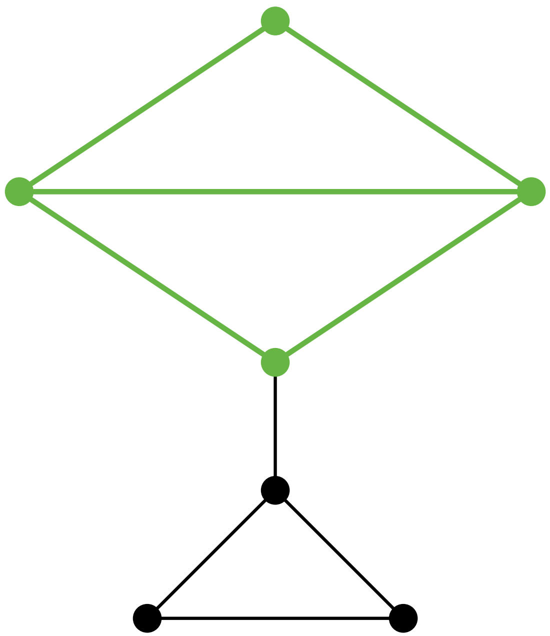

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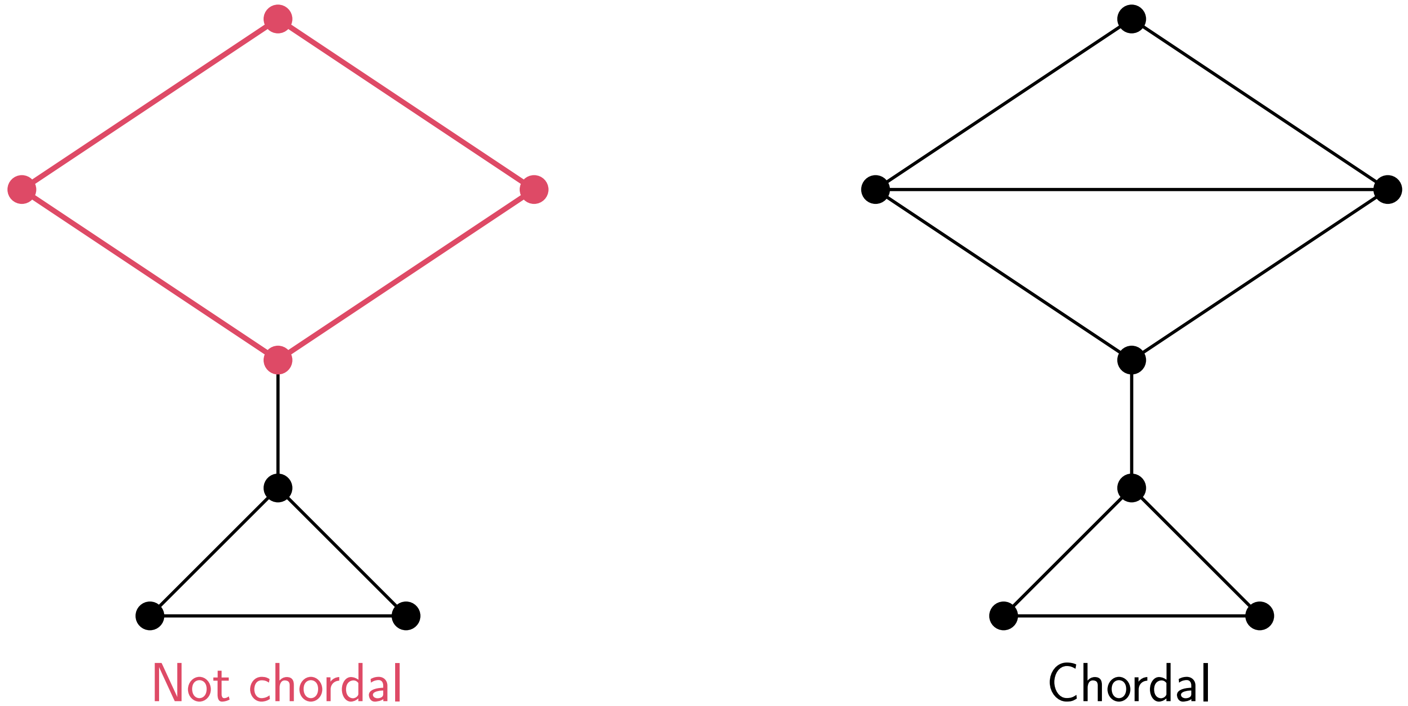

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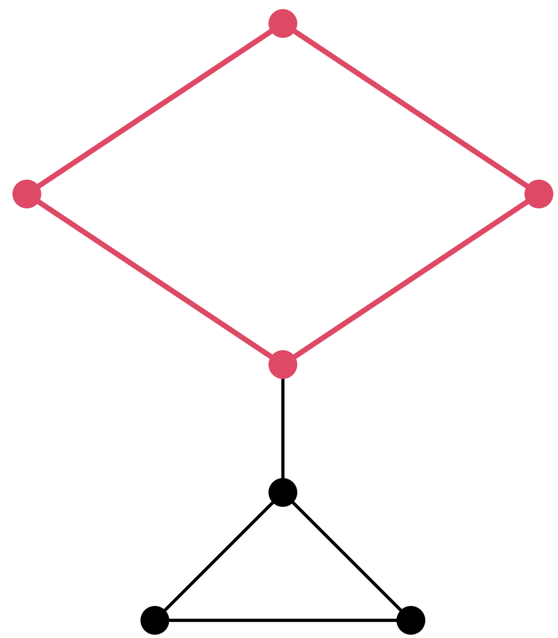

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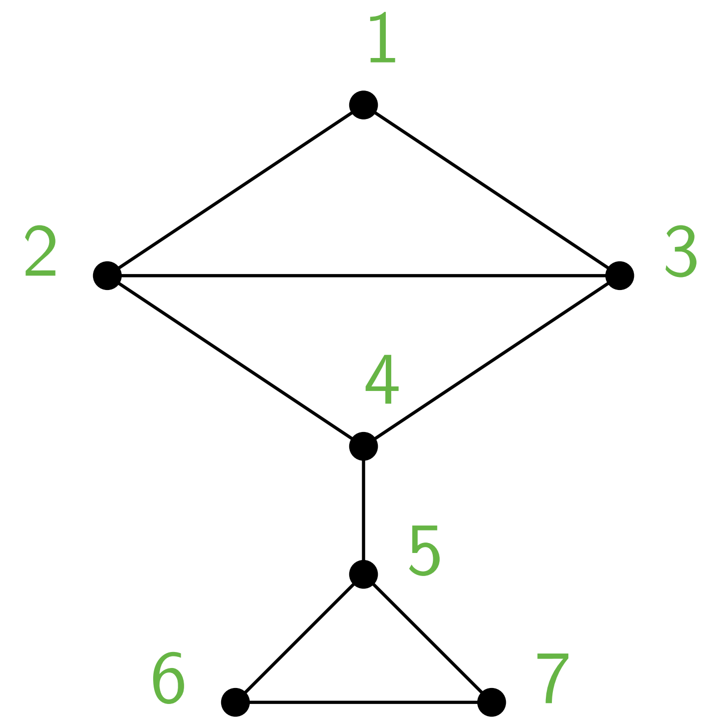

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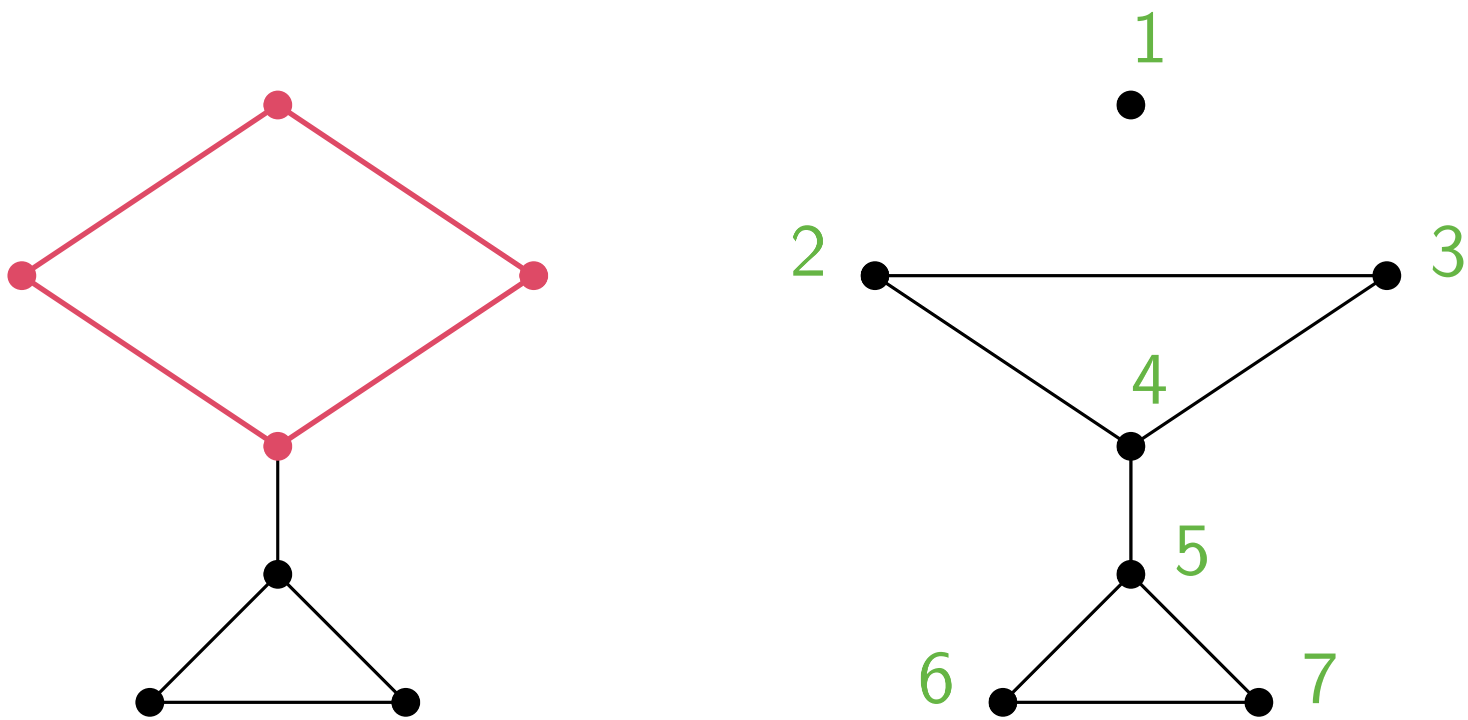

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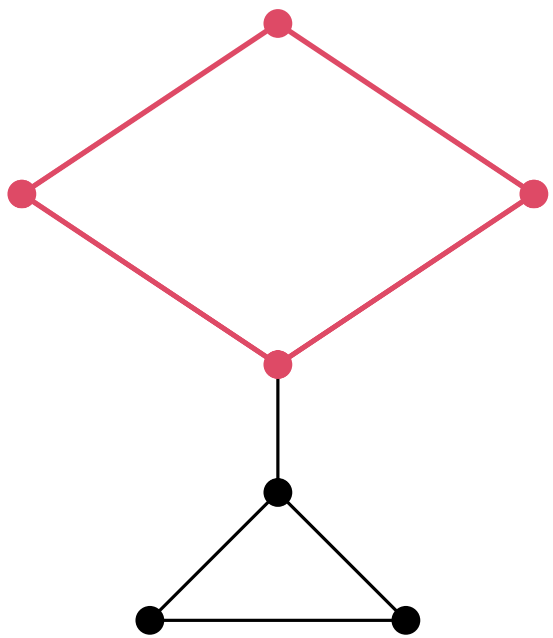

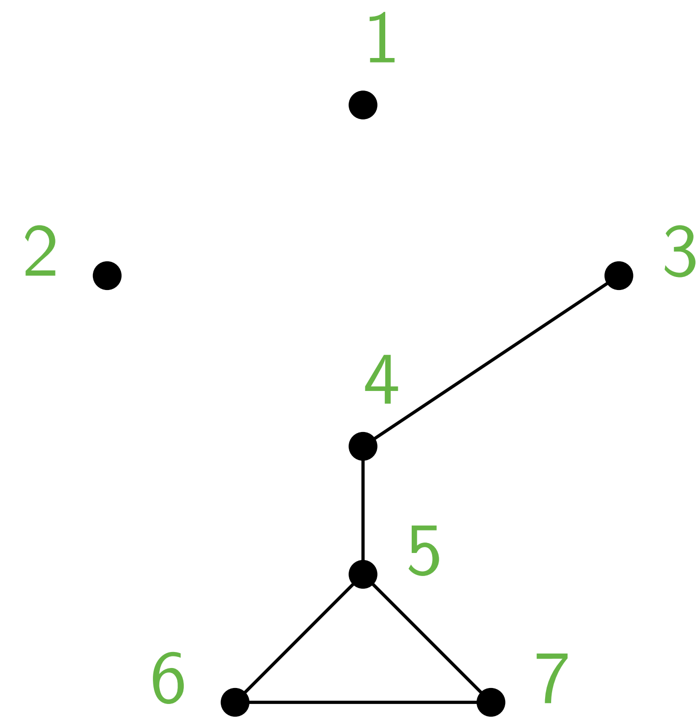

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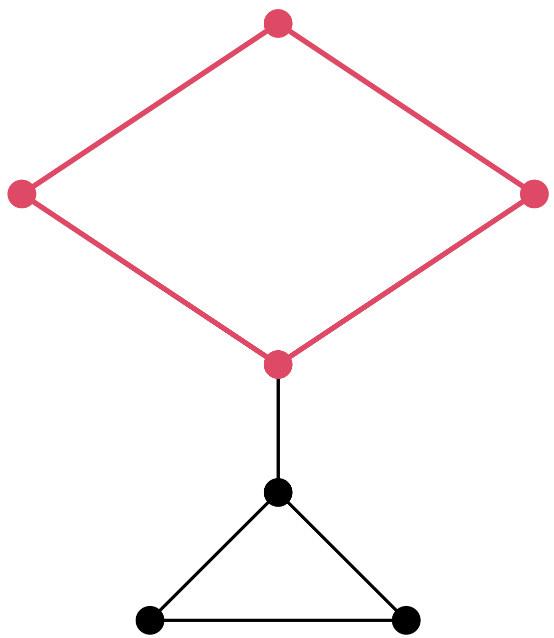

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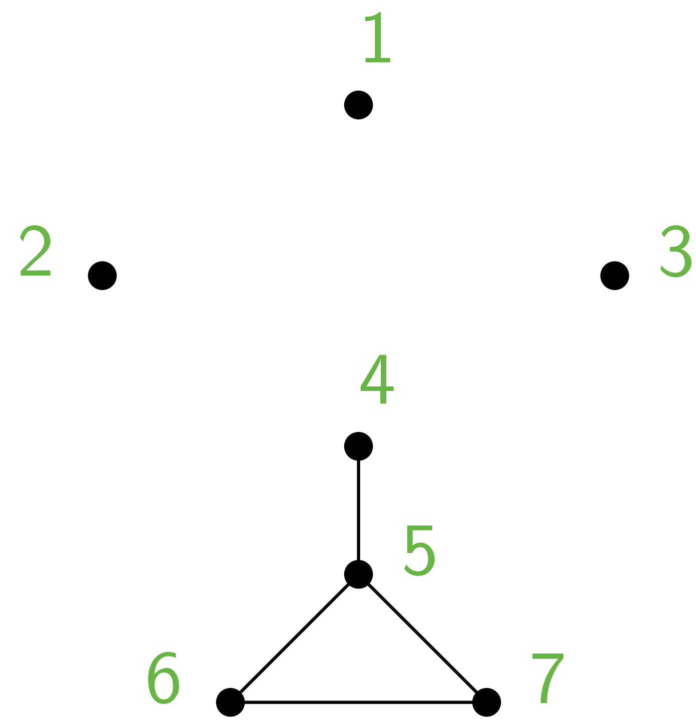

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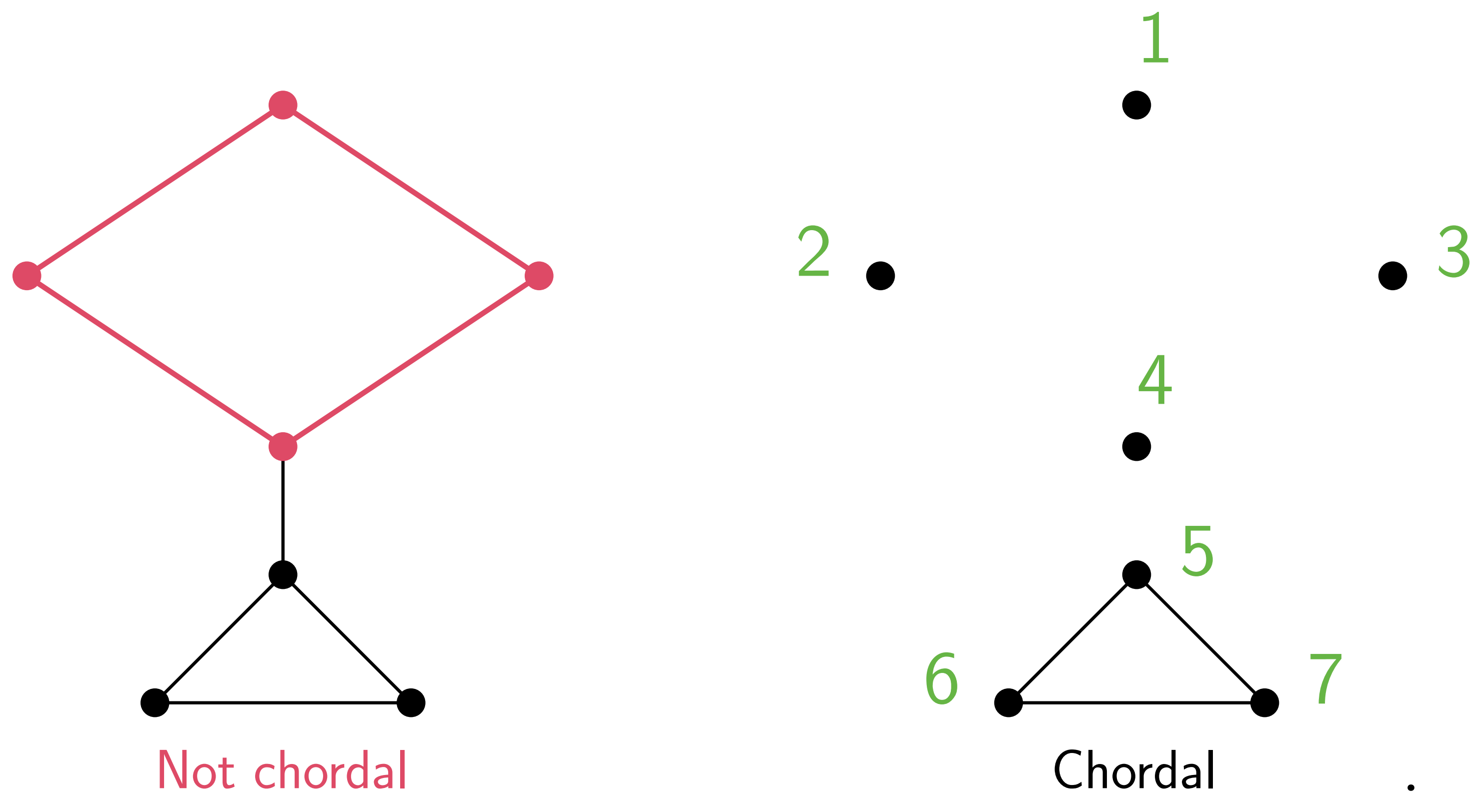

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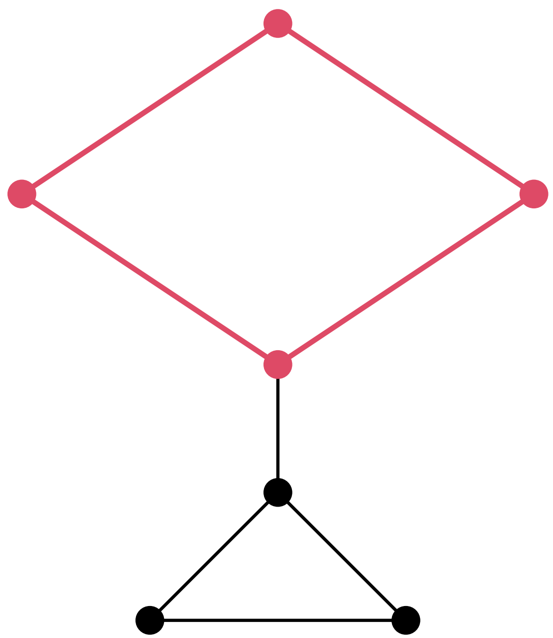

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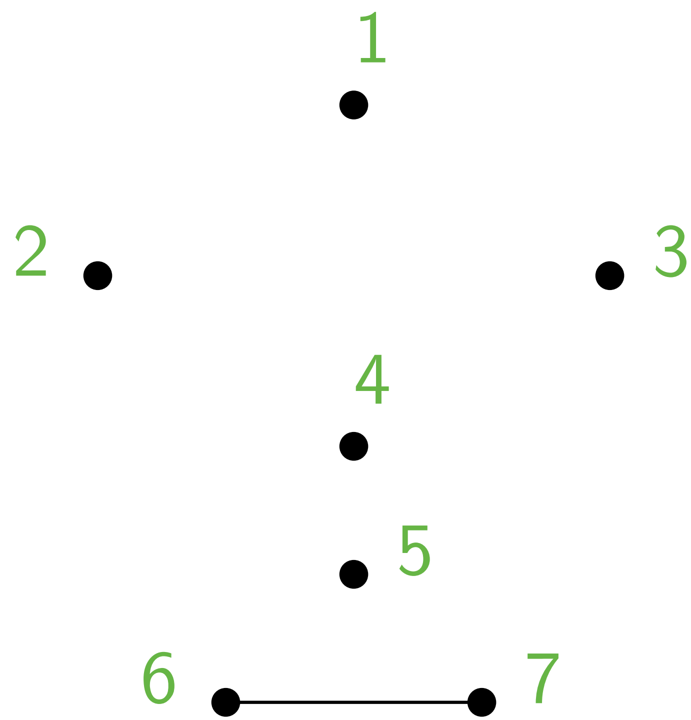

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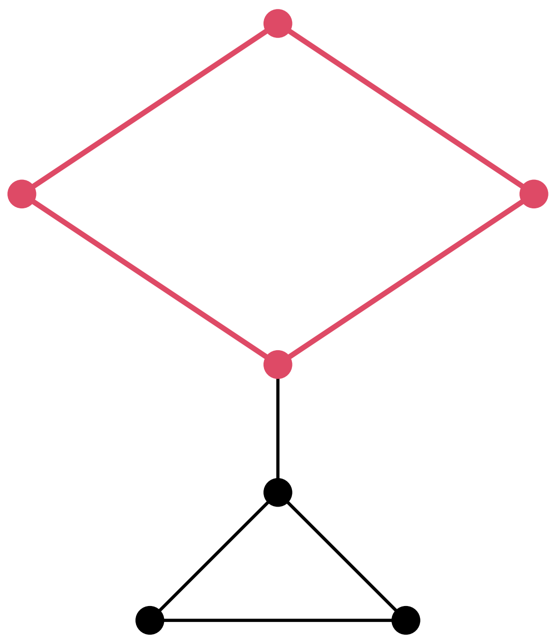

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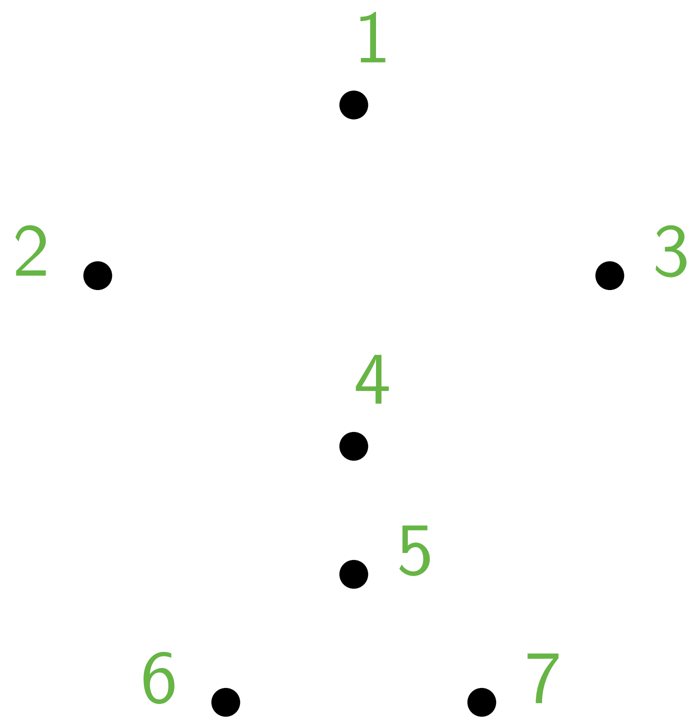

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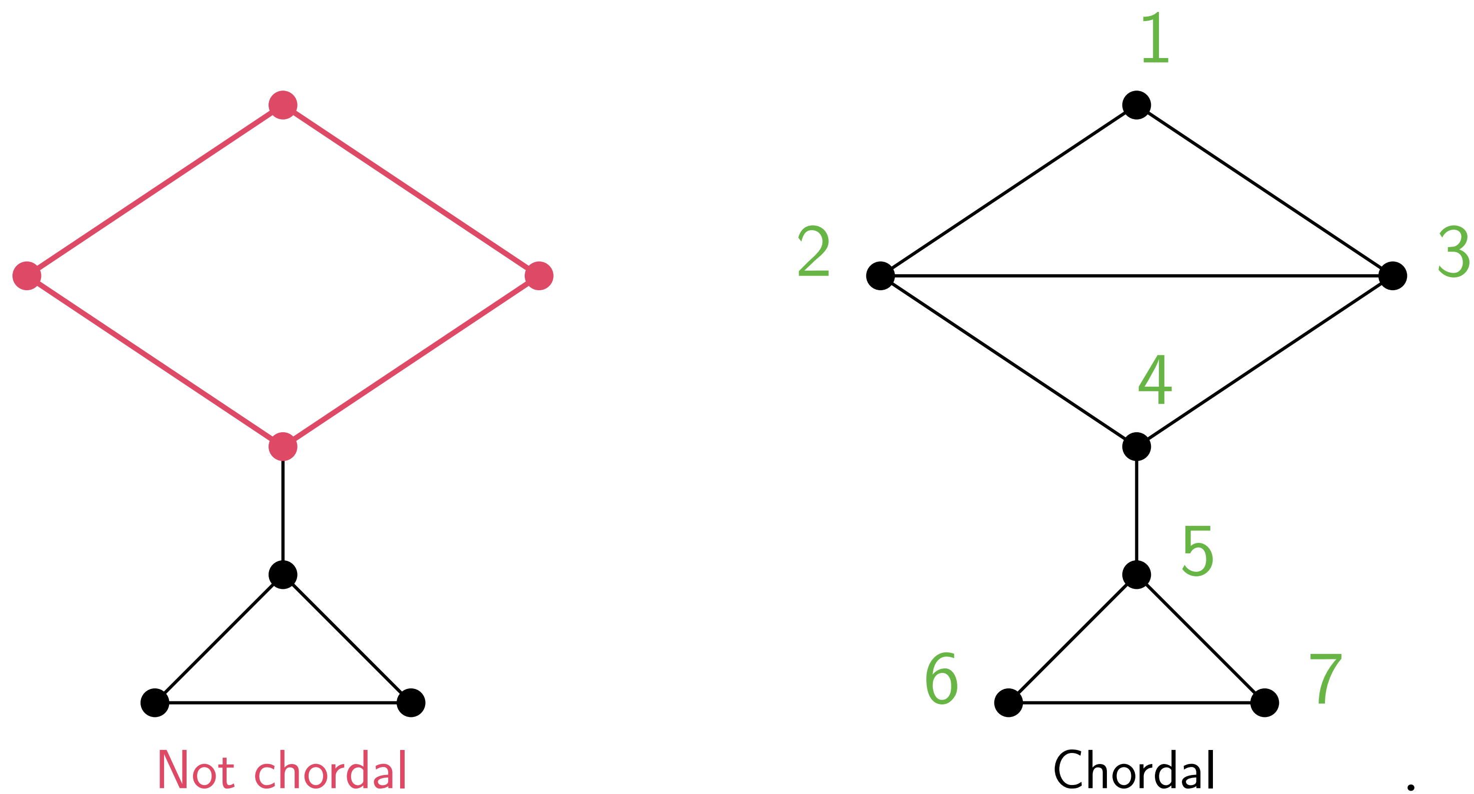

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Networks: parallel compositions of series compositions and 3-connected components.

$\mathcal{U} \subset \mathcal{B}$: 3-connected members of the class

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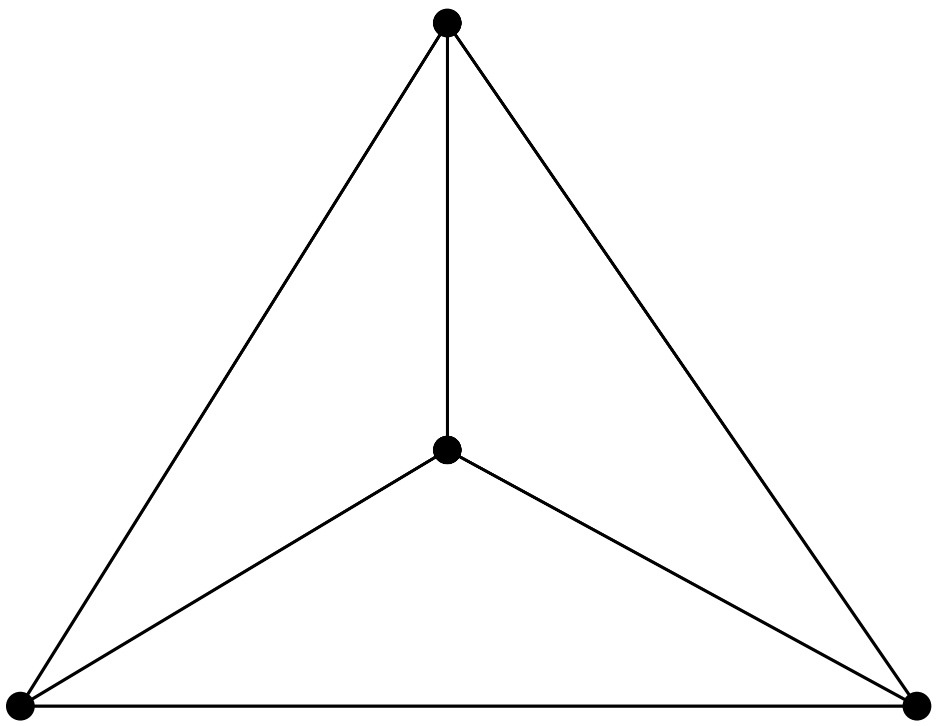

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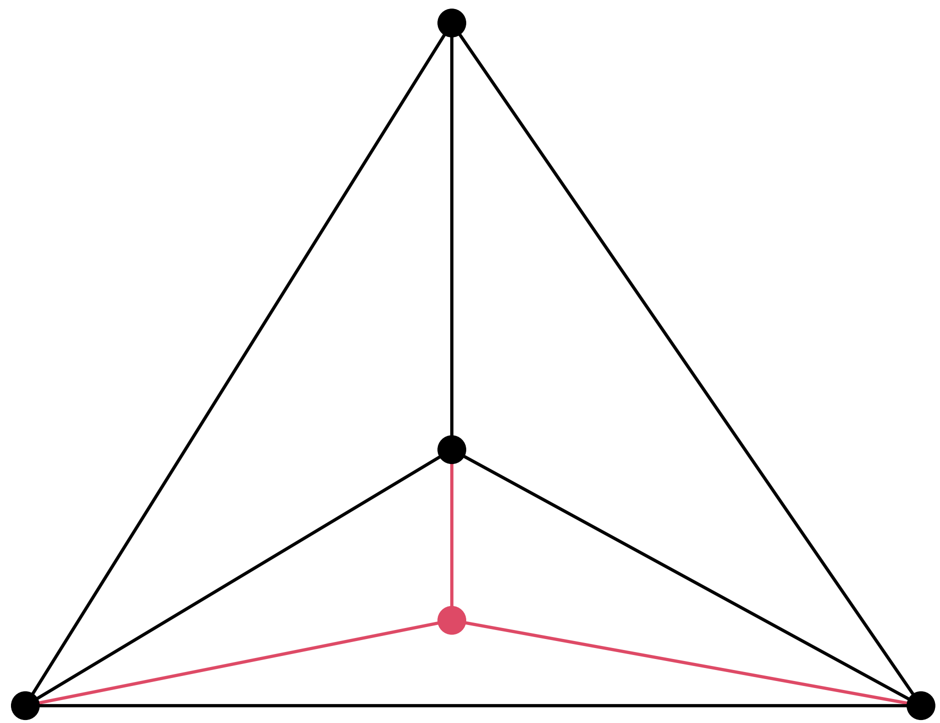

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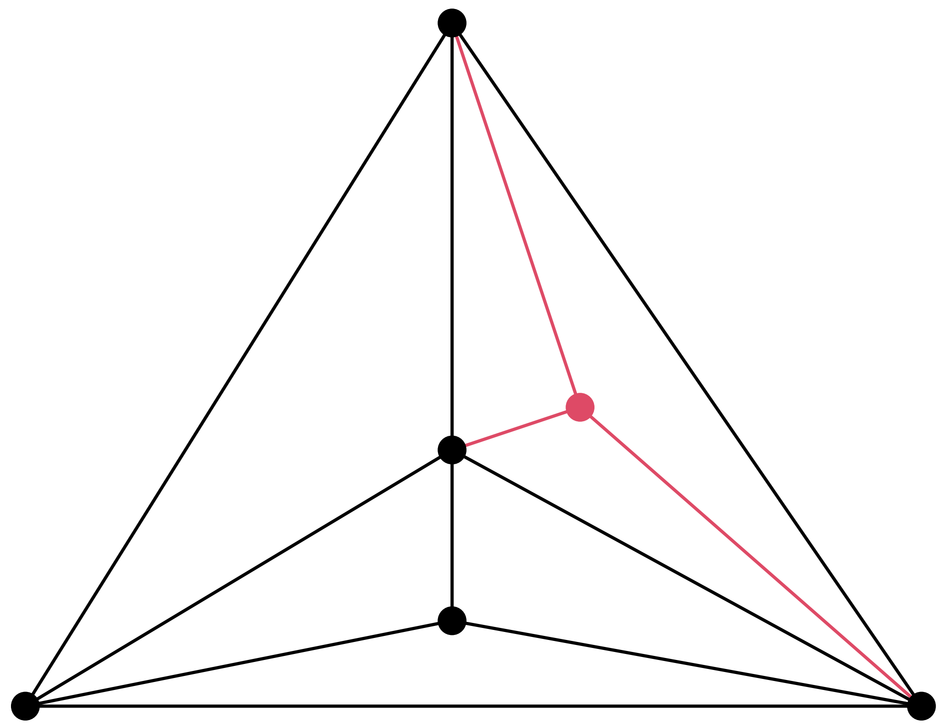

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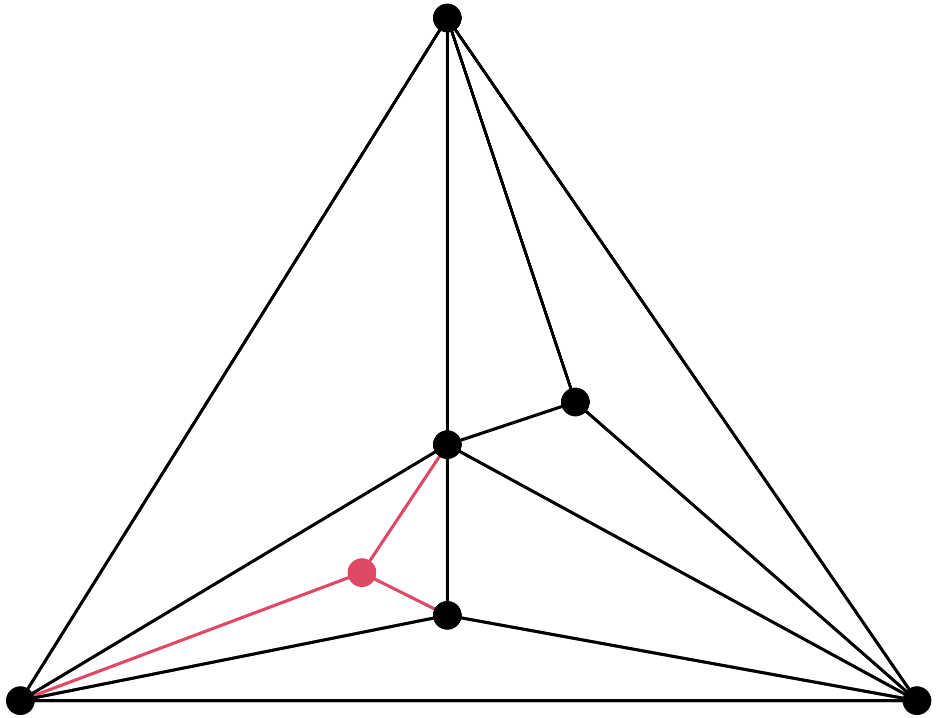

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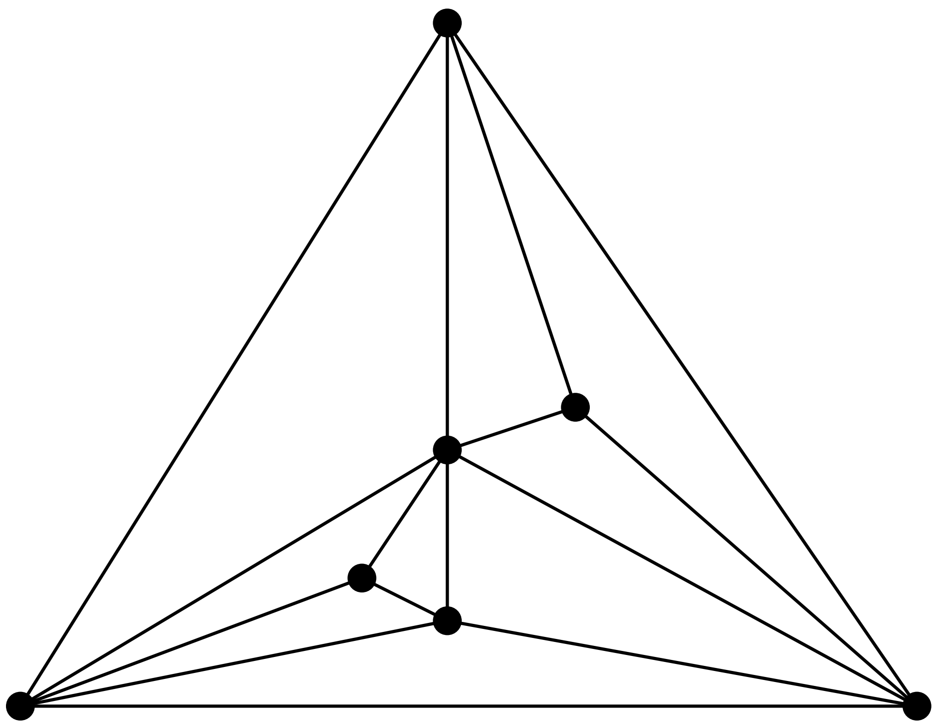

To show this, we use the perfect elimination ordering.

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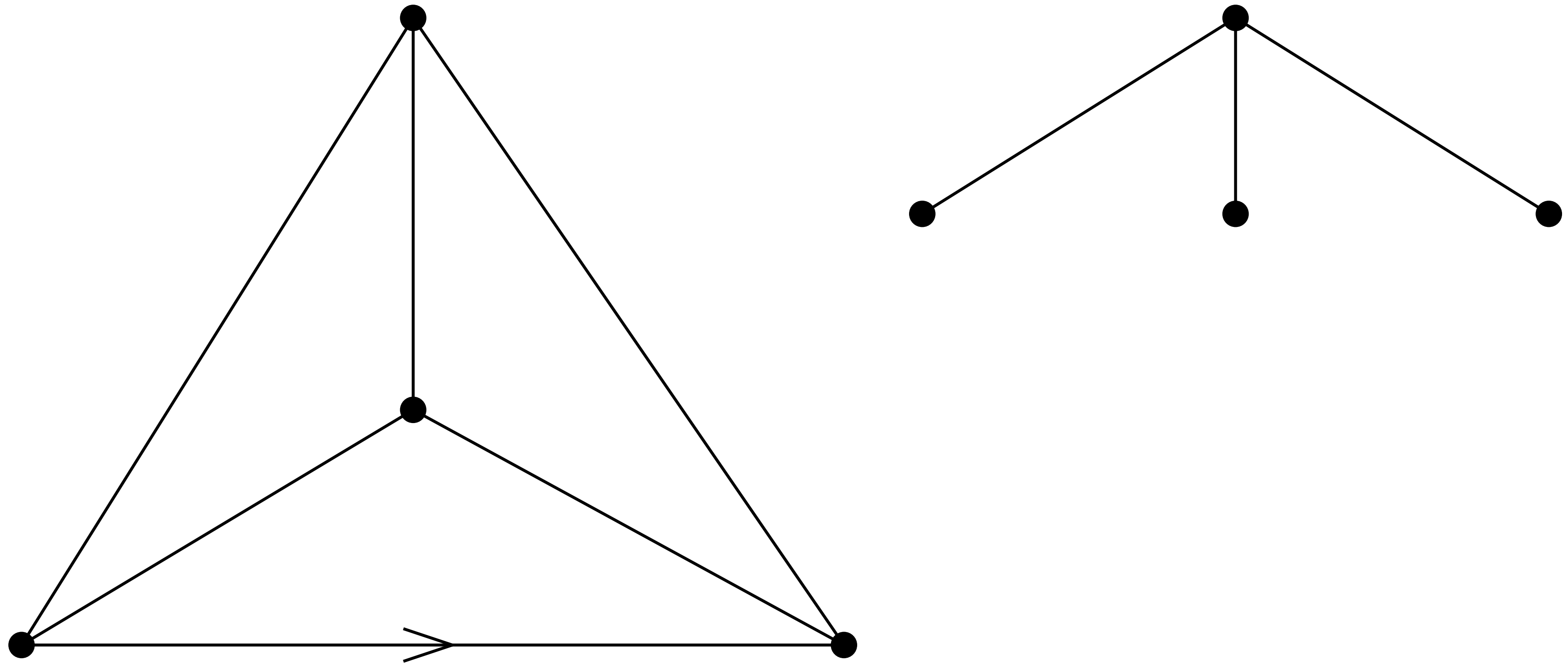

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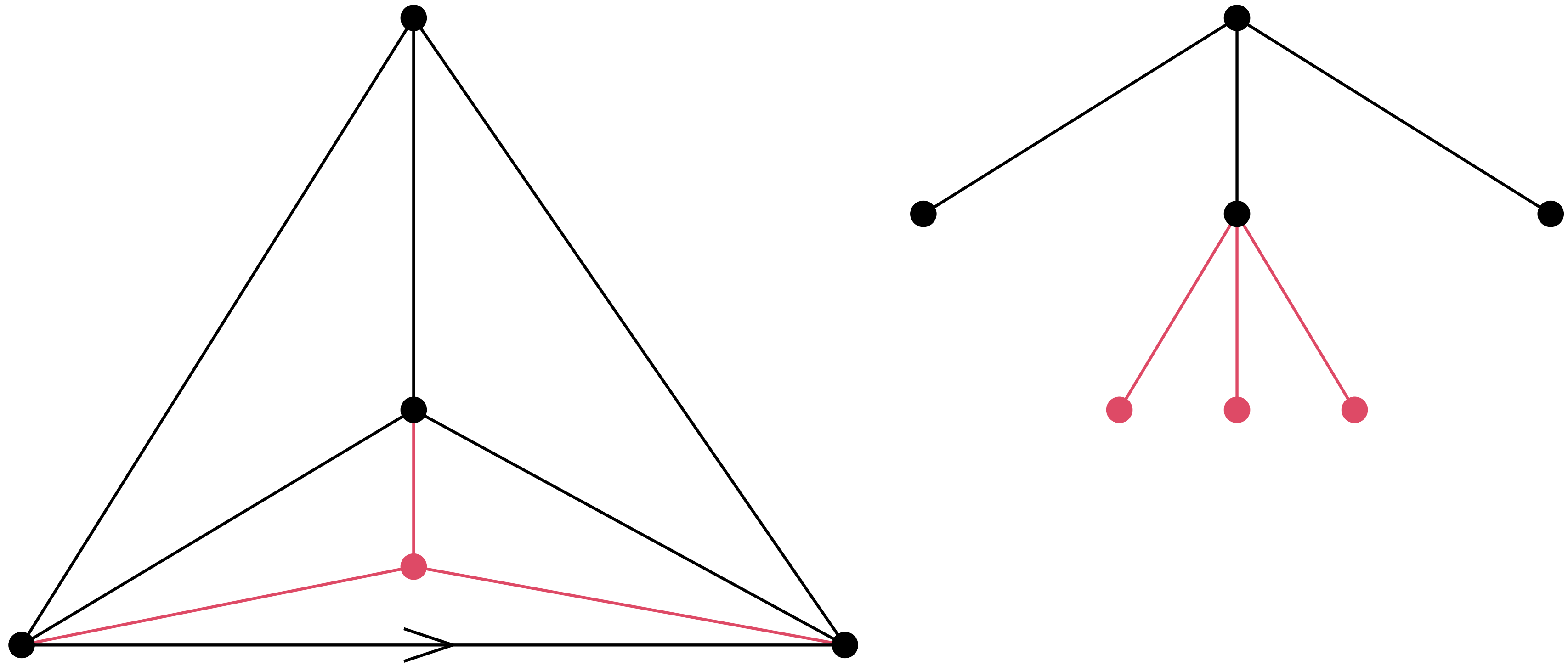

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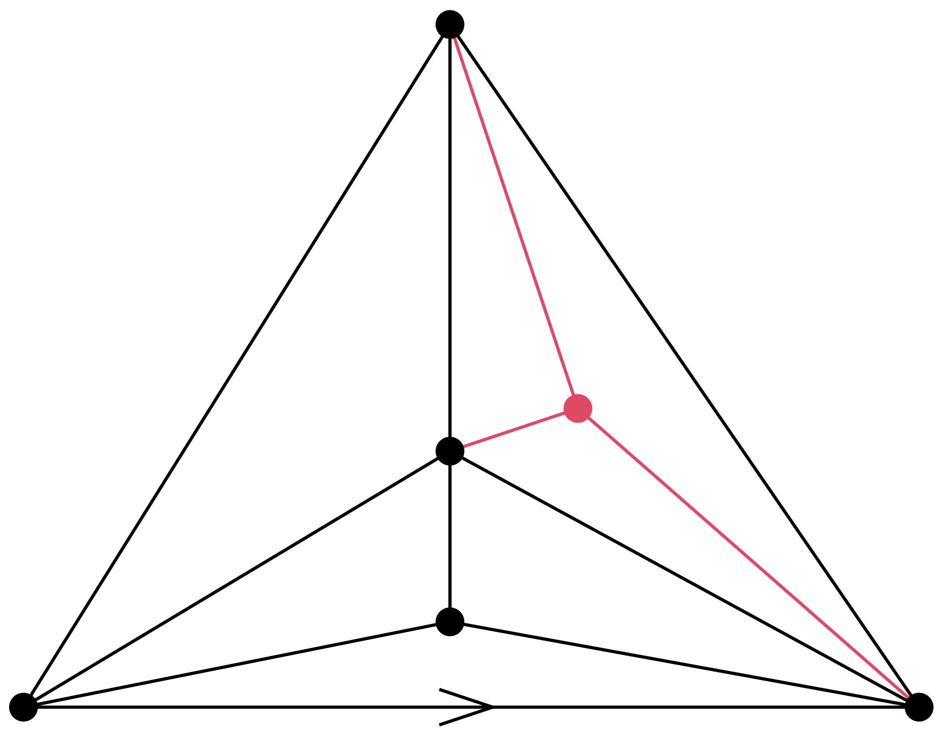

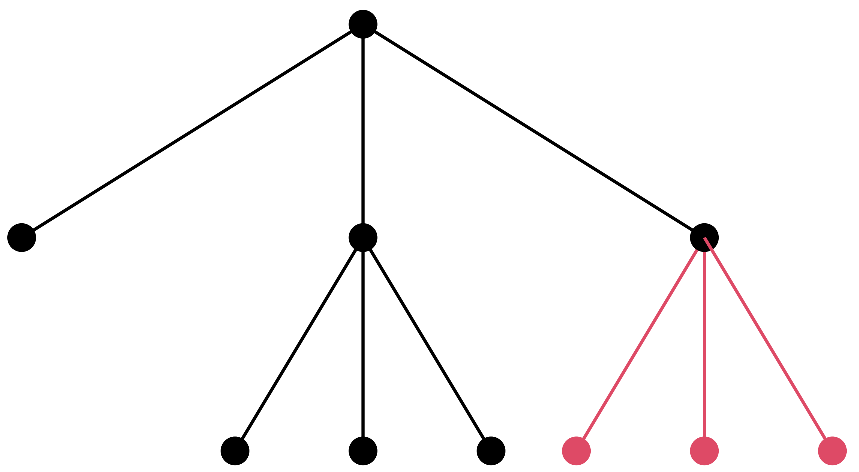

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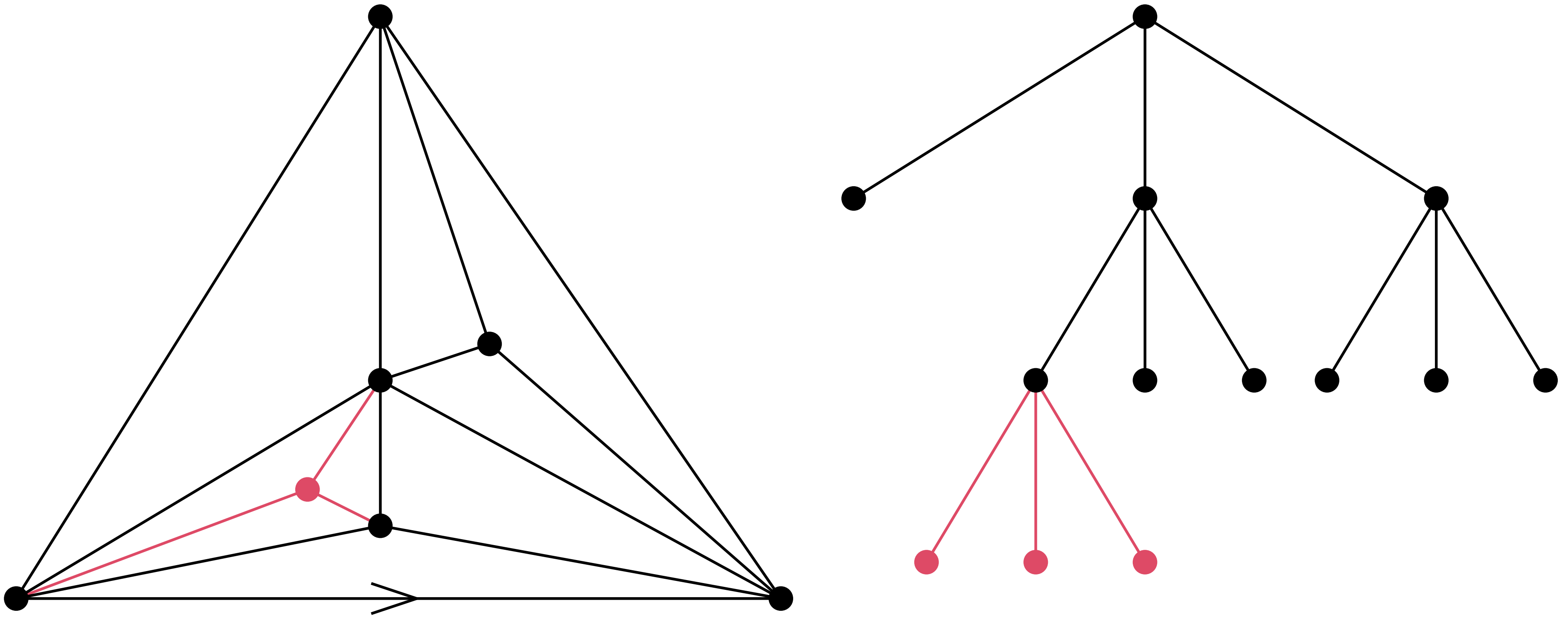

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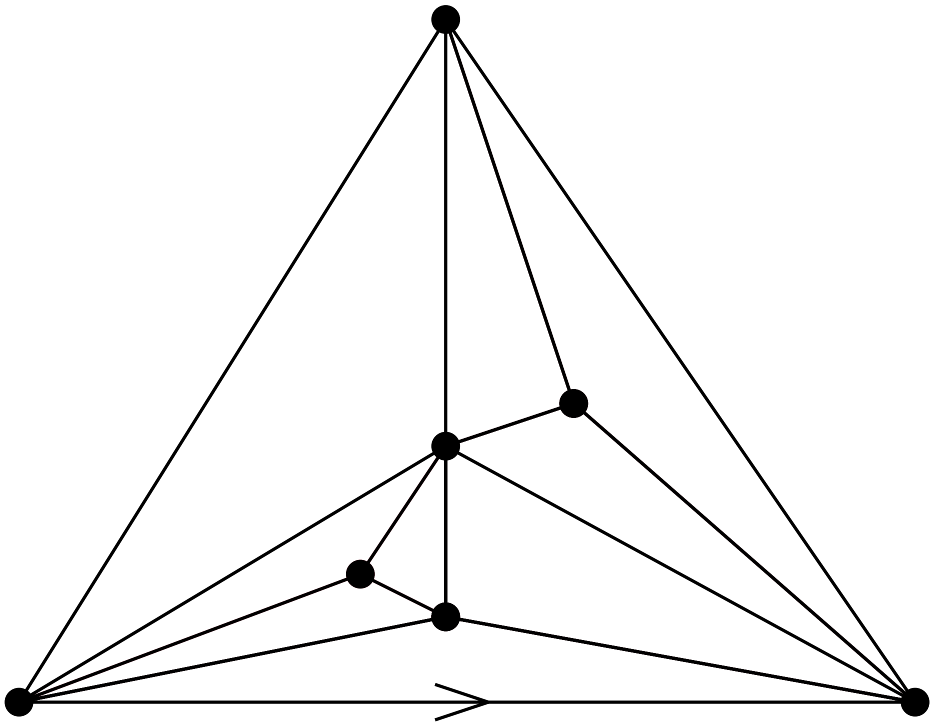

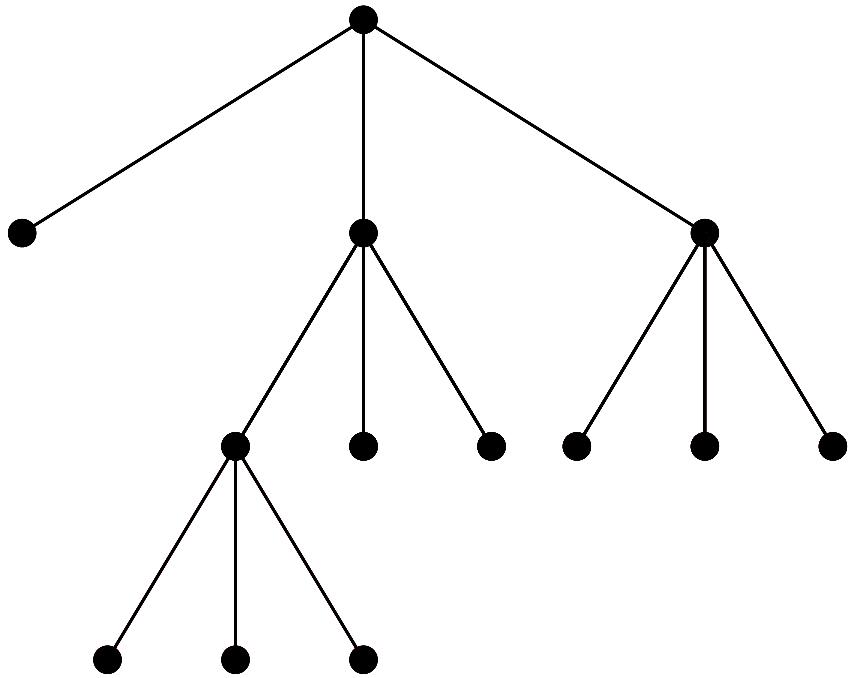

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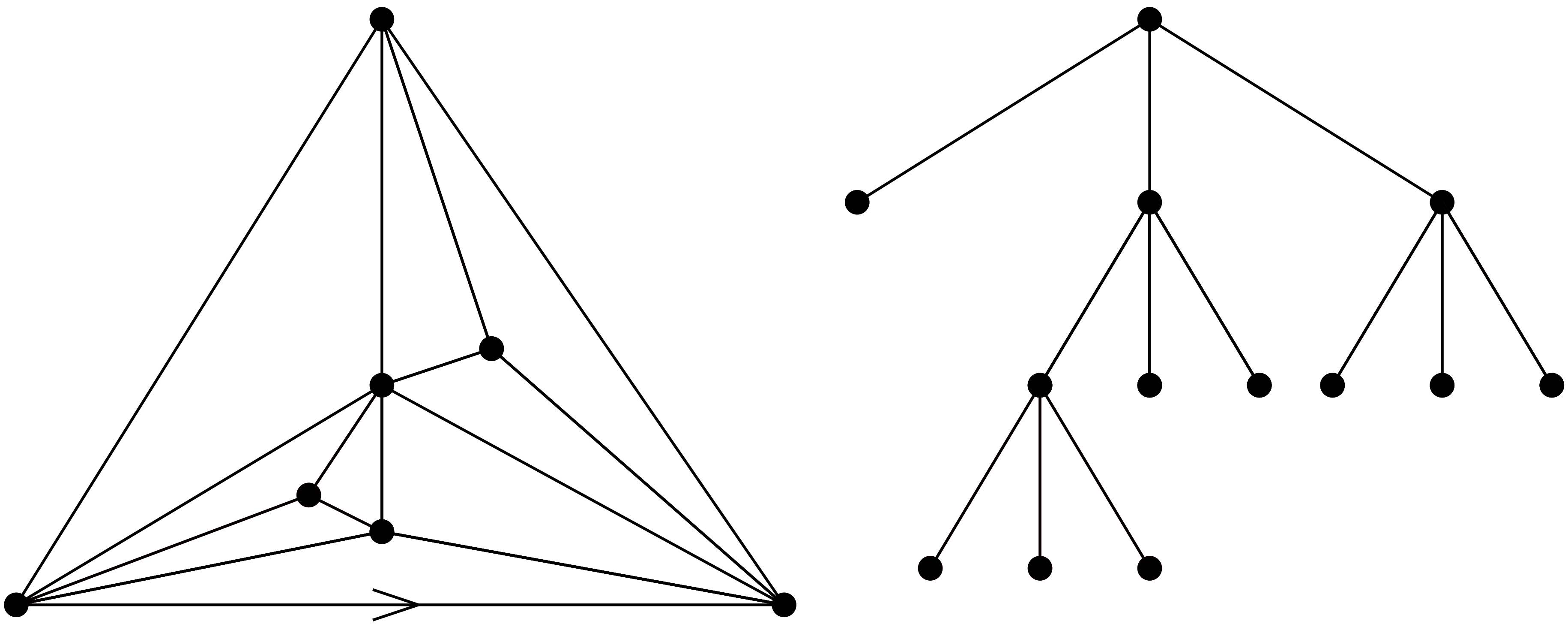

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From the bijection, the generating function of labelled chordal triangulations **rooted at a directed edge** is $T(x) = \frac{xS(x)}{2}$, where x counts #vertices-2.

Counting chordal triangulations

To obtain the generating function of **unrooted** chordal triangulations, we could take into account the number of edges in the previous equation and then integrate algebraically.

$$\frac{x^2}{2}T(x,y) = \frac{\partial}{\partial y}U(x,y)$$

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Instead, we choose to use the dissymmetry theorem and keep our proofs combinatorial.

The dissymmetry theorem

Theorem. Let $\mathcal{A}$ be a class of trees. Then,

$$\mathcal{A} + \mathcal{A}^{\bullet \to \bullet} \simeq \mathcal{A}^{\bullet} + \mathcal{A}^{\bullet - \bullet}$$

where $\simeq$ is a bijection preserving the number of nodes.

Proof sketch. Oriented edges towards the center of the tree correspond to vertices and the others correspond to nonoriented edges.

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This theorem can be applied to tree-decomposable classes of graphs.

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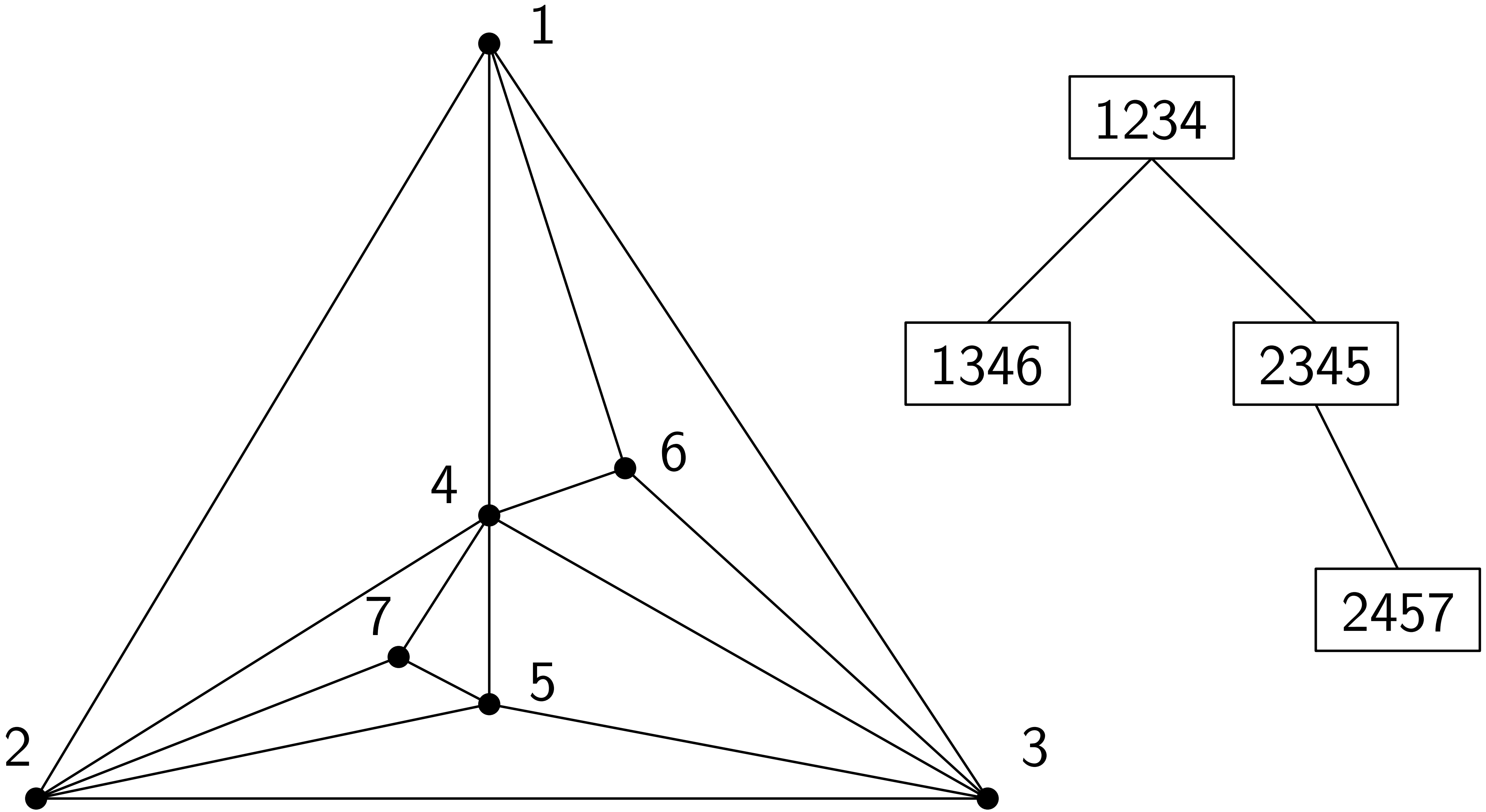

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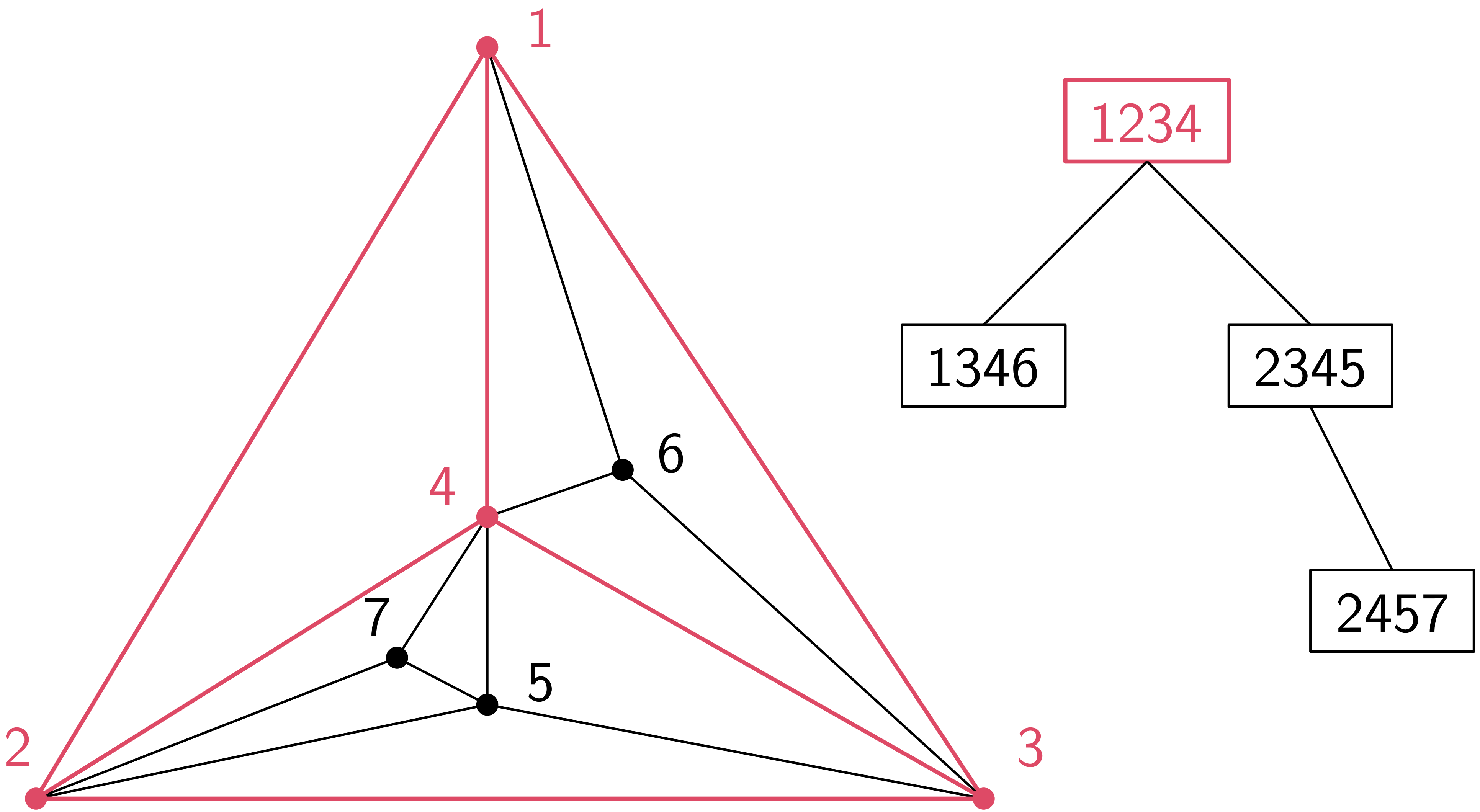

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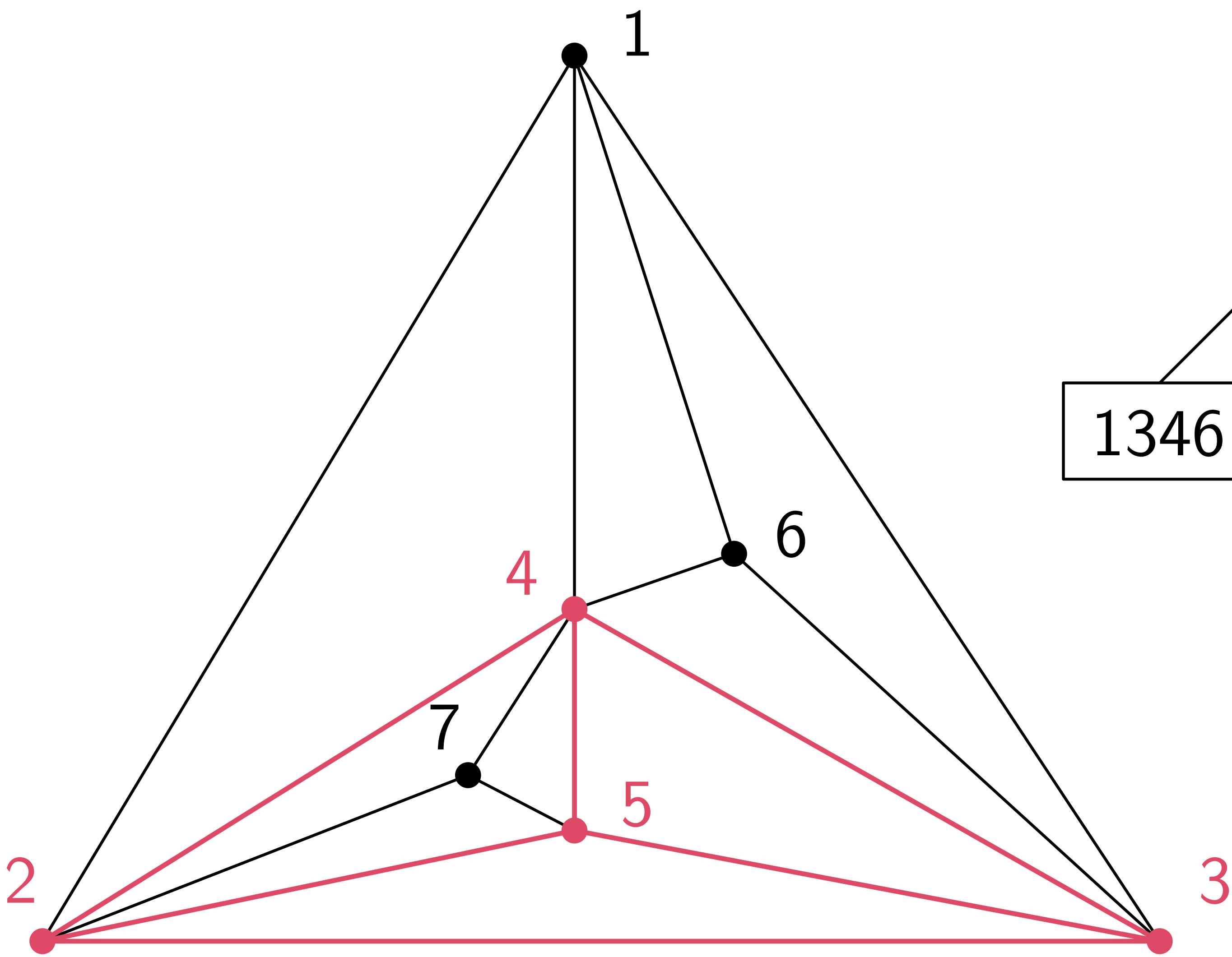

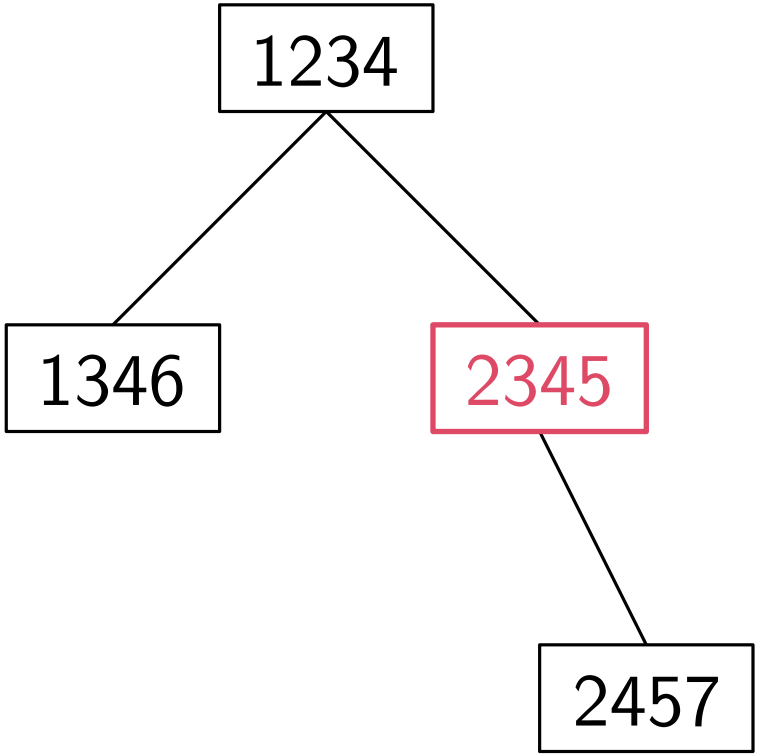

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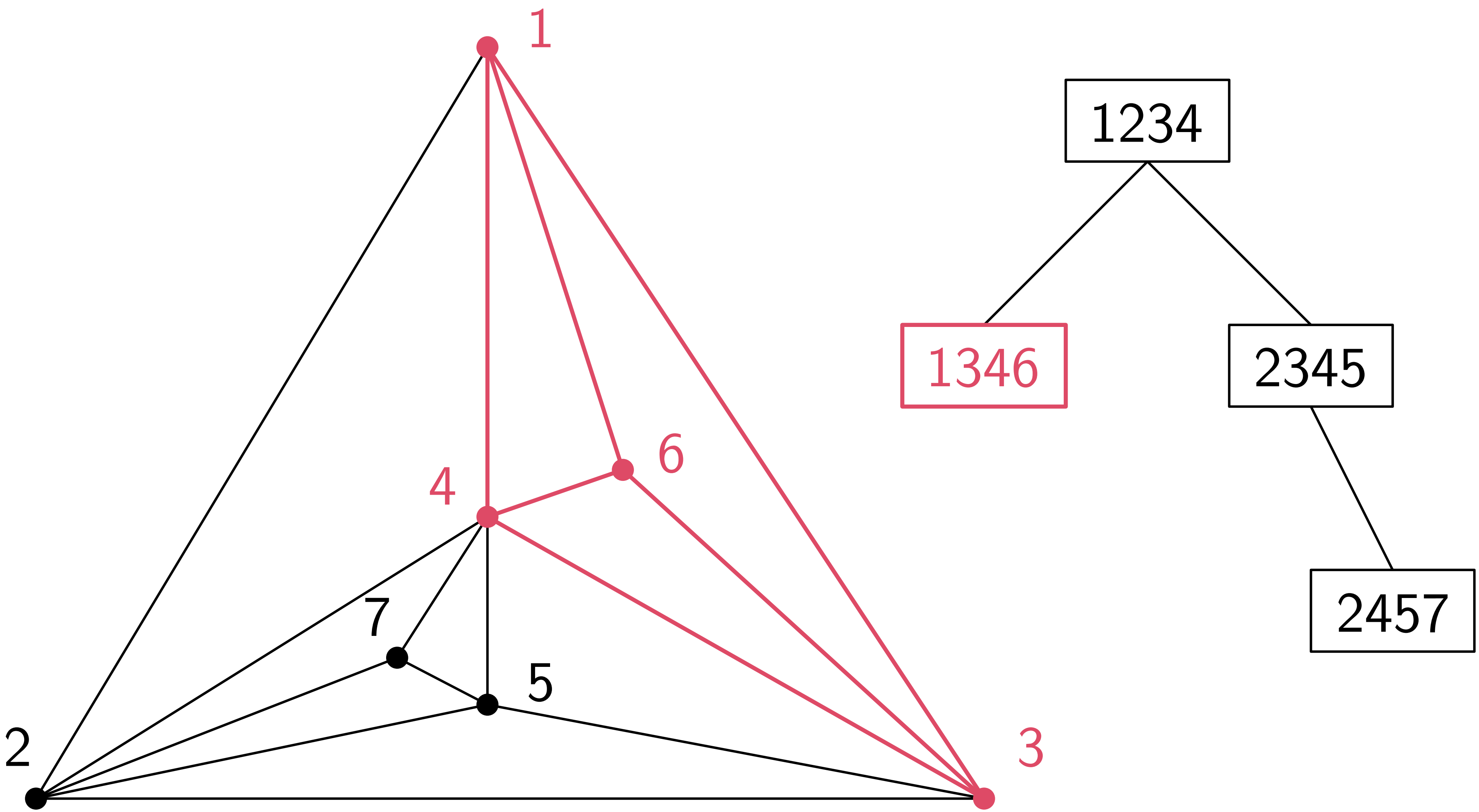

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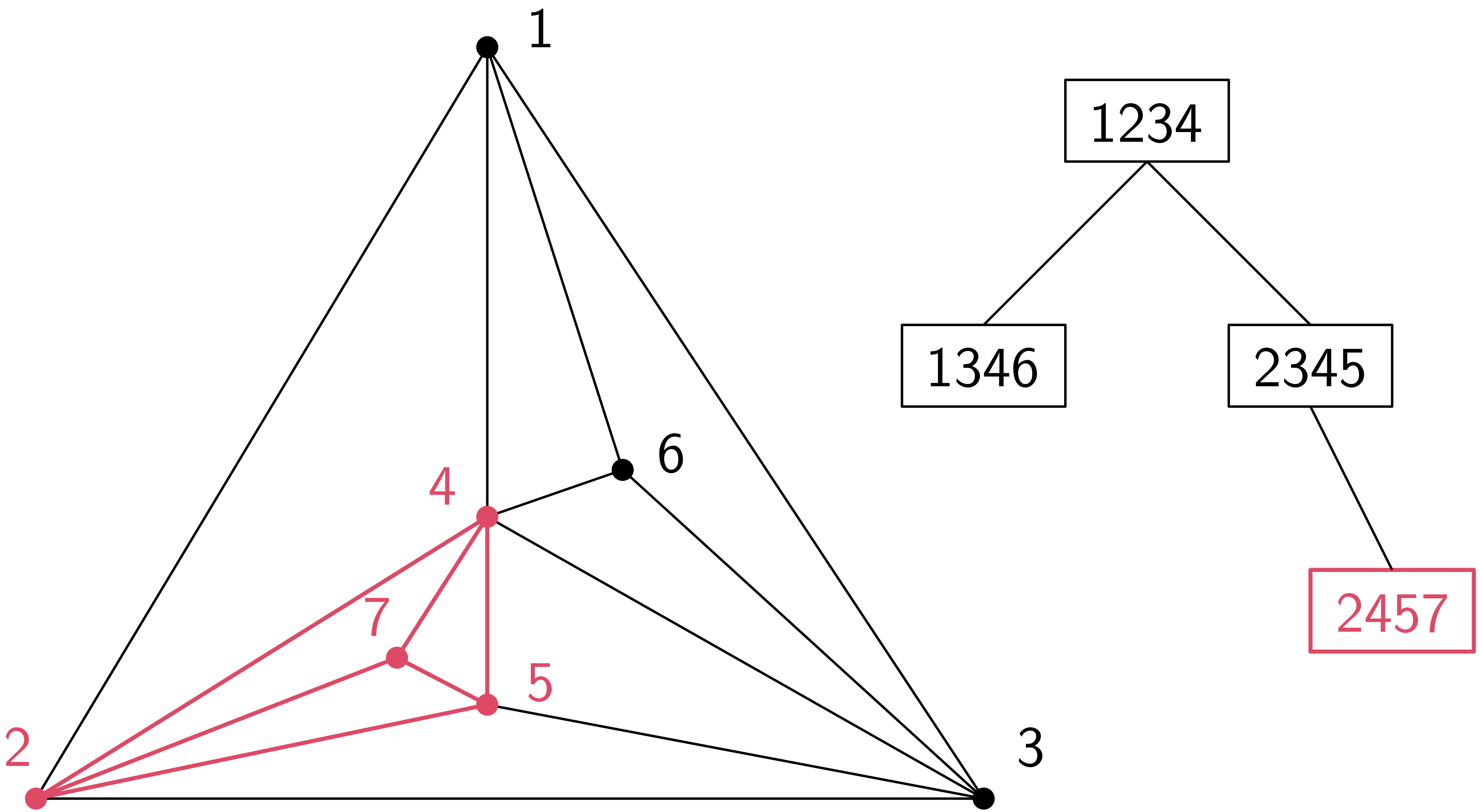

Counting chordal triangulations

This decomposition yields the following equations:

$$A^{\bullet} = \frac{x^4}{24}(1 + S(x))^4$$

$$A^{\bullet - \bullet} = \frac{x^3}{12}S(x)^2$$

$$A^{\bullet \to \bullet} = 2A^{\bullet - \bullet}$$

$$U(x) = A = A^{\bullet} - A^{\bullet - \bullet} = \frac{x^3}{24}(S(x) - S(x)^2)$$

2-connected graphs

Definition. A **network** is a 2-connected graph rooted at a directed edge whose vertices are unmarked.

If $B(x, y)$ is the generating function of 2-connected graphs and $E(x, y)$ is the generating function of networks, where x marks vertices and y marks edges, one has

$$E(x, y) = \frac{2y}{x^2} B_y(x, y).$$

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Following the classical decomposition [Giménez, Noy, Rué '13], networks are **parallel compositions of series compositions and 3-connected components**, recursively substituting edges by networks.
In our context, 3-connected components are exactly chordal triangulations.

2-connected graphs

How do our networks look like?

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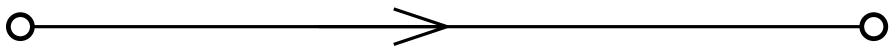

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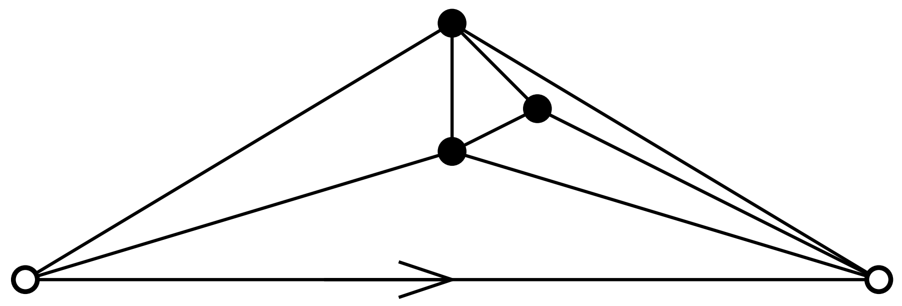

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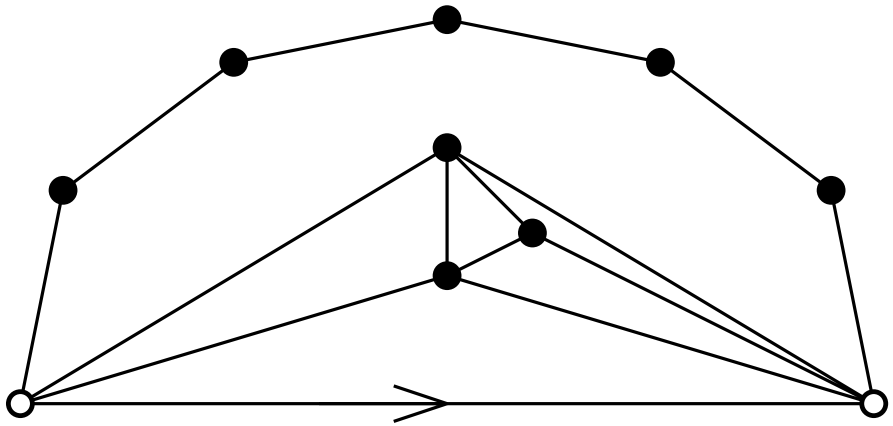

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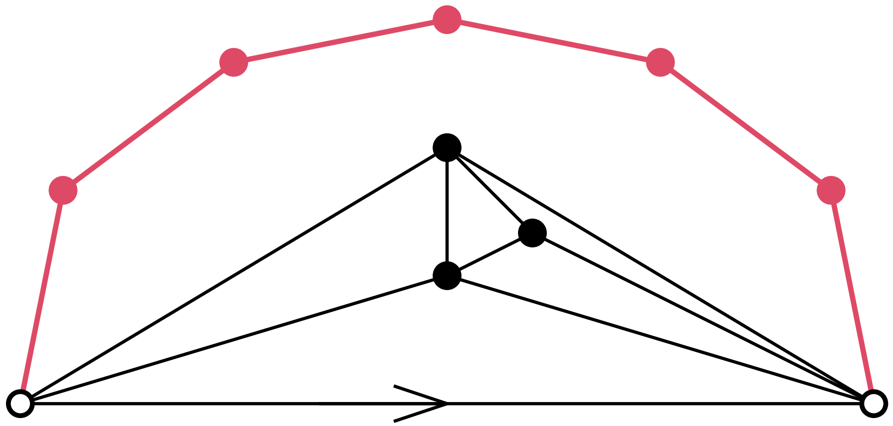

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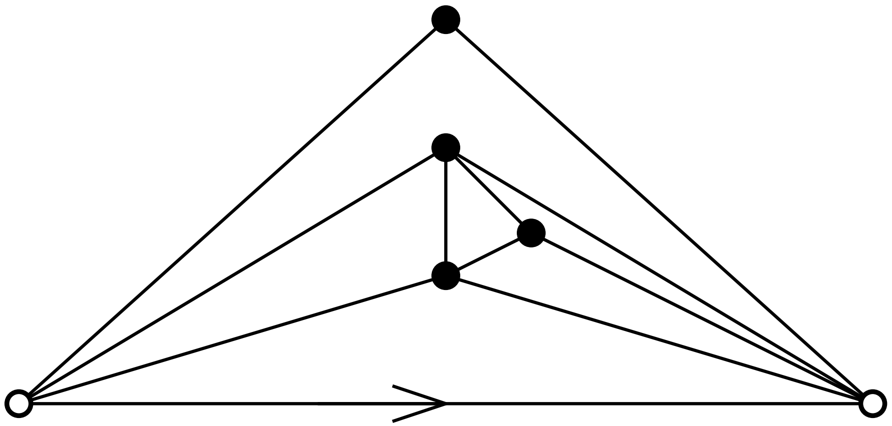

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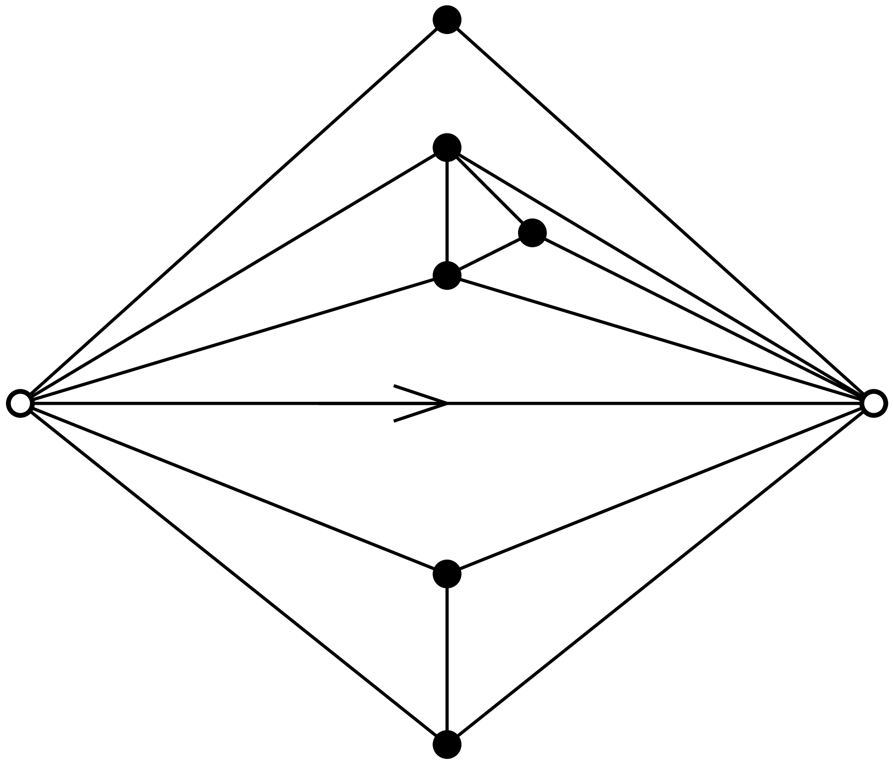

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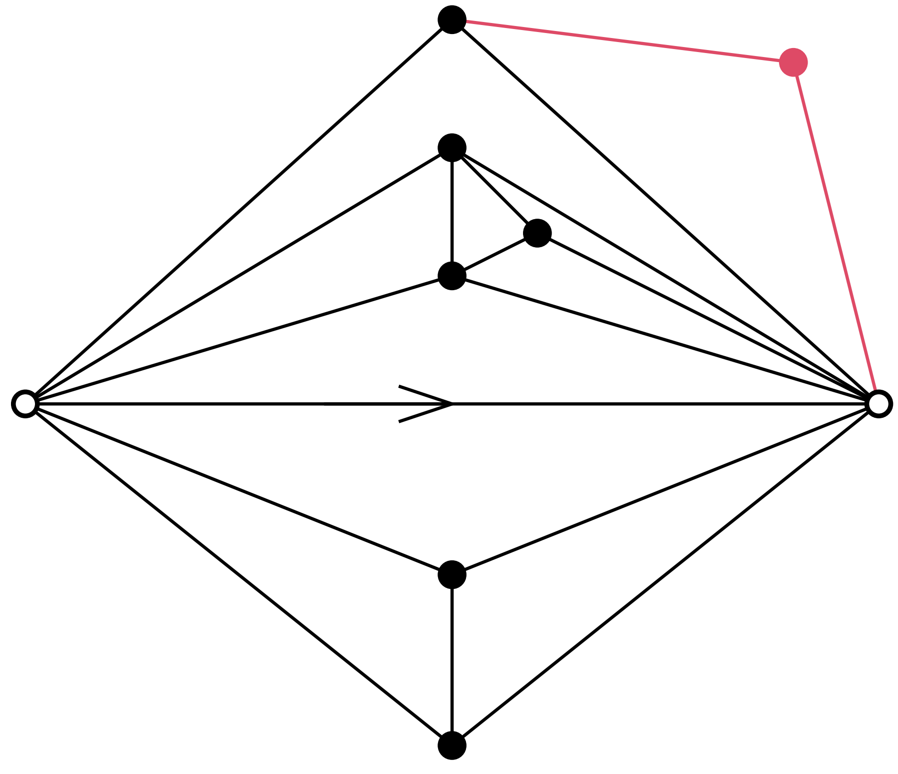

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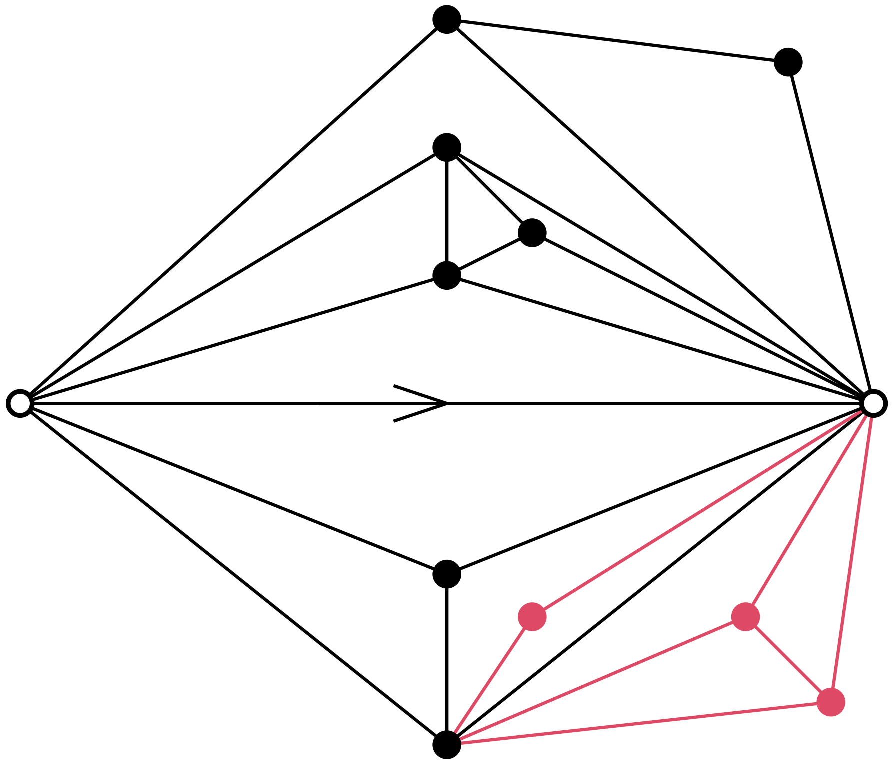

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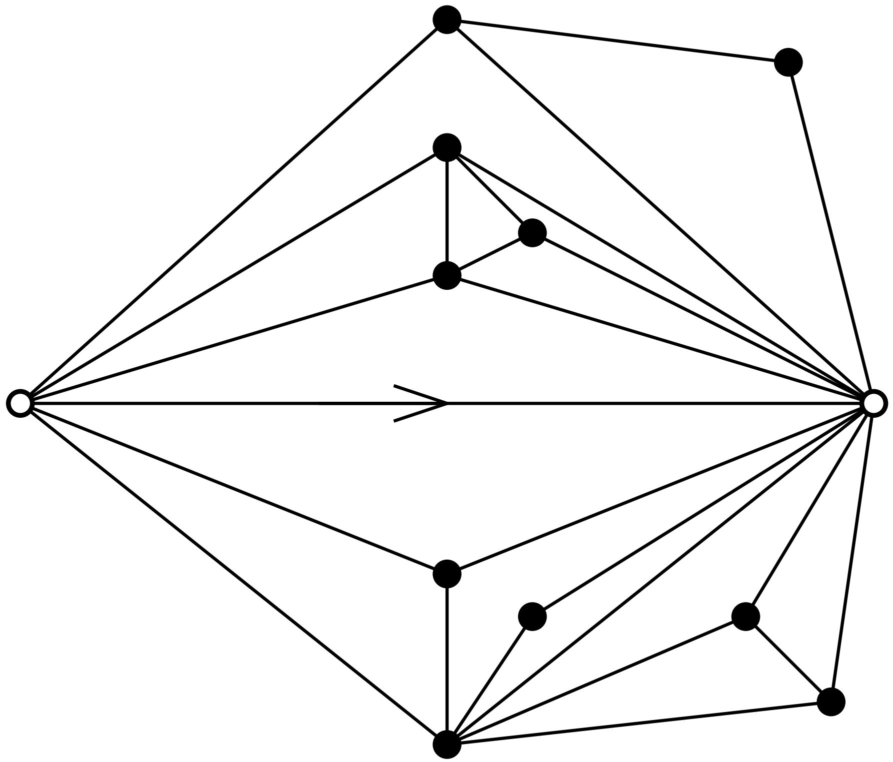

We obtain the following equation:

$$E(x,y) = y \exp\left(xE(x,y)^2 + \frac{T(xE(x,y)^3)}{E(x,y)} \right).$$

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To get the unrooted graphs, we need to compute the integral

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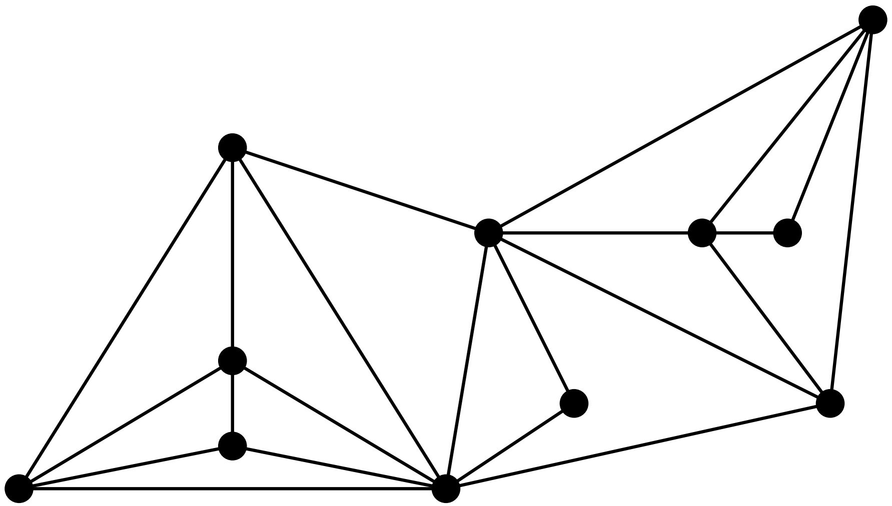
Observe that 2-connected graphs are the result of **gluing chordal triangulations and triangles through edges**.

Therefore, we can encode them using trees whose nodes have 3 possible types: **e** (edge), **s** (series/triangle) and **t** (triangulation). Notice that the edges can only be of type **s-e** or **t-e**.

2-connected graphs



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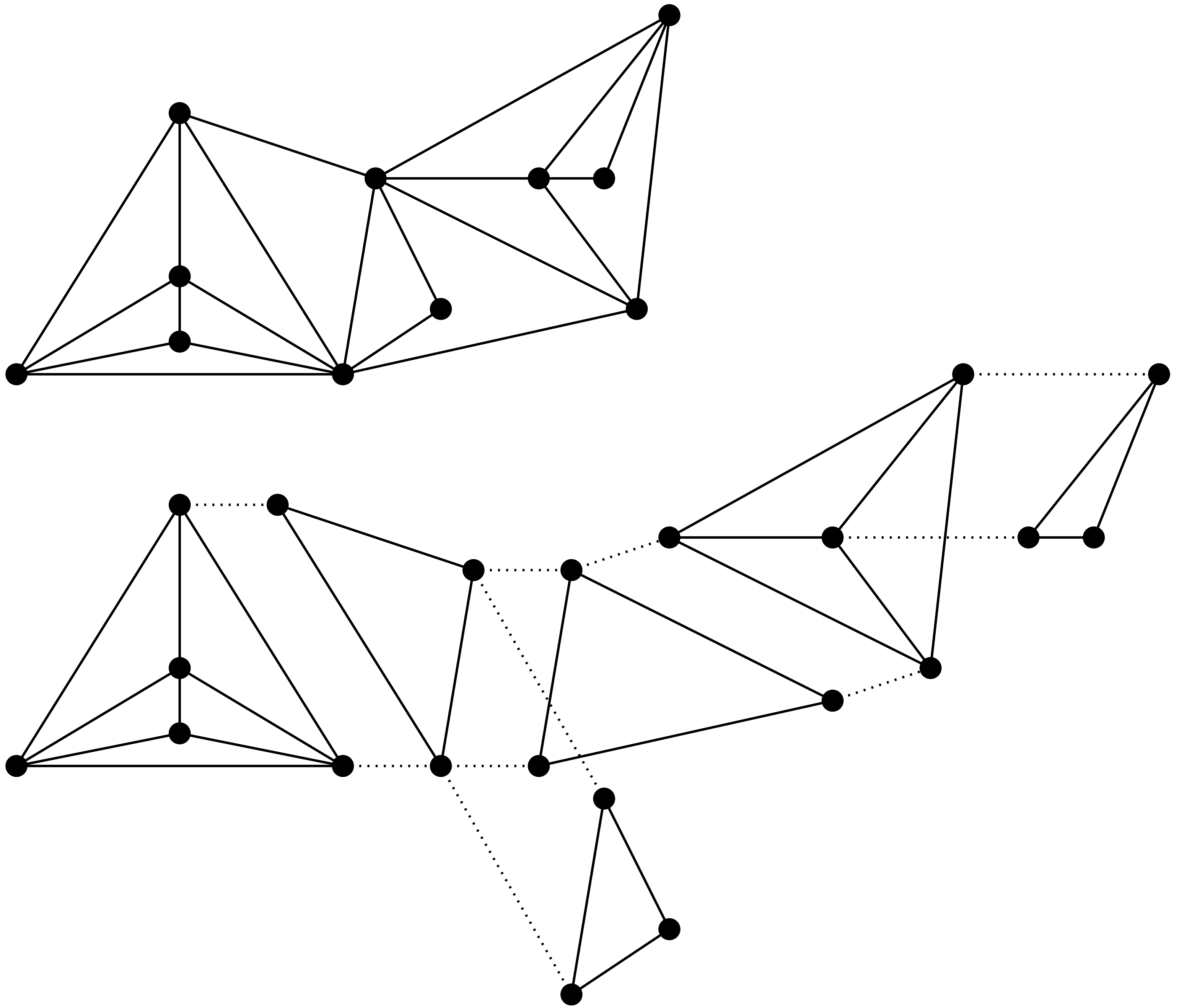

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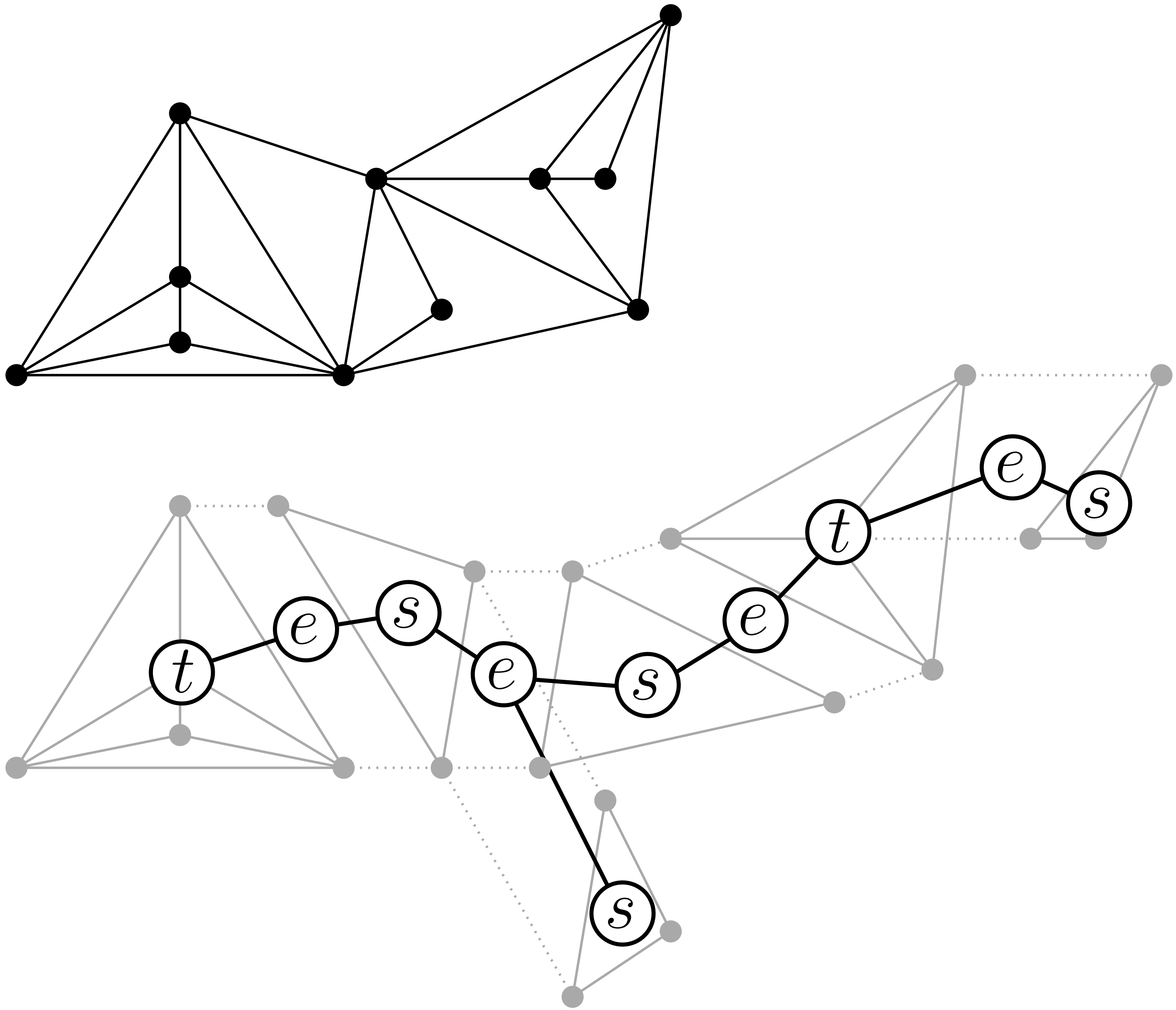

2-connected graphs

This tree decomposition yields the following equations:

$$R^e(x) = \frac{x^2}{2}(E(x) - xE(x)^2 - T(xE(x)^3)/E)$$

$$R^s(x) = \frac{x^3}{6}E(x)^3 \qquad R^t(x) = \frac{U(xE(x)^3)}{E(x)^3}$$

$$R^{s-e} = \frac{x^3}{2}E(x)^2(E(x) - 1)$$

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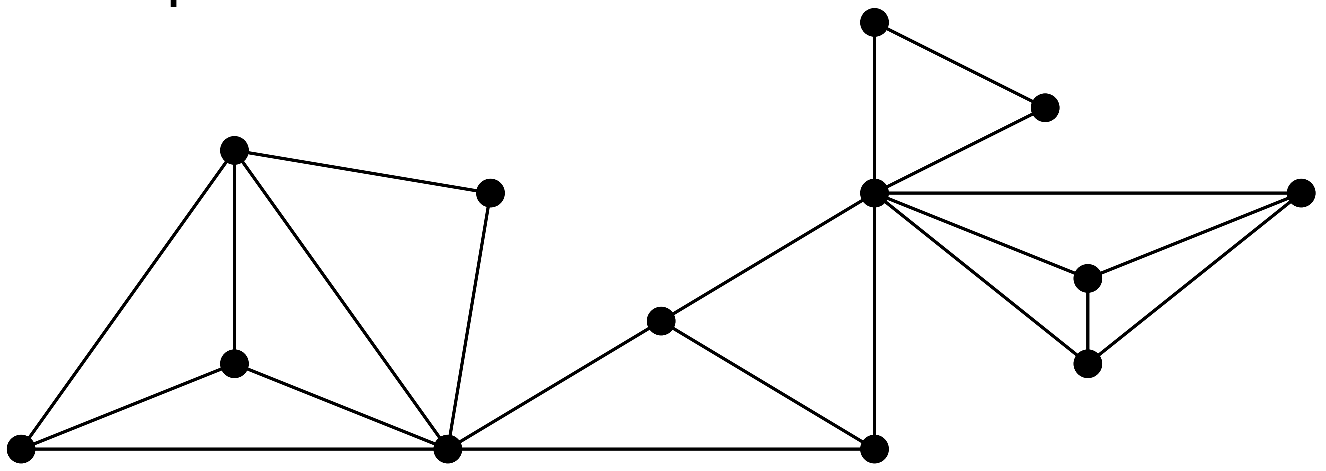
Putting everything together,

$$B(x) = R^s(x) + R^t(x) + R^e(x) - R^{s-e}(x) - R^{t-e}(x)$$
$$= \frac{x^2}{2}\left(E(x) - \frac{xE(x)^3}{12}\left(S(xE(x)^3)^2 + 5S(xE^3(x)) + 8\right)\right)$$

Connected and arbitrary graphs

We use the classical decomposition of a connected graph into 2-connected components.



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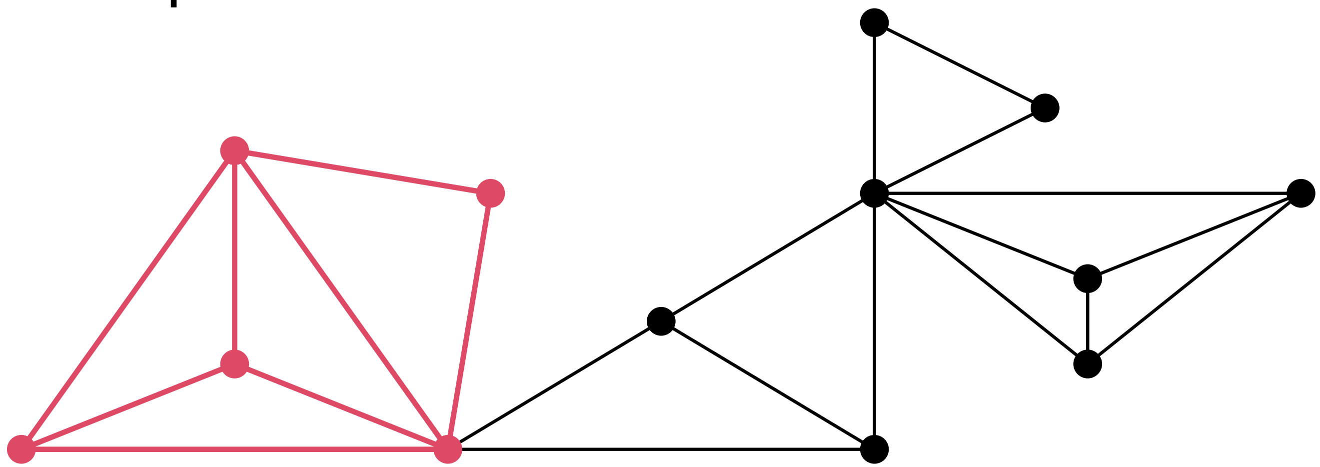

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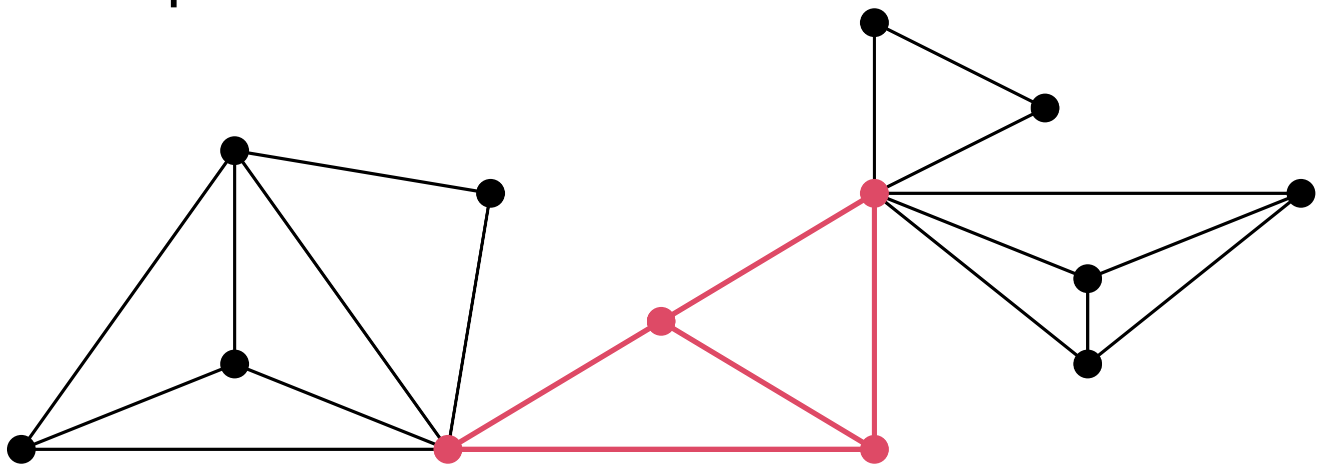

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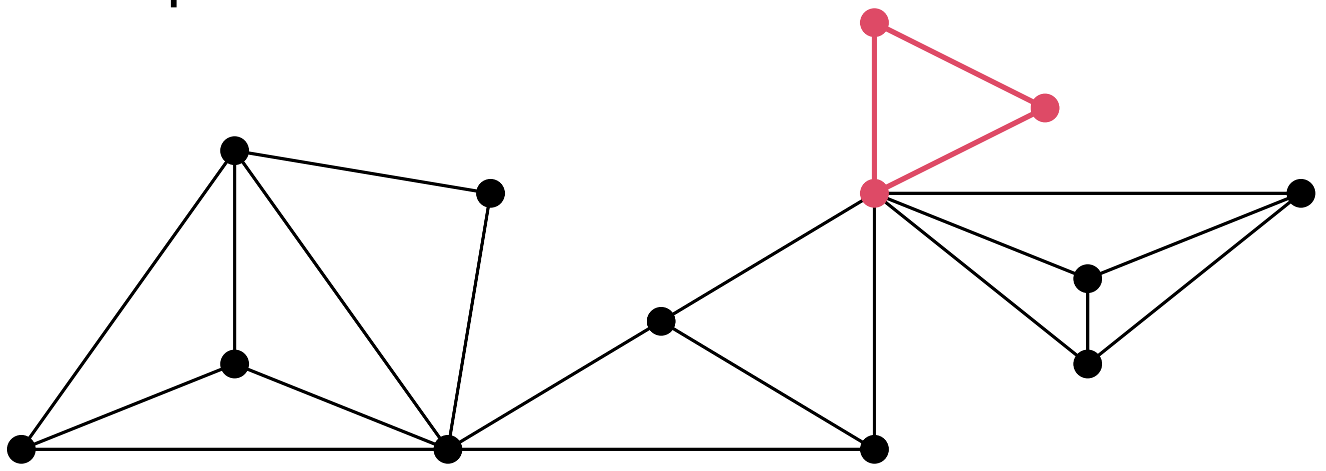

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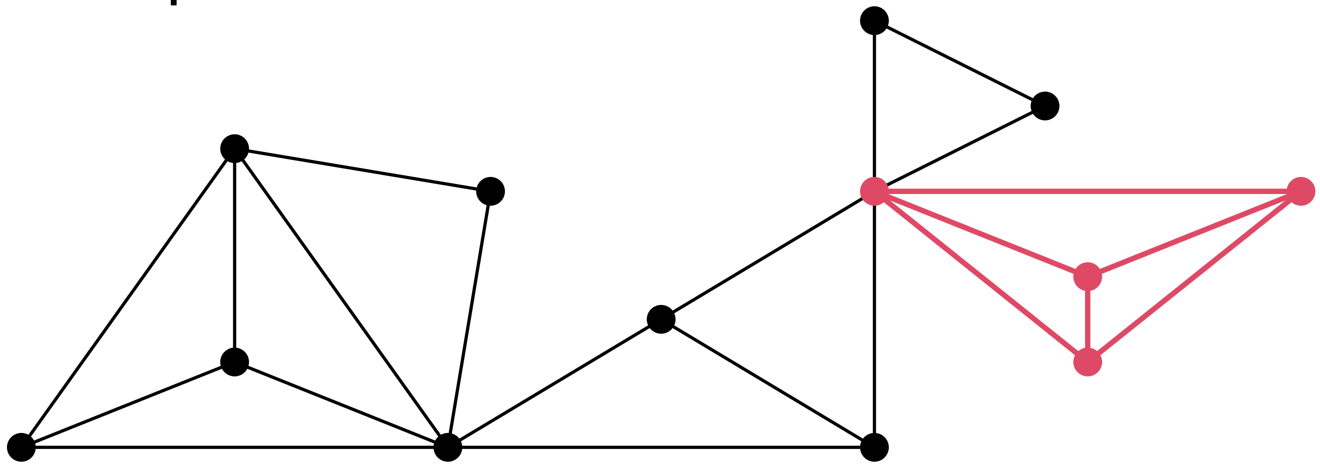

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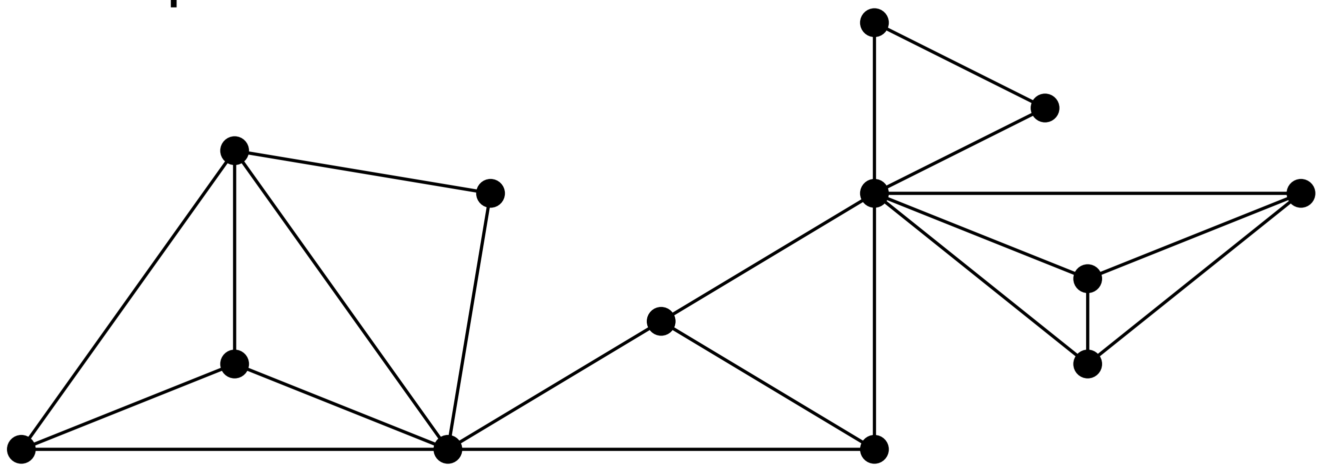

The equation associated to the decomposition is

$$C^{\bullet}(x) = x \exp(B'(C^{\bullet}(x))),$$

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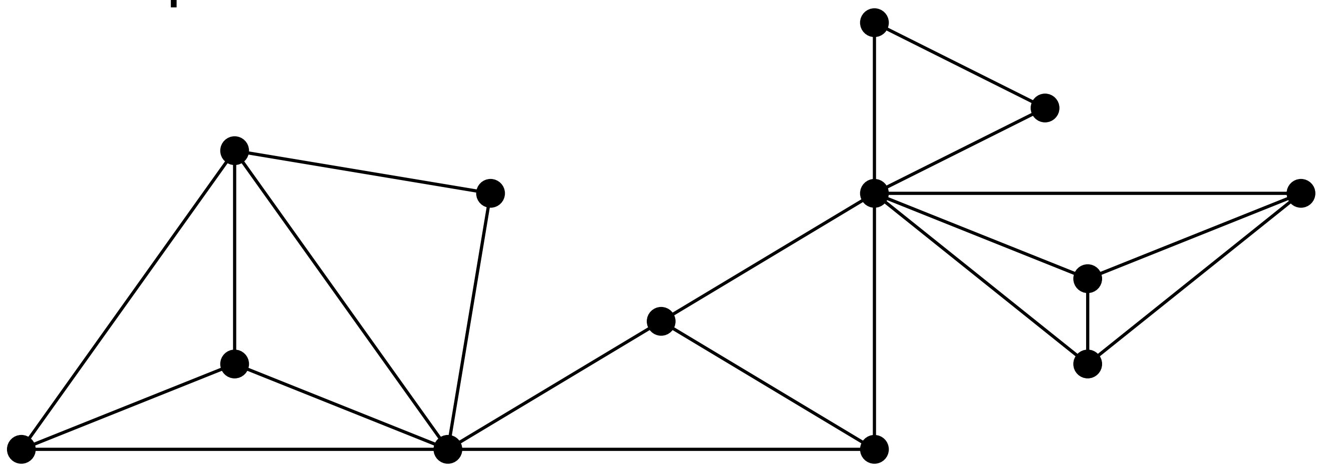

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Finally, arbitrary graphs are given by $G(x) = \exp(C(x))$.

Singularity analysis of 2-connected graphs

We have the system

$$\begin{cases} E(x) = \exp\left(xE(x)^2 + \frac{xE(x)^2 S(xE(x)^3)}{2}\right) \\ S(xE(x)^3) = xE(x)^3(1 + S(xE(x)^3))^3 \end{cases}$$

Omitting the arguments of S and E,

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This system is amenable to the Drmota-Lalley-Woods theorem.

Singularity analysis of 2-connected graphs

We obtain that $\rho_b \approx 0.092859$ is the unique dominant singularity of $E(x)$, and $E(x)$ admits an analytic continuation in a Δ-domain of the form $\Delta(R_b, \phi_b)$, for some $R_b > \rho_b$ and $0 < \phi_b < \pi/2$:

$$E(x) = E_0 + E_1 \sqrt{1 - \frac{x}{\rho_b}} + O\left(1 - \frac{x}{\rho_b}\right), \quad \text{for } x \sim \rho_b,$$

where $E_0 \approx 1.16454$ and $E_1 \approx 0.092354$.

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Also note that $\rho_b E_0^3 \approx 0.14665 < 4/27$, where $4/27$ is the dominant singularity of $S(z)$. This implies that the composition scheme $S(xE(x)^3)$ is **subcritical**.

Singularity analysis of 2-connected graphs

It follows that $B(x)$ also has ρ_b as its unique dominant singularity.

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We also show that $B(x)$ admits an analytic continuation in $\Delta(R_b, \phi_b)$:

$$B(x) = B_0 - B_2 \left(1 - \frac{x}{\rho_b}\right) + B_3 \left(1 - \frac{x}{\rho_b}\right)^{3/2} + O\left(1 - \frac{x}{\rho_b}\right)^2,$$

where $B_0 \approx 0.0044796, B_2 \approx 0.0085328$ and $B_3 \approx 0.00038321.$

Singularity analysis of connected graphs

For the connected graphs, the composition scheme

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is also **subcritical** because $B''(\rho_b) \to \infty$. Therefore, the singularities of $C^{\bullet}$ come from a branch point and not from the singularities of B.

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We now solve the equation obtained by differentiating the expression above:

$$\tau B''(\tau) = 1$$

And find the unique dominant singularity of $C^{\bullet}$

$$\rho = \tau e^{-B'(\tau)} \approx 0.084088.$$

Singularity analysis of connected graphs

As before, $C(x)$ admits an analytic continuation in a Δ-domain $\Delta(R, \phi)$, for some $R > \rho$ and $0 < \phi < \pi/2$:

$$C(x) = C_0 - C_2\left(1 - \frac{x}{\rho}\right) + C_3\left(1 - \frac{x}{\rho}\right)^{3/2} + O\left(1 - \frac{x}{\rho}\right)^2,$$

where $C_0 \approx 0.00037470, C_2 \approx 0.092859$ and $C_3 \approx 0.00027194$.

Singularity analysis of arbitrary graphs

Since $G(x) = \exp(C(x))$, the dominant singularity of $G(x)$ is also ρ and again $G(x)$ admits an analytic continuation in $\Delta(R, \phi)$:

$$G(x) = e^{C_0}\left(1 - C_2\left(1 - \frac{x}{\rho}\right) + C_3\left(1 - \frac{x}{\rho}\right)^{3/2}\right) + O\left(1 - \frac{x}{\rho}\right)^2.$$

Therefore, $G_0 = e^{C_0} \approx 1.00037, G_2 = C_2 e^{C_0} \approx 0.092894$ and $G_3 = C_3 e^{C_0} \approx 0.00027205$.

Main theorem

Theorem. Let g_n be the number of labelled chordal planar graphs with n vertices, c_n those which are connected, and b_n those which are 2-connected. Then as $n \to \infty$ we have

1. $g_n \sim g \cdot n^{-5/2}\gamma^n n!, \qquad \gamma \approx 11.89235, \;\; g \approx 0.00027205$
2. $c_n \sim c \cdot n^{-5/2}\gamma^n n!, \qquad c \approx 0.00027194,$
3. $b_n \sim b \cdot n^{-5/2}\gamma_b^n n!, \qquad \gamma_b \approx 10.76897, \;\; b \approx 0.00016215,$

Where $\gamma = 1/\rho$ and $\gamma_b = 1/\rho_b$.

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In fact, it is also straightforward to show [Giménez, Noy, Rué '13] that the number of connected components is asymptotically distributed as $1 + X$, where X follows a Poisson law with parameter $C_0 \approx 0.00037470$, so that $p = e^{-C_0}$.

3-connected and 2-connected maps

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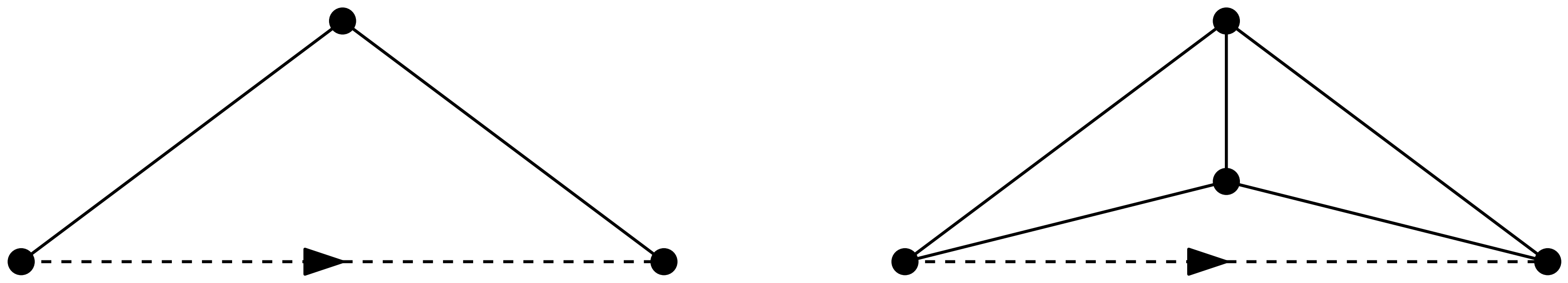

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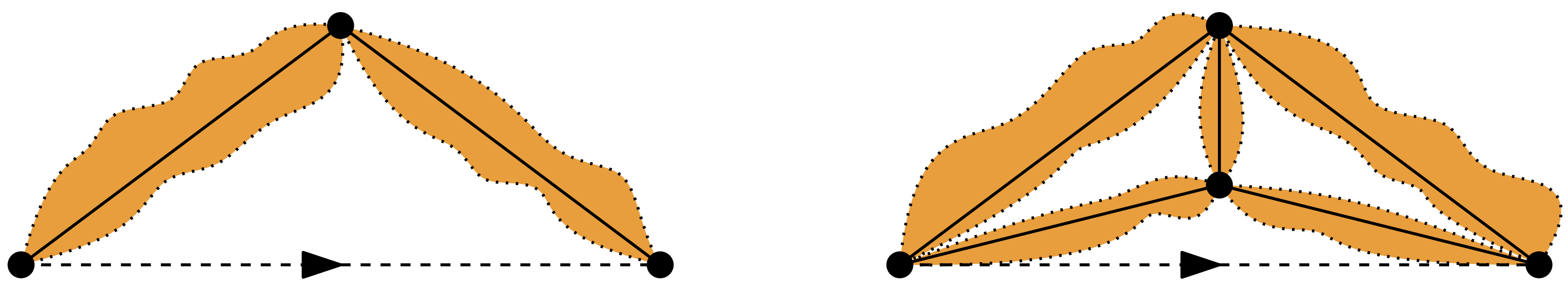

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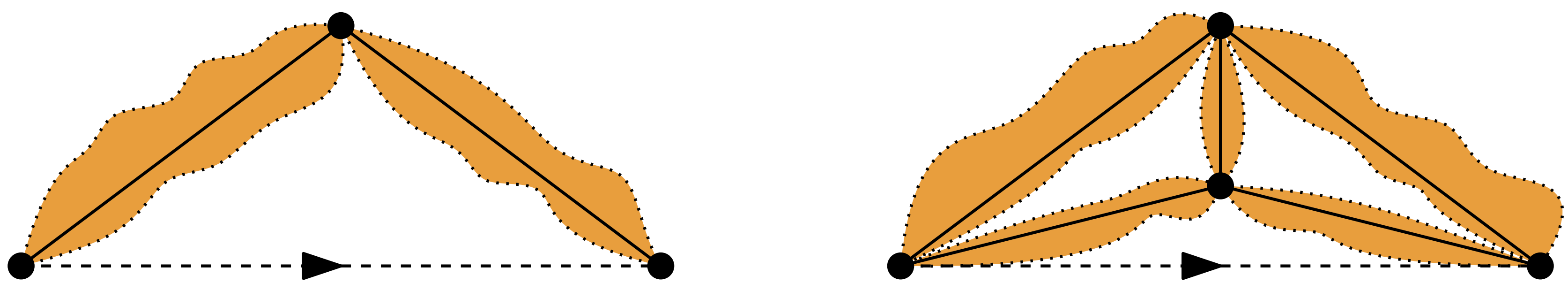

The 2-connected maps are decomposed into **sequences** of smaller maps, instead of sets. Maps grow from both sides of edges. We have

$$D(z) = \frac{1}{1 - x^2 D(x)^4 \left(1 + S\left(x^3 D(x)^6\right)\right)}$$

All maps

Let $M(x)$ be the generating function of all simple chordal maps, where x marks the total number of edges. The decomposition of a map into blocks is given by the equation

$$M(x) = B\left(x(1 + M(x))^2\right),$$

reflecting the fact that a map is obtained from its 2-connected core by attaching a (possibly empty) map at each corner.

Singularity analysis

By algebraic elimination, we can obtain irreducible polynomial equations satisfied by $B(x)$ and $M(x)$ and compute the singularities. As before, the composition scheme $M(x) = B\left(x(1+M(x))^2\right)$ is subcritical.

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Theorem. Let M_n be the number of rooted chordal simple planar maps with n edges, and B_n those which are 2-connected. Then as $n \to \infty$ we have

1. $B_n \sim b \cdot n^{-3/2} \cdot \sigma_b^{-n}$, with $b \approx 0.071674$ and $\sigma_b^{-1} \approx 3.65370$,
2. $M_n \sim m \cdot n^{-3/2} \cdot \sigma^{-n}$, with $m \approx 0.12596$ and $\sigma^{-1} \approx 6.40375$.

Future work

- Enumerate related families of chordal graphs, such as outerplanar, series-parallel graphs and planar multigraphs. Also non-planar graphs, such as $K_{3,3}$ or K_5-minor-free graphs and graphs with bounded tree-width.

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• Enumerate **non-labelled** chordal planar graphs, using Pólya's theory of counting.