Enumeration of unlabeled graph classes

Jordi Castellví (CRM)

Work in collaboration with Michael Drmota and Clément Requilé

BMS-BGSMath Junior Meeting 2026 - Cerdanyola del Vallès

Labeled vs unlabeled

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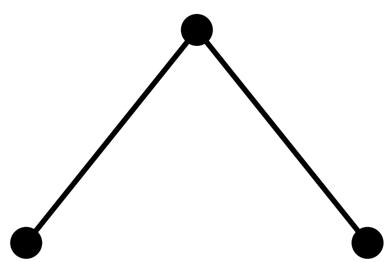

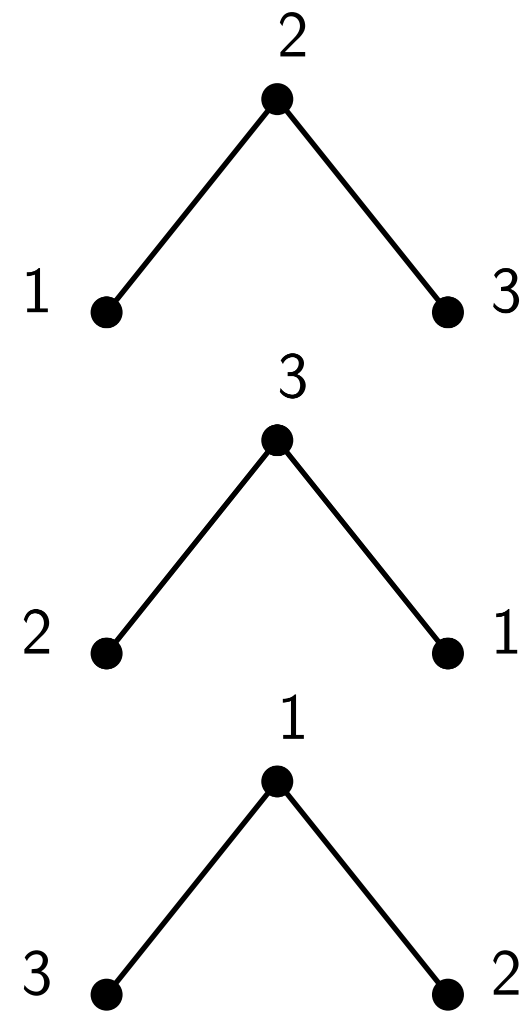

The symbolic method

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Definition. A **combinatorial class** is a pair $(\mathcal{A}, |\cdot|)$ where

- $\mathcal{A}$ is a family of combinatorial objects,
- $|\cdot| : \mathcal{A} \to \mathbb{N}$ is a size function,
- The number of objects with size n is $a_n < \infty$.

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$$A(x) = \sum_{n \geq 0} a_n x^n.$$

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$$A(x) = \sum_{n \geq 0} \frac{a_n}{n!} x^n.$$

Suitable for labeled classes.

The symbolic method

Operations between **classes** translate into relations involving their **generating functions**.

Combinatorial class	**Generating function**
$\mathcal{A} \cup \mathcal{B}$	$A(x) + B(x)$
$\mathcal{A} \times \mathcal{B}$	$A(x) \cdot B(x)$
$\mathcal{A} \circ \mathcal{B}$	$A(B(x))$
$\mathsf{Seq}(\mathcal{A})$	$\frac{1}{1-A(x)}$
$\mathsf{Set}(\mathcal{A})$	$\exp(A(x))$

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The growth of the coefficients of generating functions can be studied by considering them as complex variable functions and analysing their singularities. This is called **Analytic Combinatorics** [Flajolet & Sedgewick (2009)]

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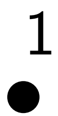

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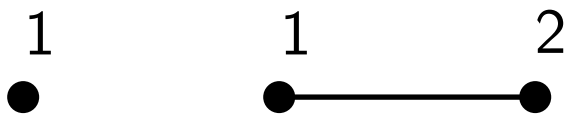

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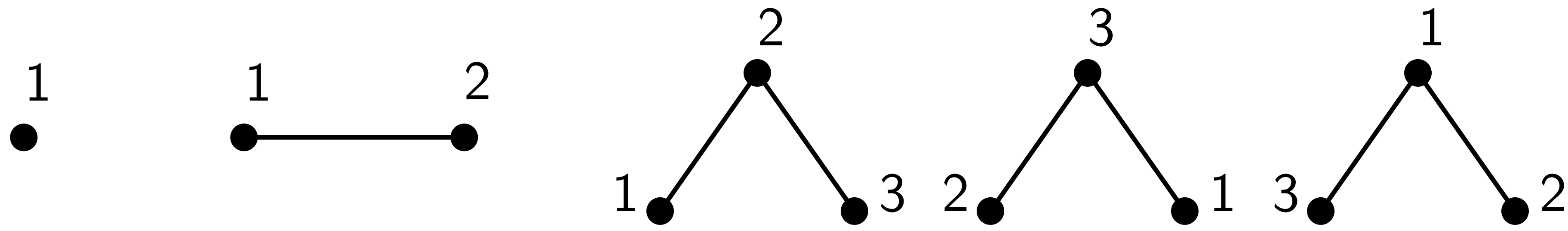

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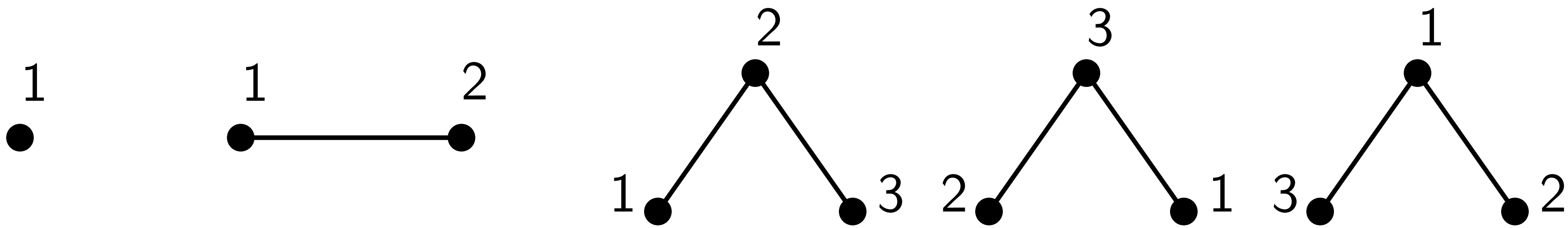

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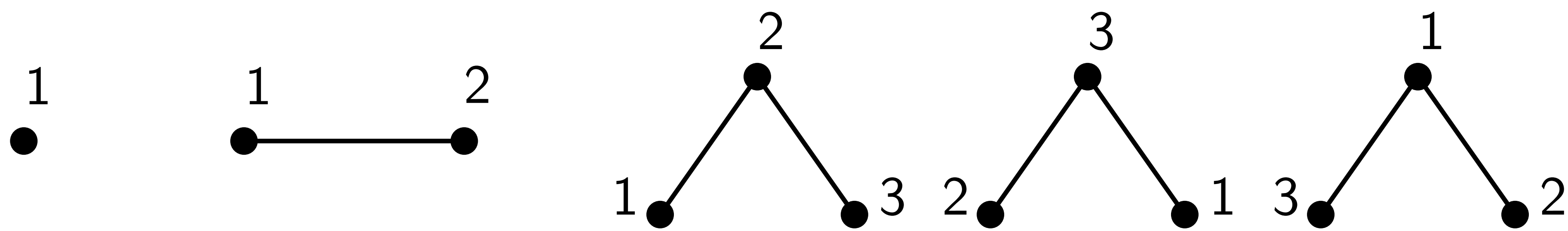

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Rooting. Let $\mathcal{T}^\bullet$ be the class of rooted labeled trees.
Since all vertices are distinguishable, there are n ways to root a tree with n vertices. Thus,

$$T^\bullet(x) = \sum_{n\geq 0} n\frac{t_n}{n!}\,x^n = xT'(x).$$

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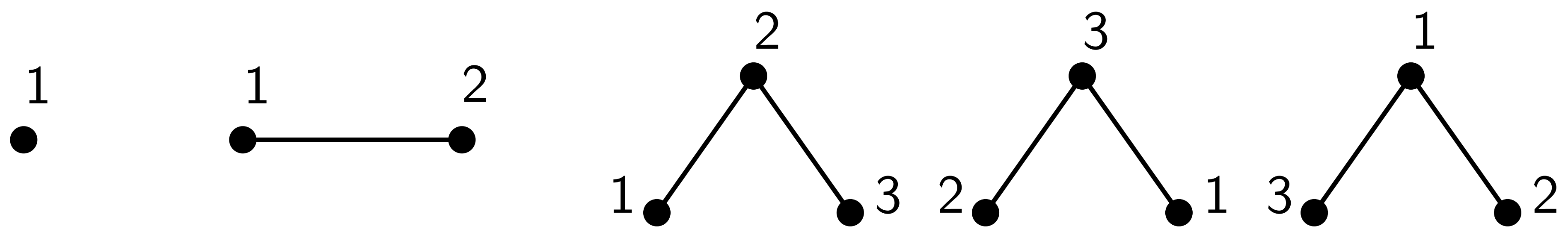

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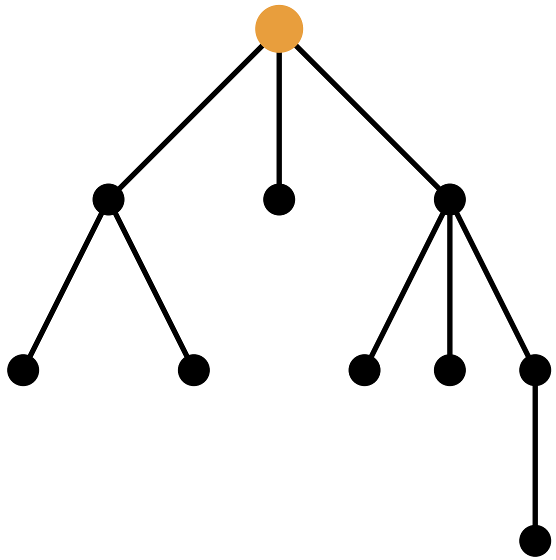
Unrooting. To do the inverse operation, we can simply integrate:

$$T(x) = \int T^\bullet(x)/x \; dx.$$

Labeled trees



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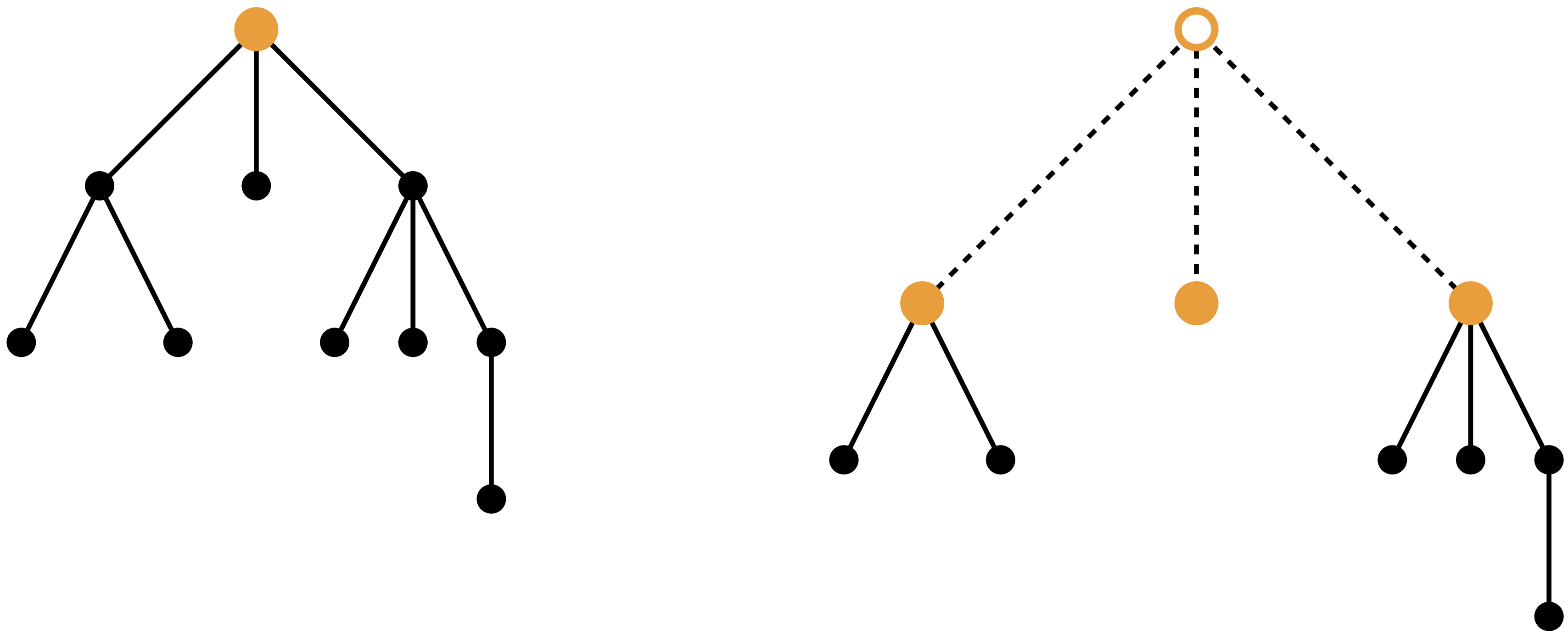

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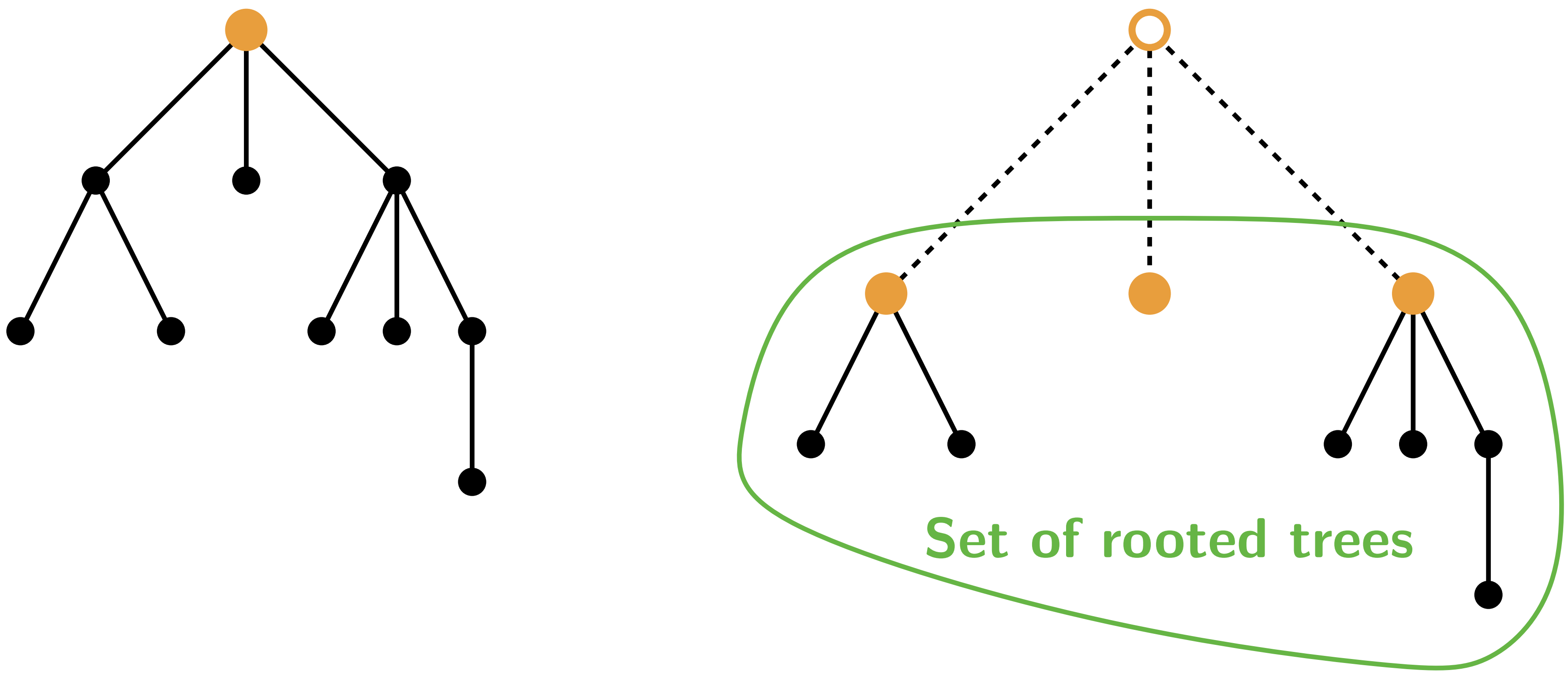

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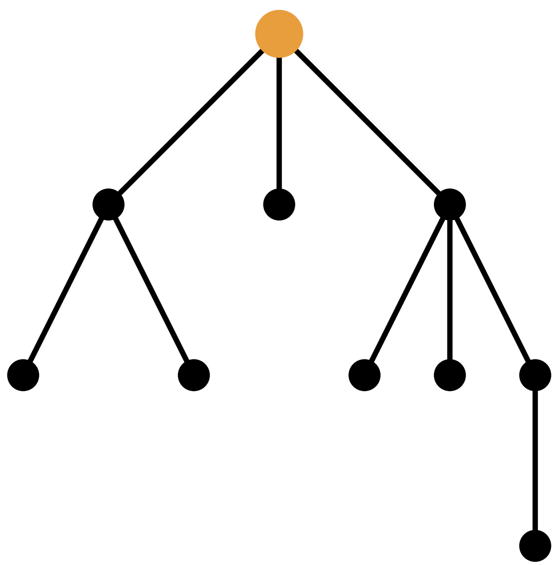

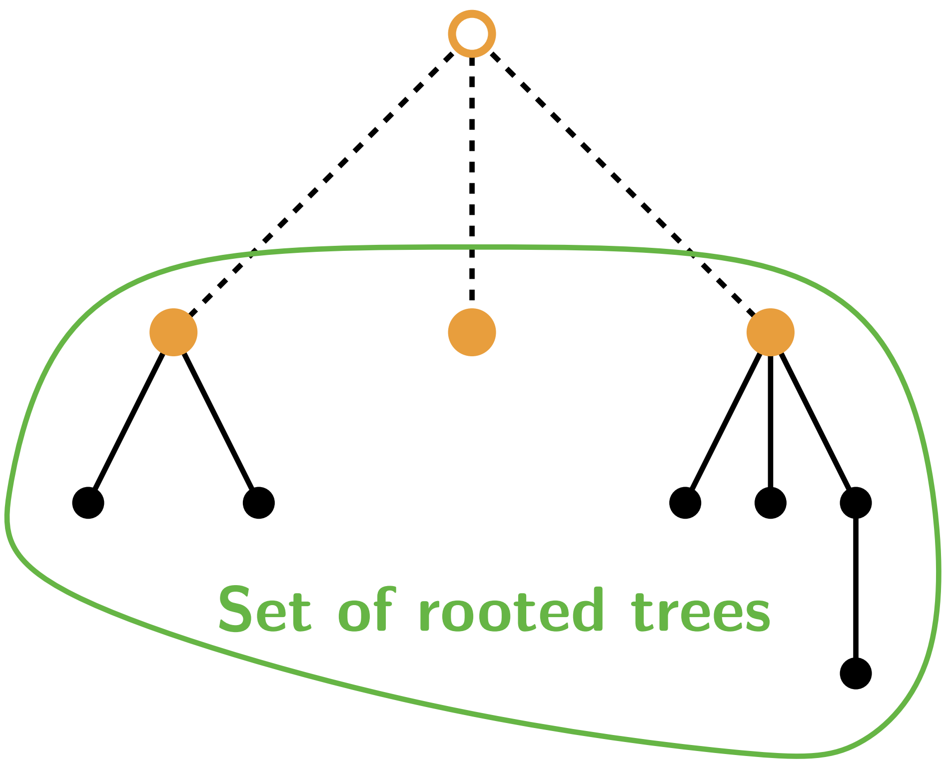

Implicit equation: $\mathcal{T}^\bullet = \{\text{root}\} \times \mathsf{Set}(\mathcal{T}^\bullet)$, which gives

$$T^\bullet(x) = x \exp\left(T^\bullet(x)\right) = x + x^2 + \frac{3x^3}{2} + \cdots .$$

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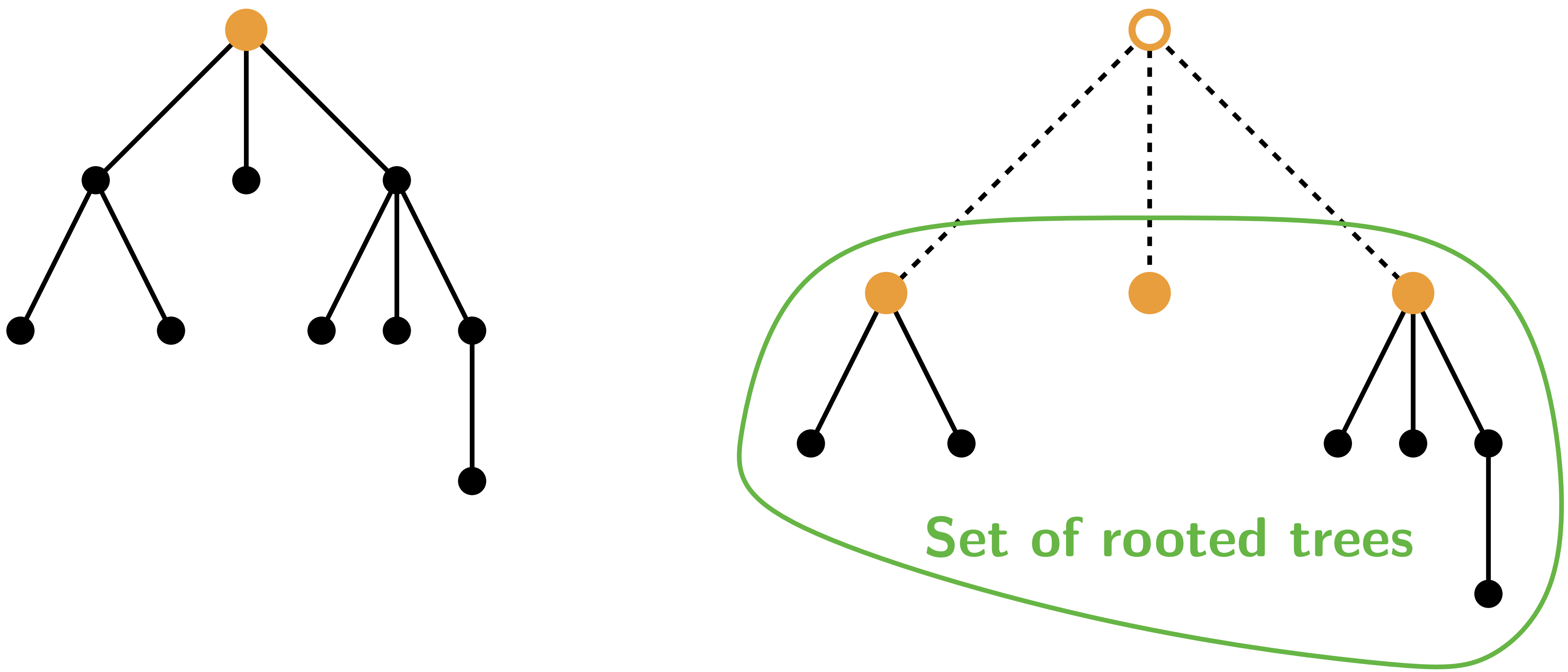

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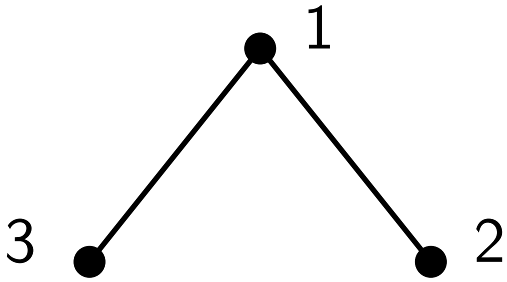
By using the **Lagrange inversion formula** we obtain:

$$|\mathcal{T}_n^{\bullet}| = n![x^n]T^{\bullet}(x) = n^{n-1} \implies |\mathcal{T}_n| = |\mathcal{T}_n^{\bullet}|/n = n^{n-2}.$$

Pólya theory



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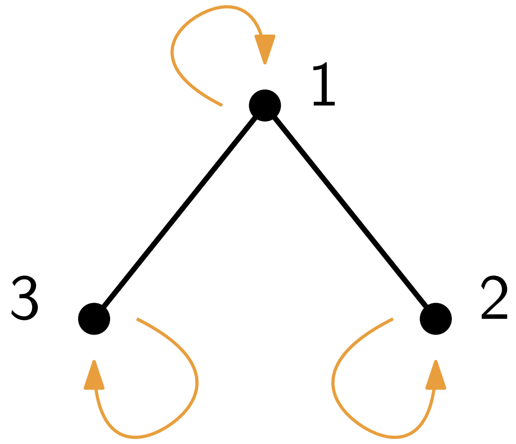

$(1)(2)(3) \longrightarrow s_1^3$

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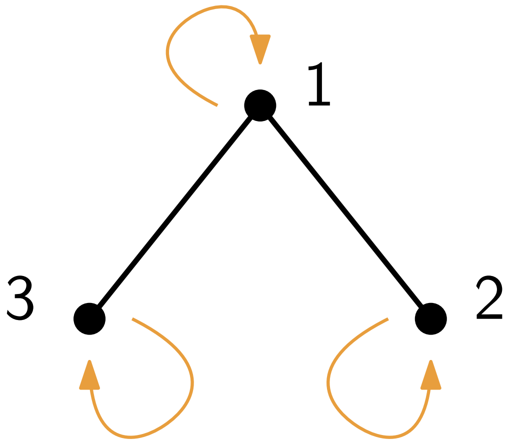

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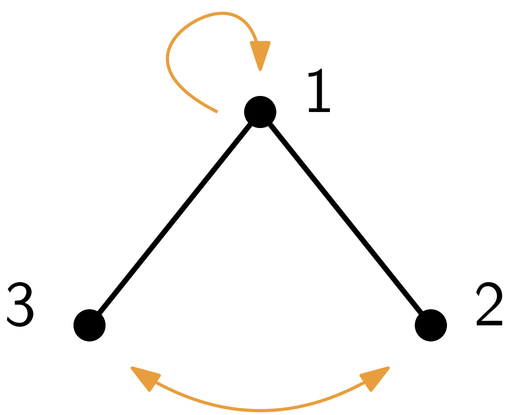

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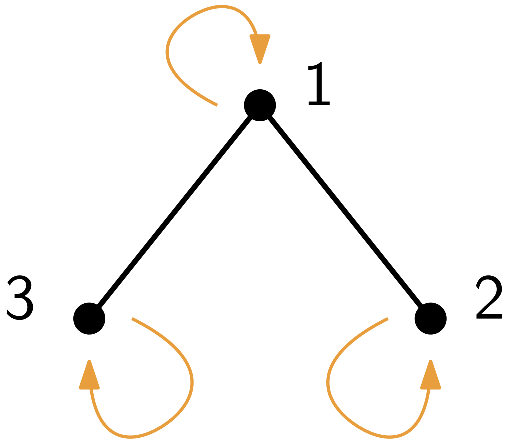

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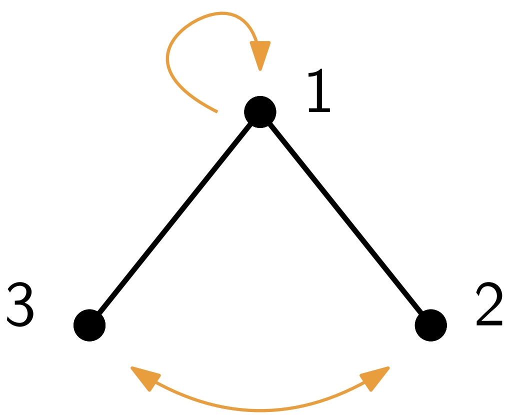

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$$\frac{1}{3!}\left(s_1^3 + s_1 s_2\right)$$

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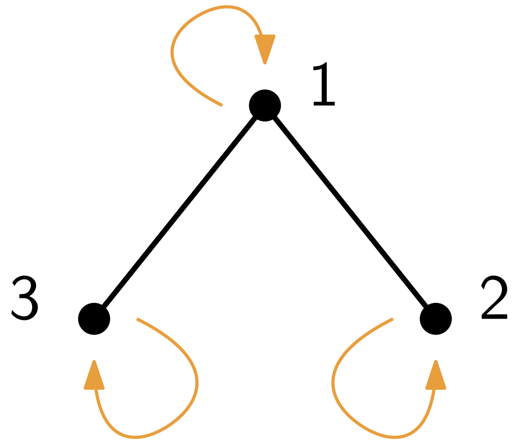

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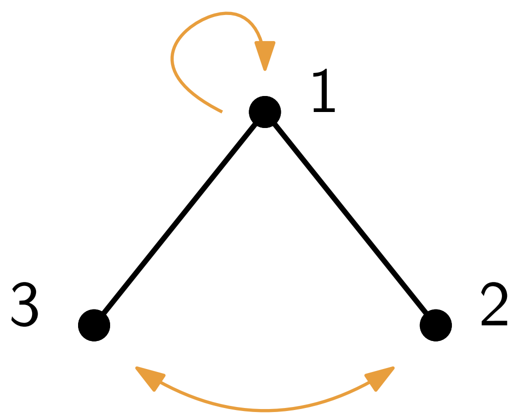

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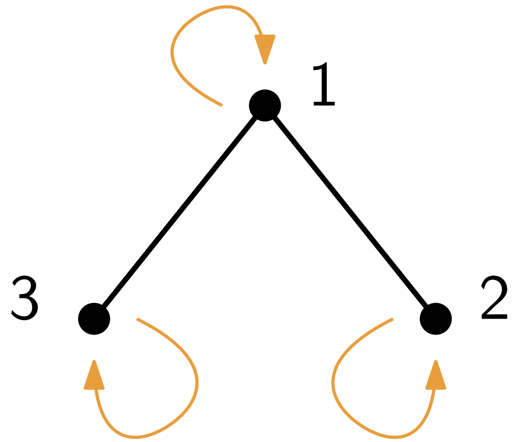

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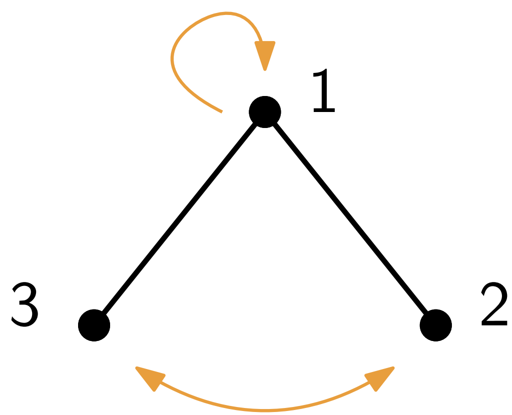

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Cycle index sum

$Z_{\mathcal{G}}(s_1, s_2, s_3, \ldots)$

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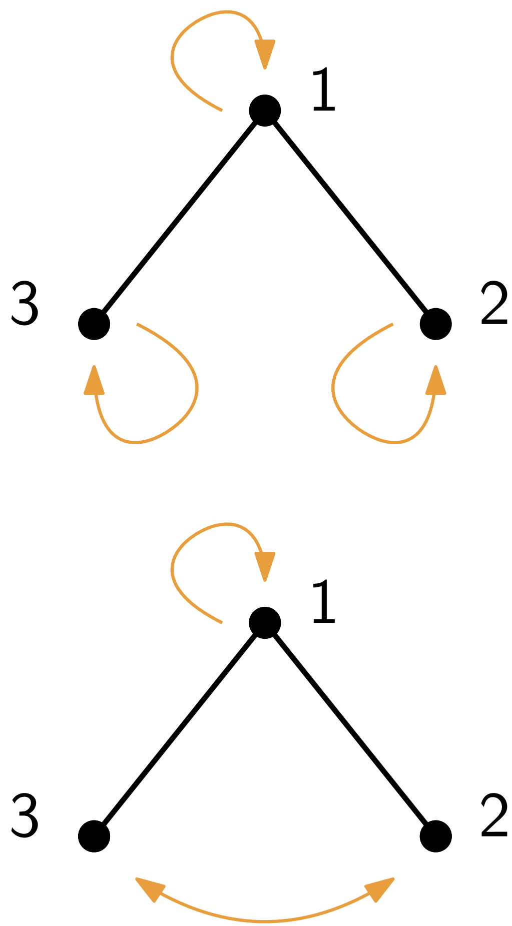

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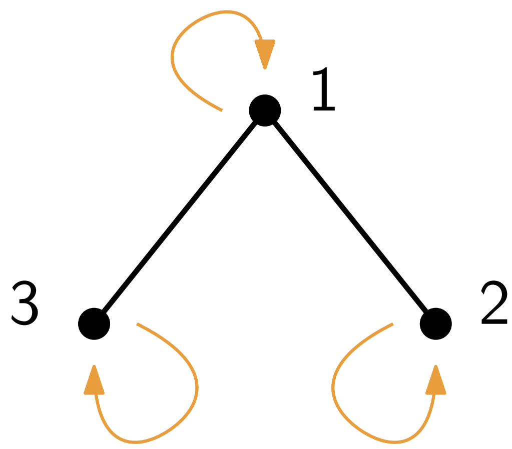
Theorem [Pólya (1937)]

The OGF of the unlabeled class $\tilde{\mathcal{G}}$ is given by

$$\tilde{G}(x) = Z_{\mathcal{G}}(x, x^2, x^3, \dots).$$

Pólya theory



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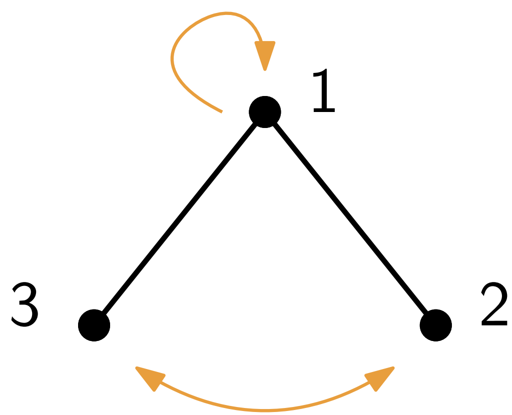

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Theorem [Pólya (1937)]
The OGF of the unlabeled class $\tilde{\mathcal{G}}$ is given by

$$\tilde{G}(x) = Z_{\mathcal{G}}(x, x^2, x^3, \dots).$$

In this example,

$$G(x) = \frac{3}{3!}(x^3 + x \cdot x^2) = x^3$$

Pólya theory

Operations between **classes** translate into relations involving their **cycle index sums**.

Class	Cycle index sum
$\mathcal{A} \cup \mathcal{B}$	$Z_{\mathcal{A}}(s_1, s_2, s_3, \dots) + Z_{\mathcal{B}}(s_1, s_2, s_3, \dots)$
$\mathcal{A} \times \mathcal{B}$	$Z_{\mathcal{A}}(s_1, s_2, s_3, \dots) \cdot Z_{\mathcal{B}}(s_1, s_2, s_3, \dots)$
$\mathcal{A} \circ \mathcal{B}$	$Z_{\mathcal{A}}(Z_{\mathcal{B}}(s_1, s_2, s_3, \dots), Z_{\mathcal{B}}(s_2, s_4, s_6, \dots), \dots)$
Seq$(\mathcal{A})$	$\frac{1}{1-Z_{\mathcal{A}}(s_1,s_2,s_3,\dots)}$
Set$(\mathcal{A})$	$\exp\left(\sum_{k\geq 1} \frac{1}{k} Z_{\mathcal{A}}(s_k, s_{2k}, s_{3k}, \dots)\right)$

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Theorem. [Pólya (1937)]
The OGF $P(x)$ of Pólya trees is given by

$$P(x) = x \exp\left(P(x) + \frac{P(x^2)}{2} + \frac{P(x^3)}{3} + \dots\right).$$

As $n \to \infty$ we have

$$[x^n]P(x) \sim \frac{b\sqrt{\rho}}{2\sqrt{\pi}} \cdot n^{-3/2} \cdot \rho^{-n},$$

with $b \approx 2.681127$ and $\rho \approx 0.338219$.

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What about unrooted unlabeled trees?

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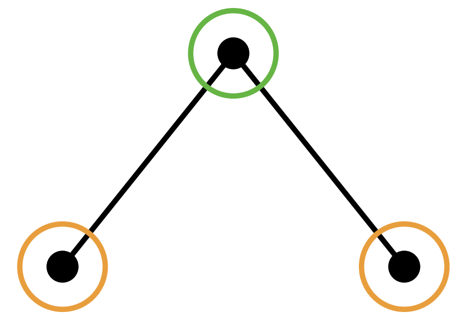
Not every unlabelled graph of size n gives rise to n rooted graphs.



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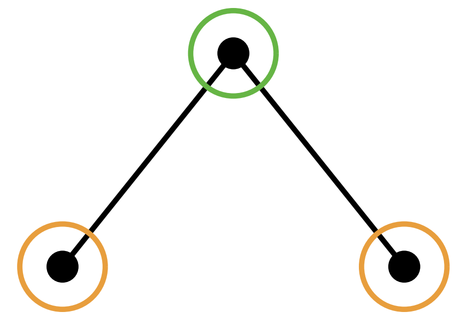

Theorem. [Otter (1948)]
The OGF $U(x)$ of unlabeled trees is given by

$$U(x) = P(x) + \frac{1}{2}(P(x^2) - P(x)^2).$$

As $n \to \infty$ we have

$$[x^n]P(x) \sim \frac{b^3 \rho^{3/2}}{4\sqrt{\pi}} \cdot n^{-3/2} \cdot \rho^{-n},$$

with $b \approx 2.681127$ and $\rho \approx 0.338219$.

Proof. Using the **dissymmetry theorem**.

The dissymmetry theorem

Theorem. Let $\mathcal{T}$ be a class of trees. Then,

$$\mathcal{T} + \mathcal{T}^{\bullet \rightarrow \bullet} \simeq \mathcal{T}^{\bullet} + \mathcal{T}^{\bullet - \bullet}$$

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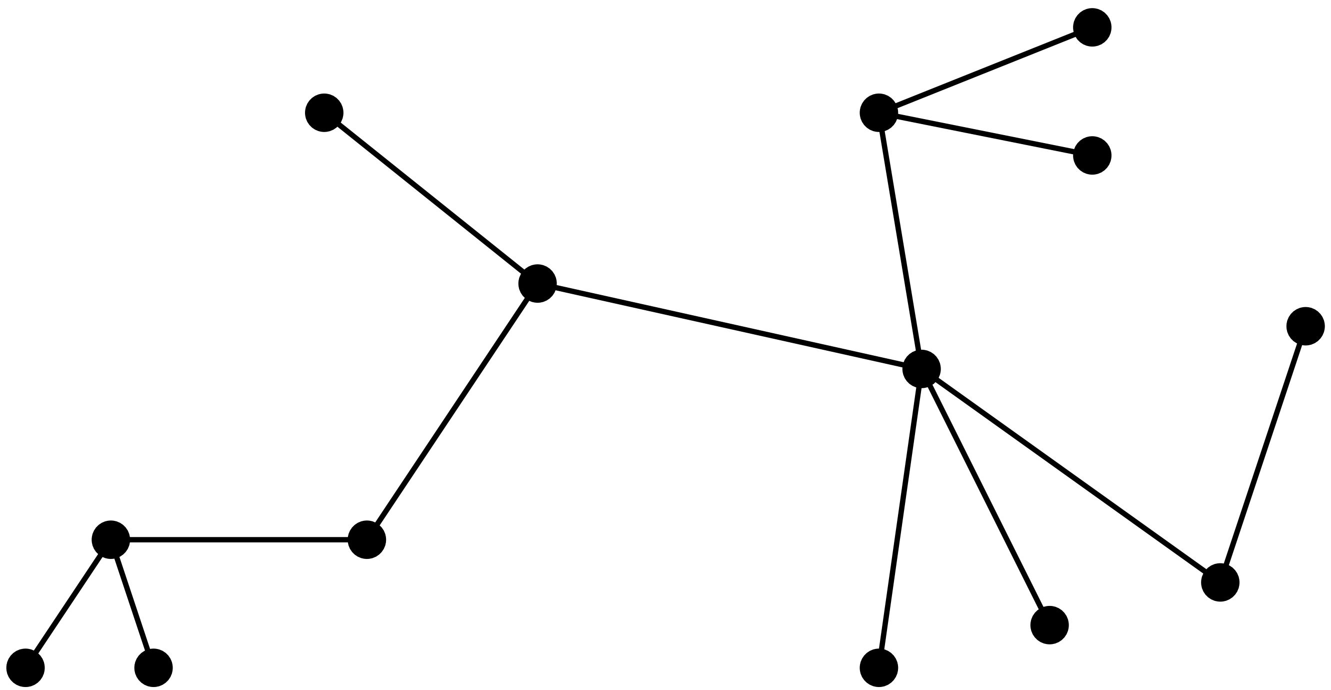

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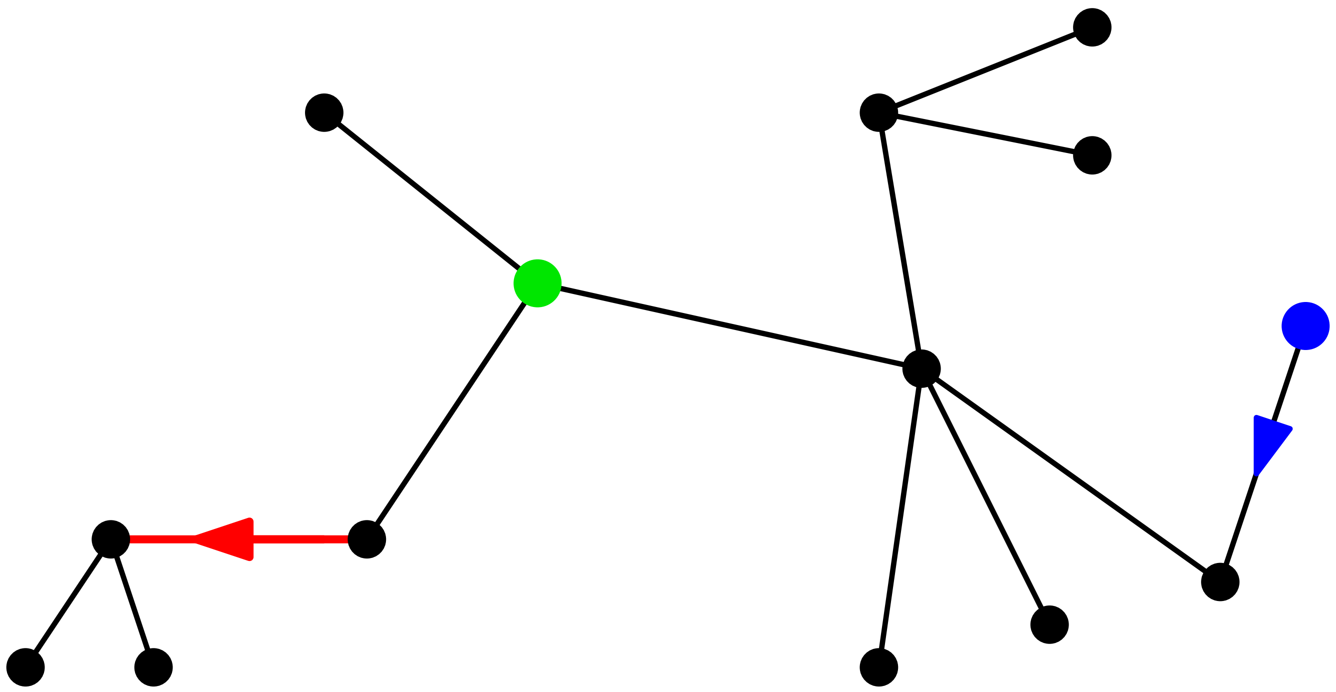
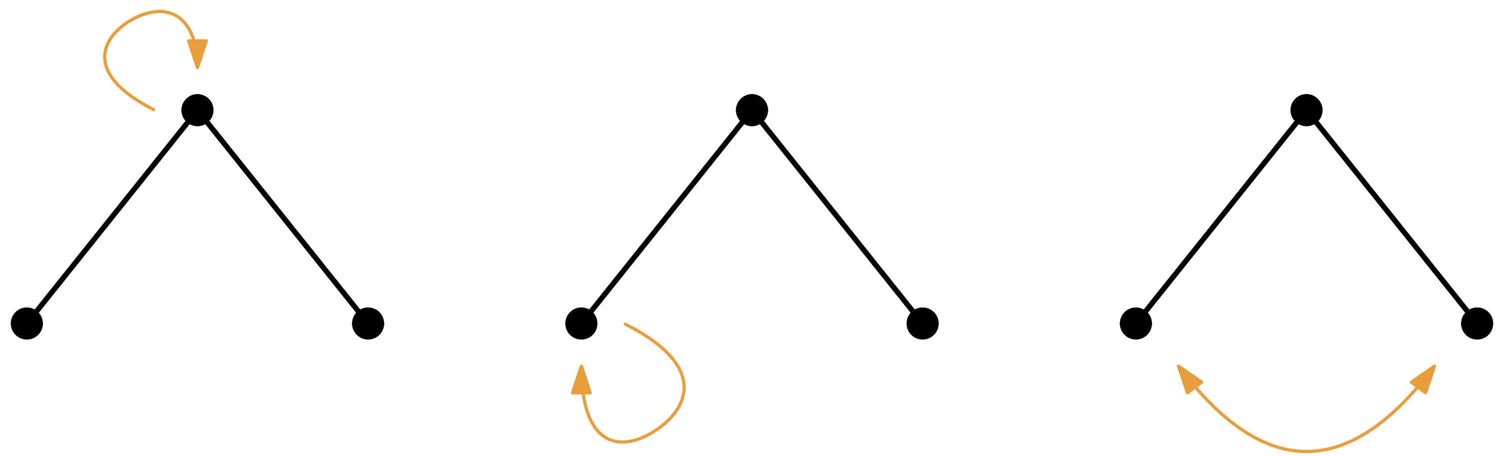


Cycle-pointing



Definition. A **cycle-pointed graph** is a pair (G, c) where $G \in \mathcal{G}$ is an unlabelled graph and c is a cycle of some automorphism of G.

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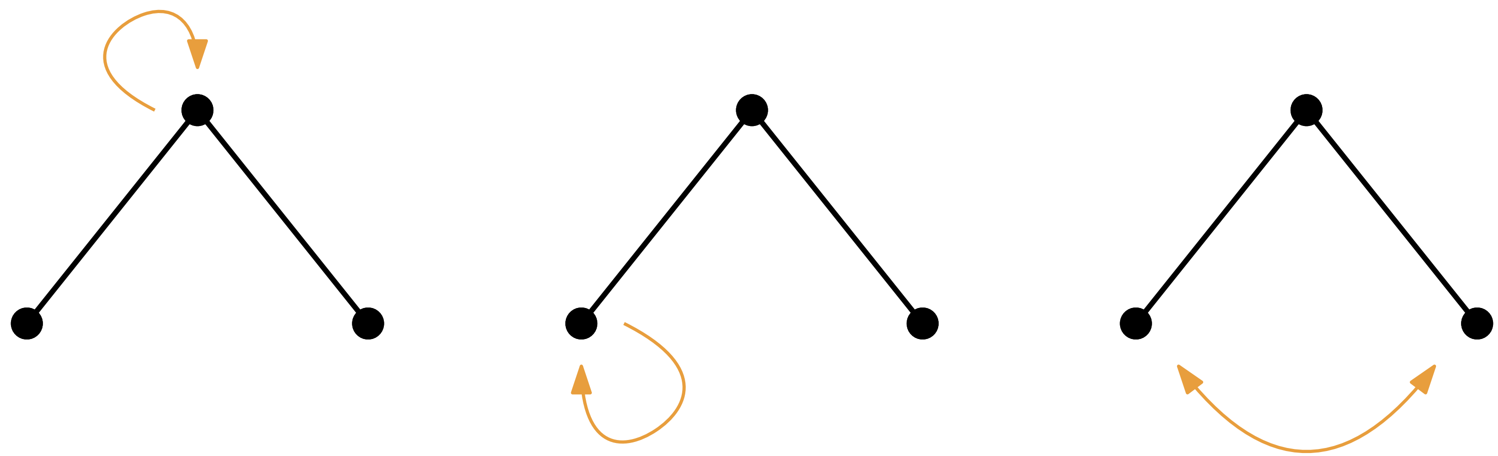

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Every unlabelled graph $G \in \mathcal{G}$ of size n admits exactly n cycle-pointings.

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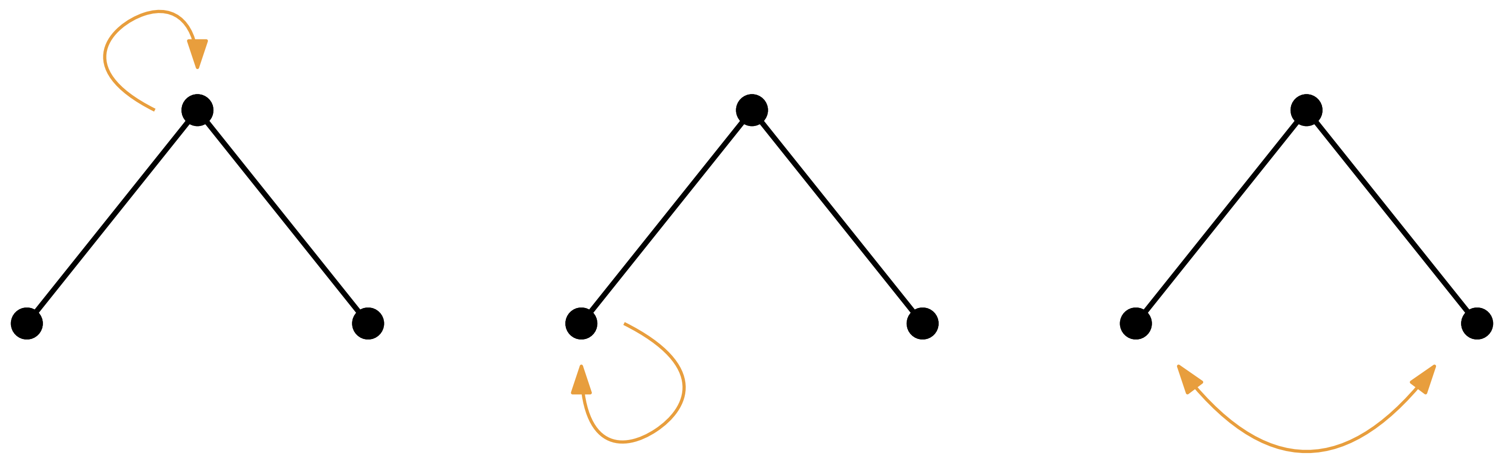

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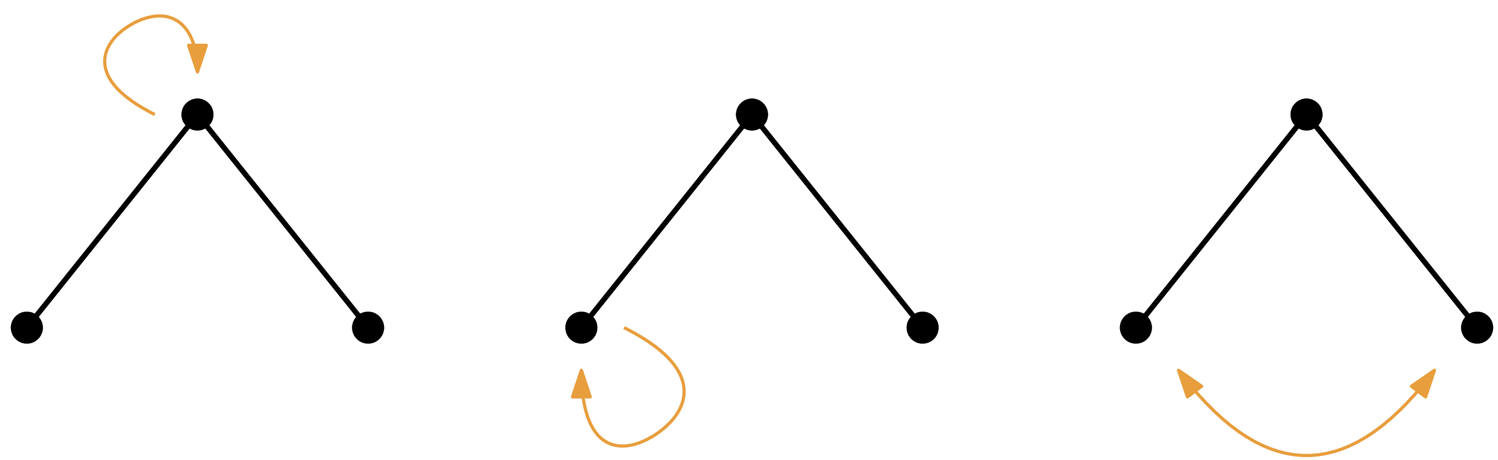

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An unbiased rooting (pointing) operator!

They extend Pólya theory to cycle-pointed graphs. In particular, they manage to unroot Pólya trees via cycle-pointing and they recover Otter's formula.

Beyond trees

What is a tree?

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Iteratively add a new vertex connected to an existing vertex.

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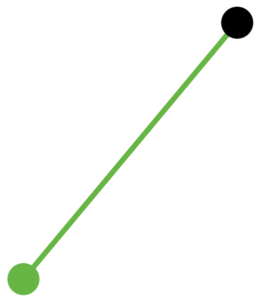

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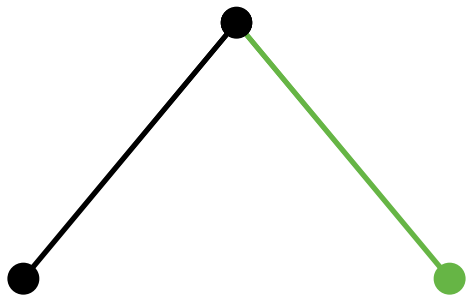

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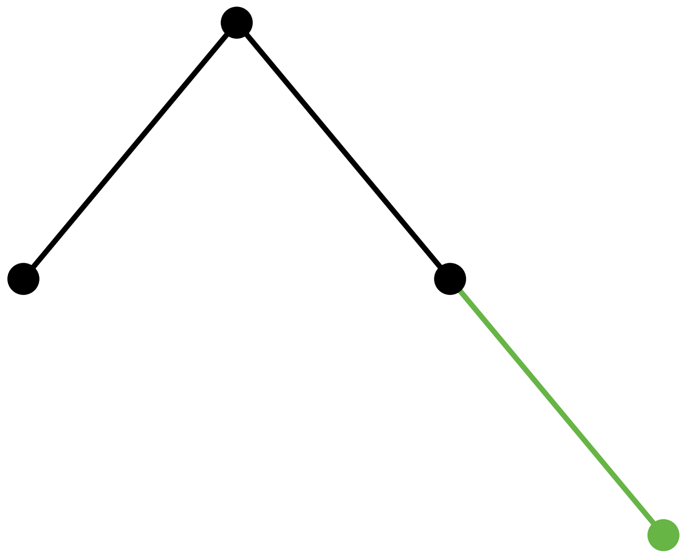

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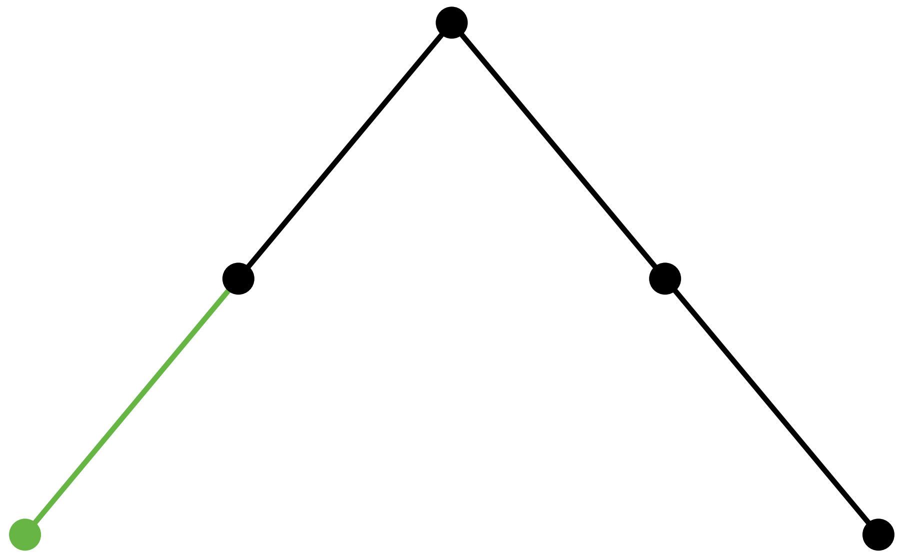

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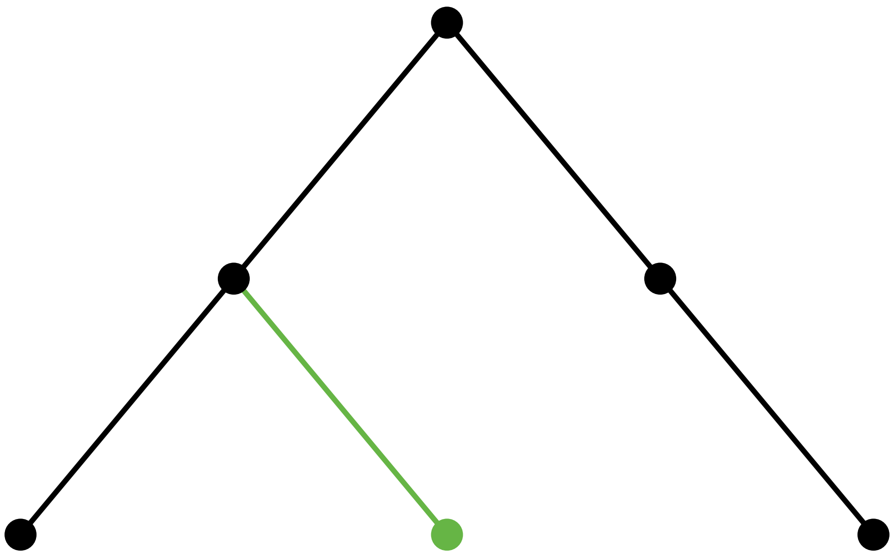

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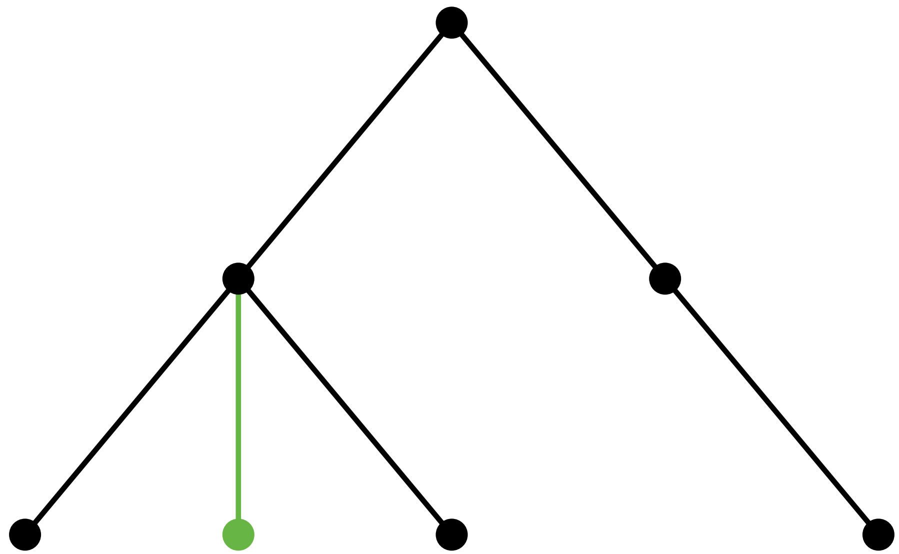

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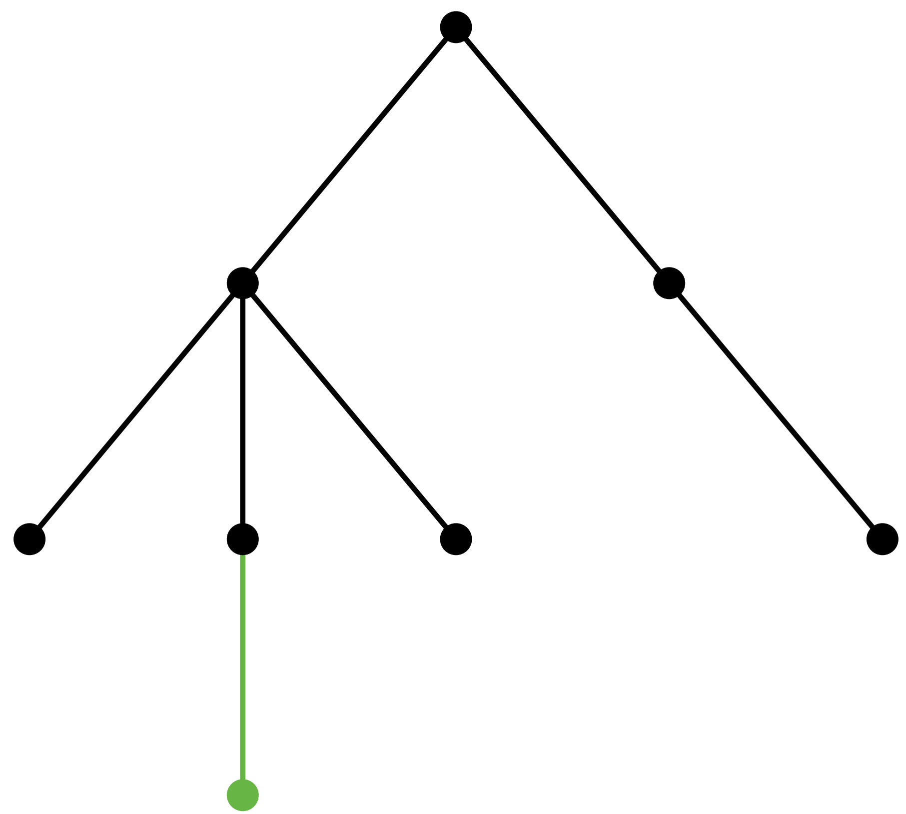

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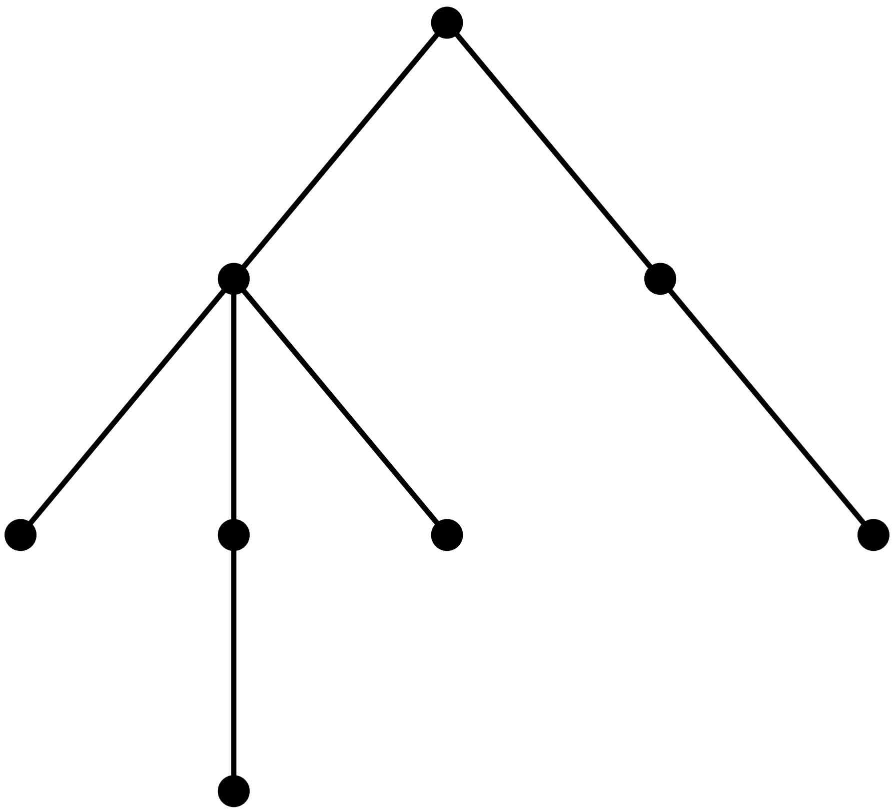

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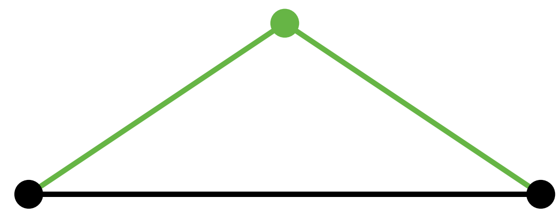

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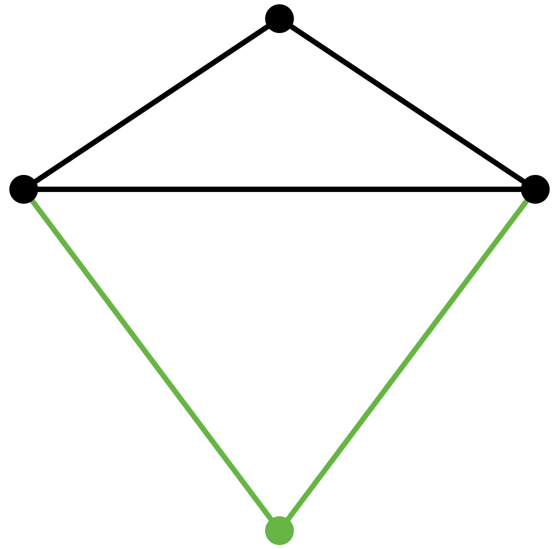

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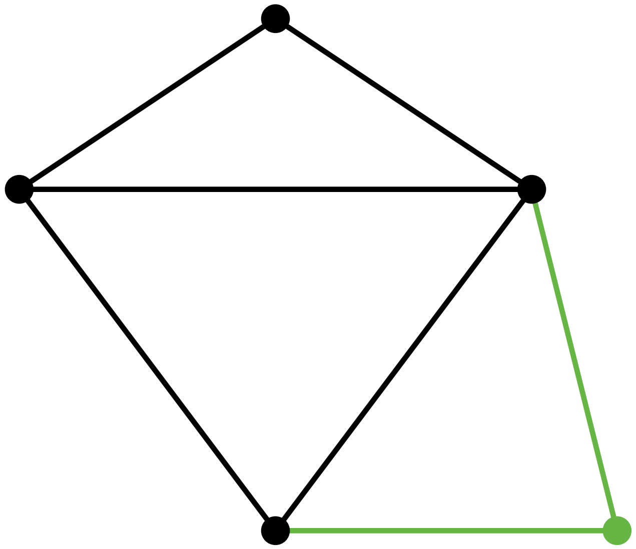

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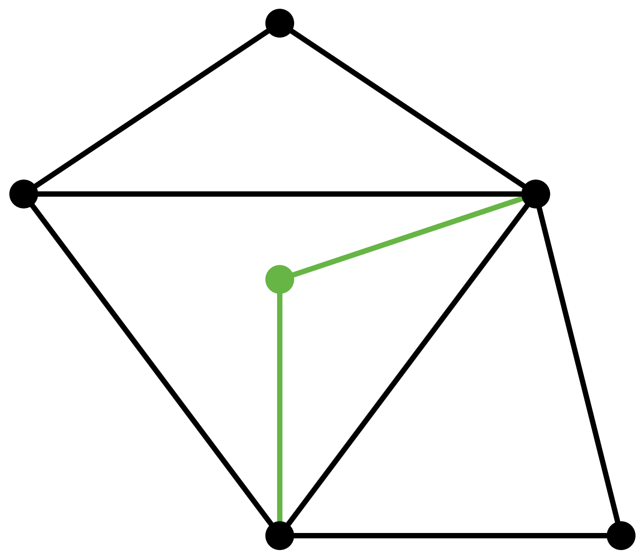

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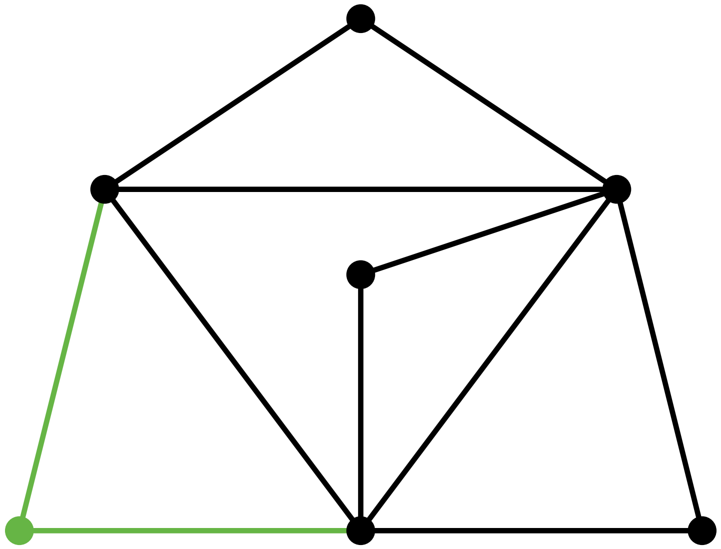

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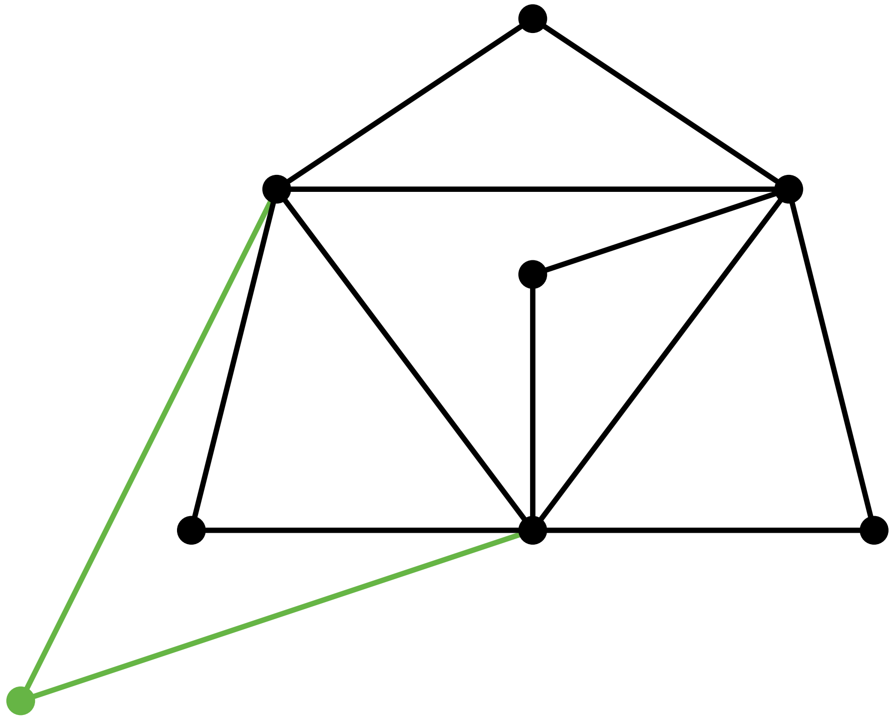

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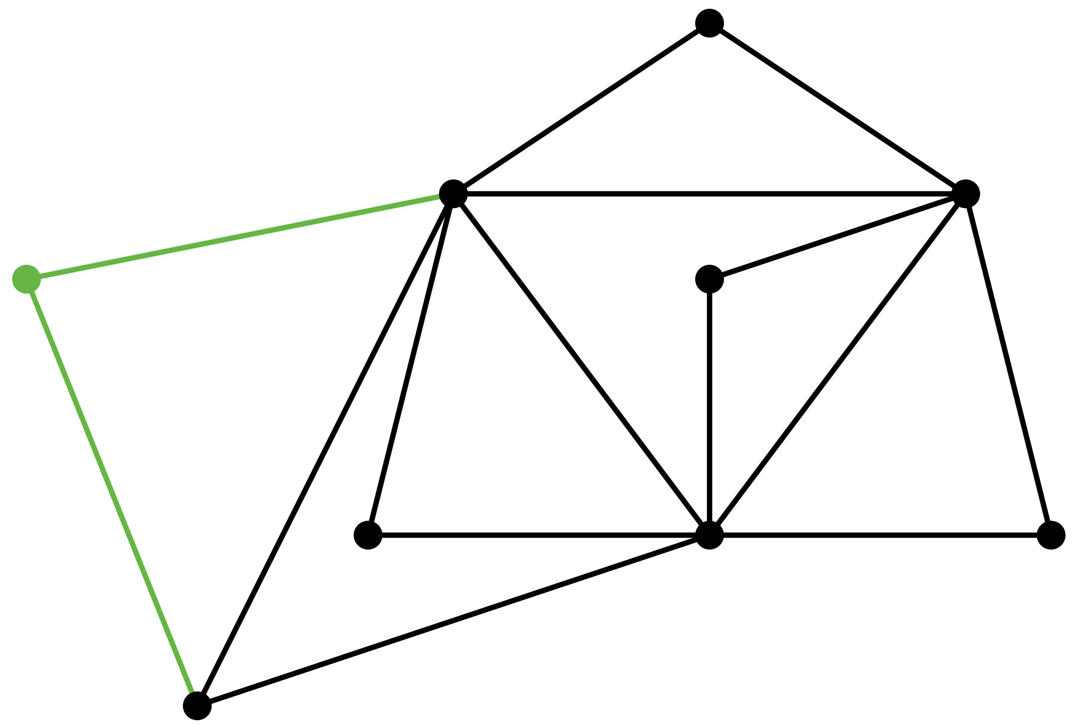

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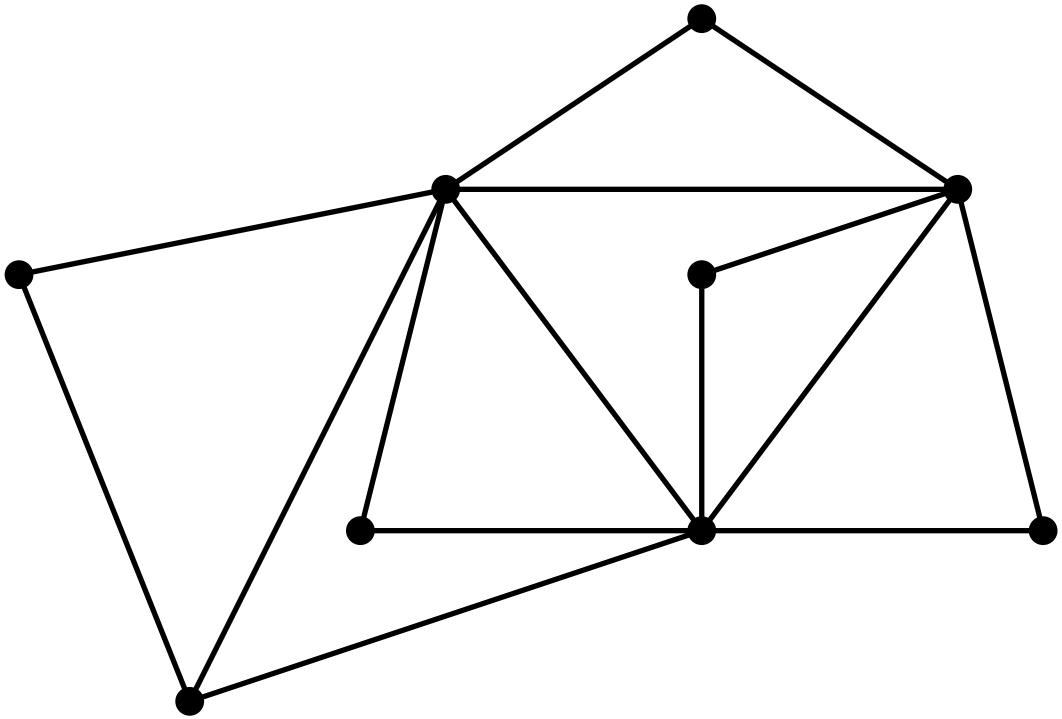

2-trees

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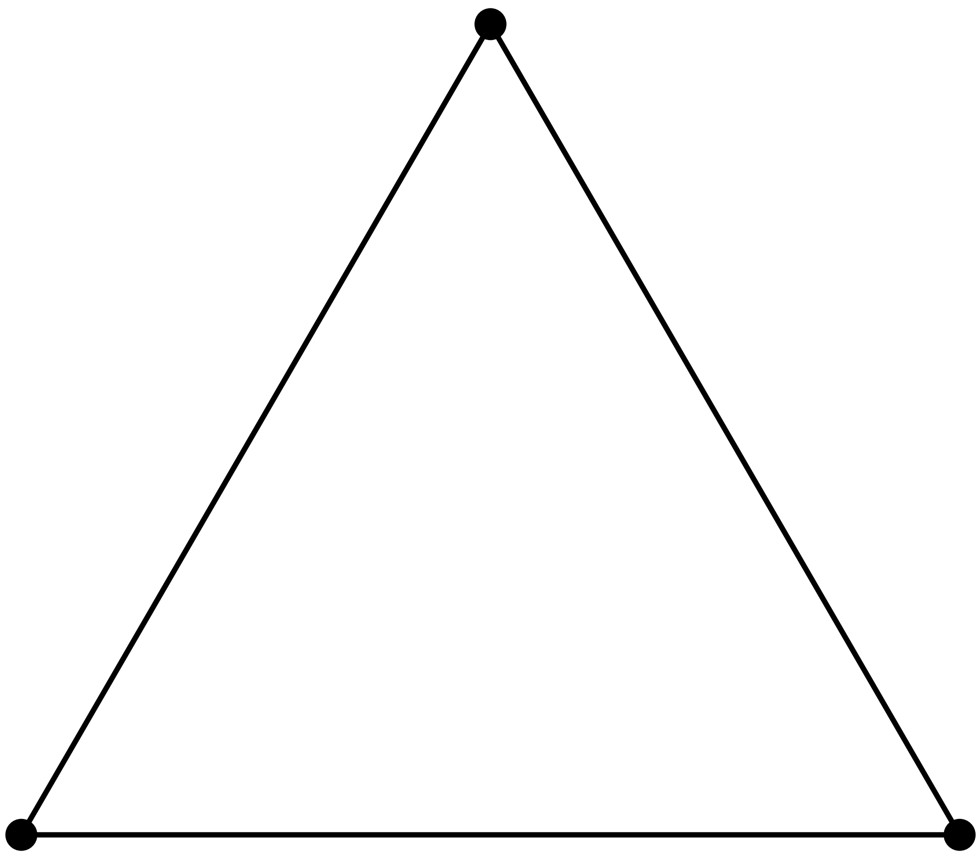

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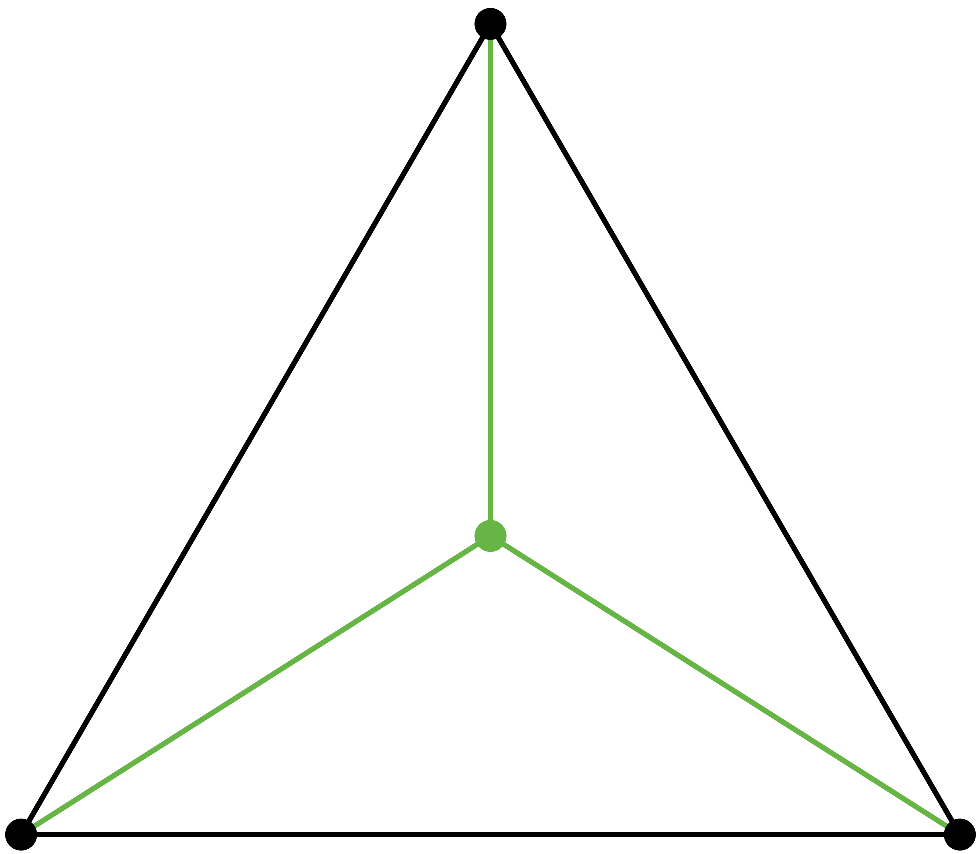

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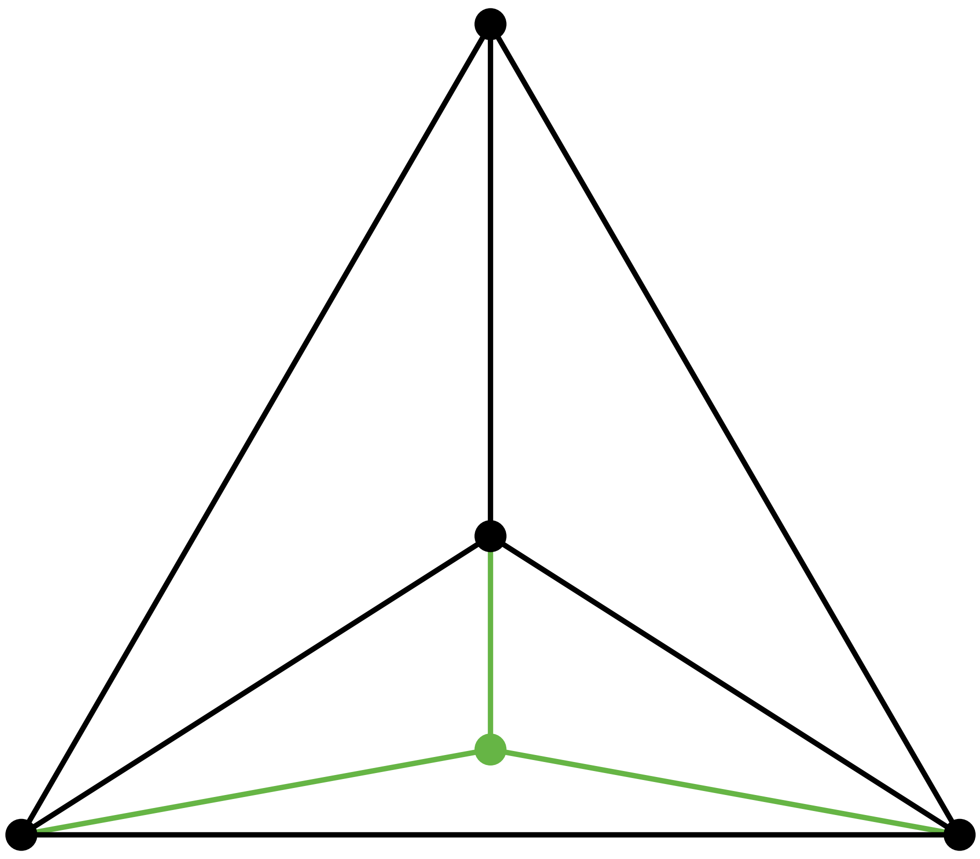

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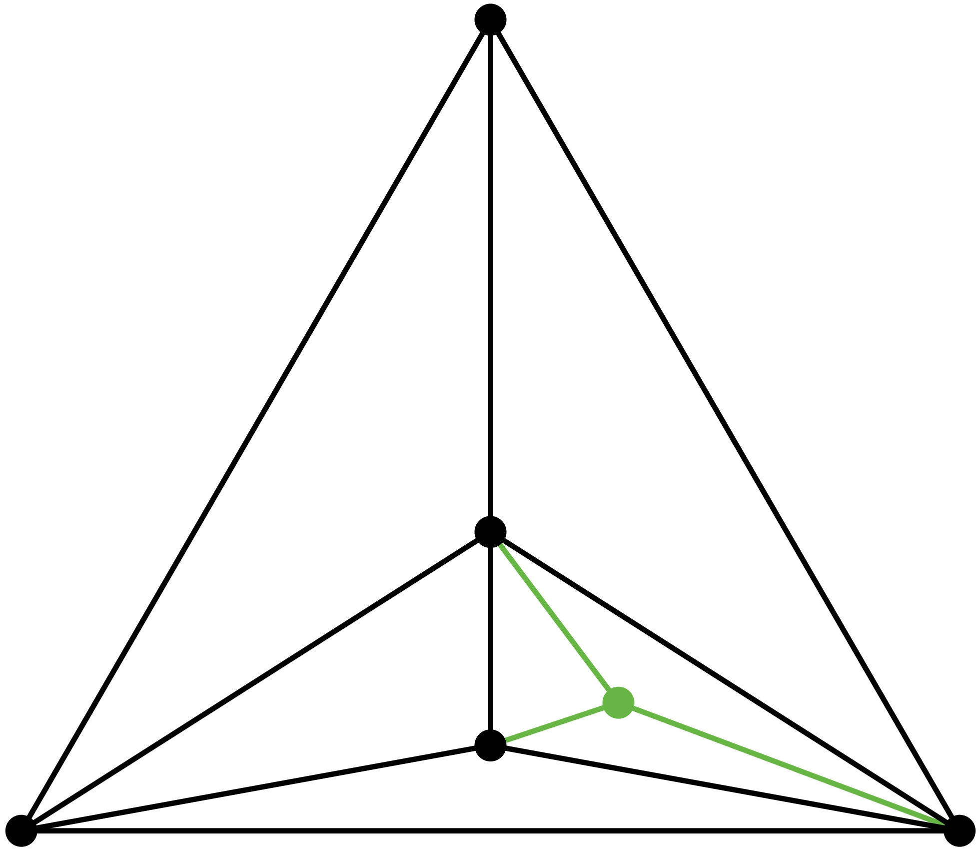

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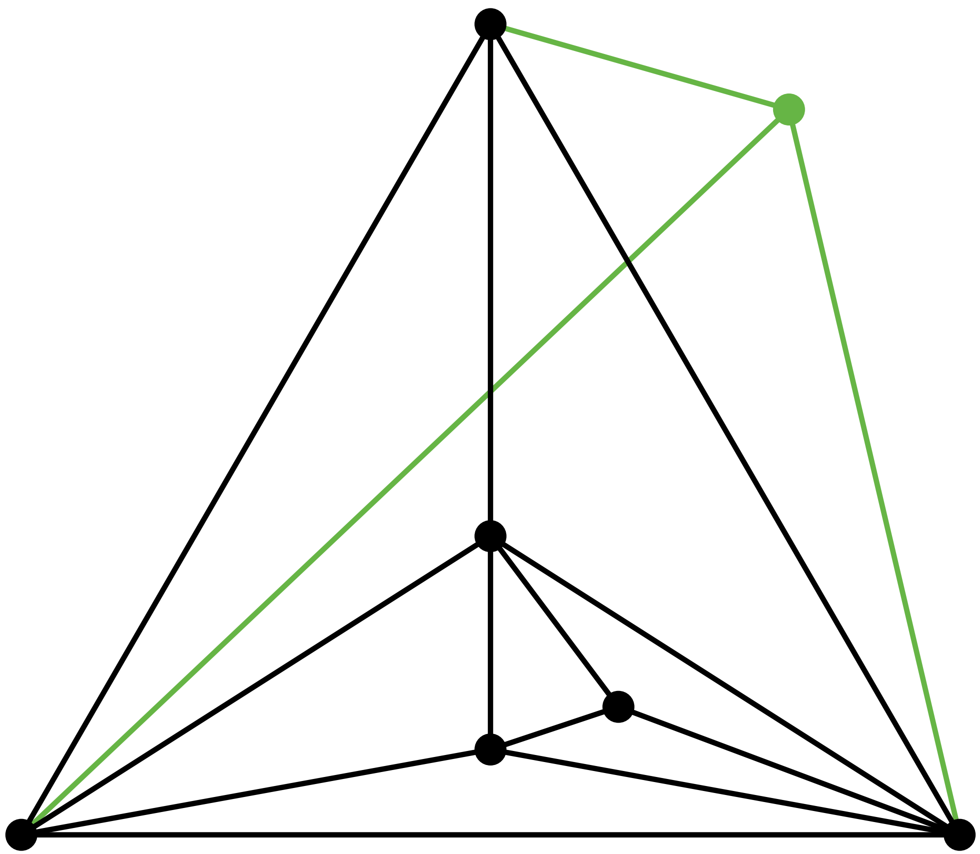

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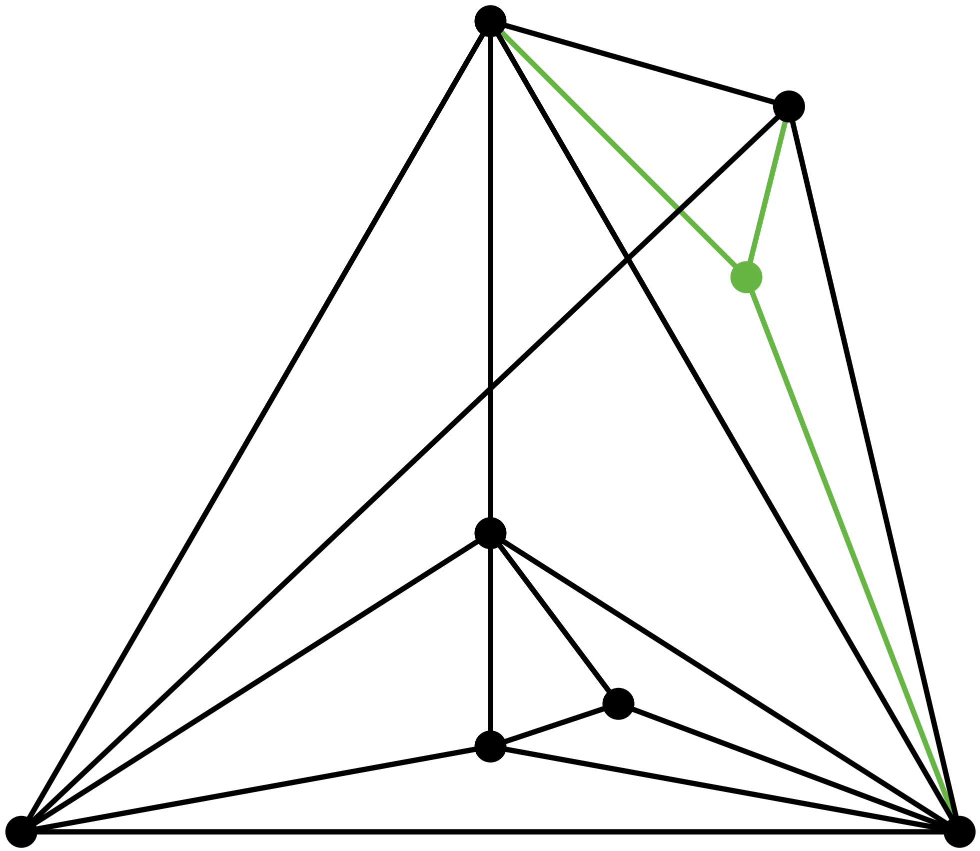

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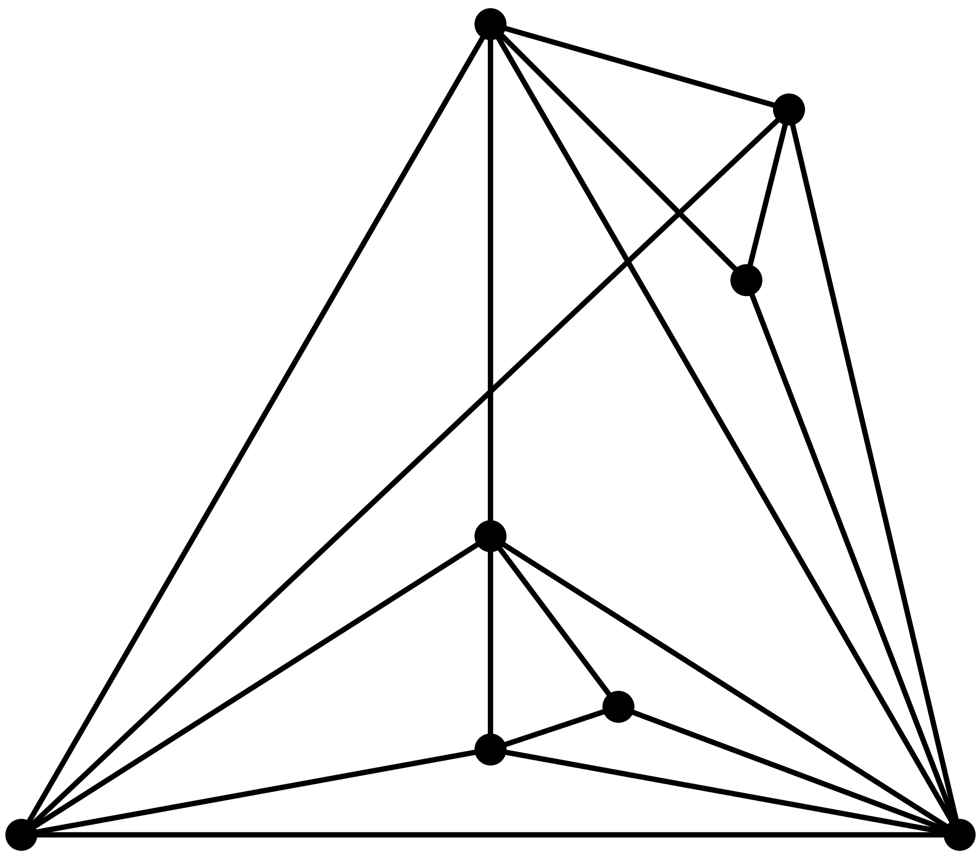

3-trees

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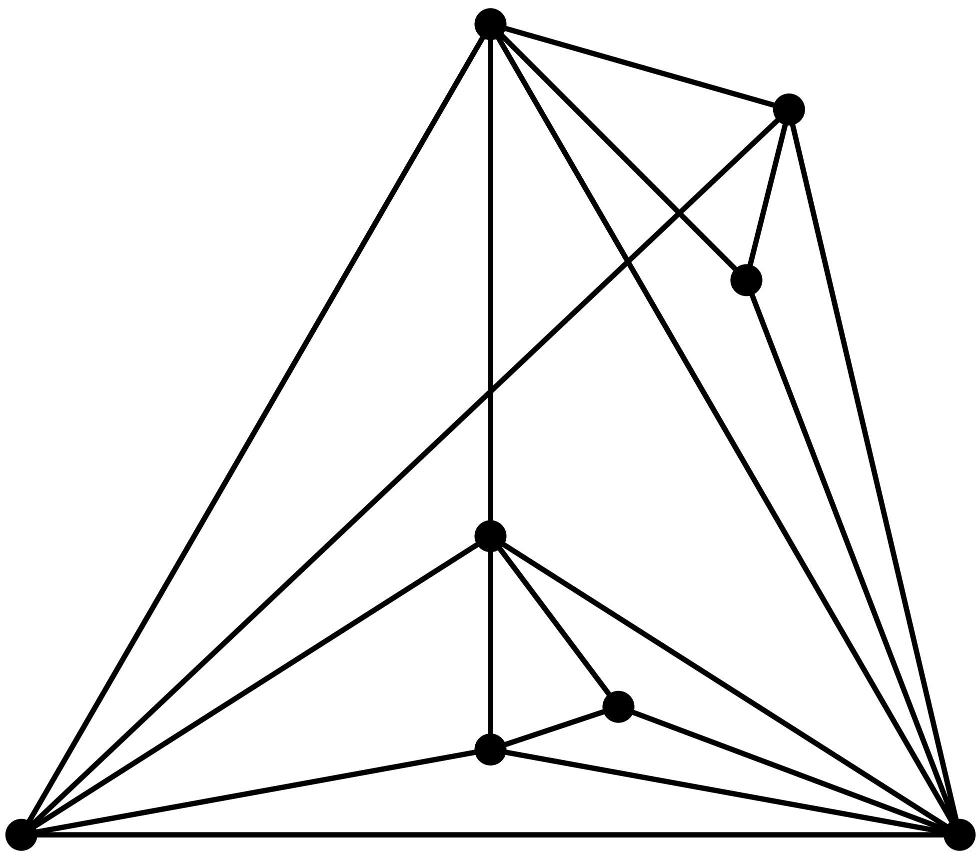

3-trees

Definition. A k-**tree** is a graph obtained from a $(k+1)$-clique by successively adding a new vertex connected to all vertices of an existing k-clique.

Chordal graphs

Iteratively add a new vertex connected to the vertices of an existing **clique** (complete subgraph).

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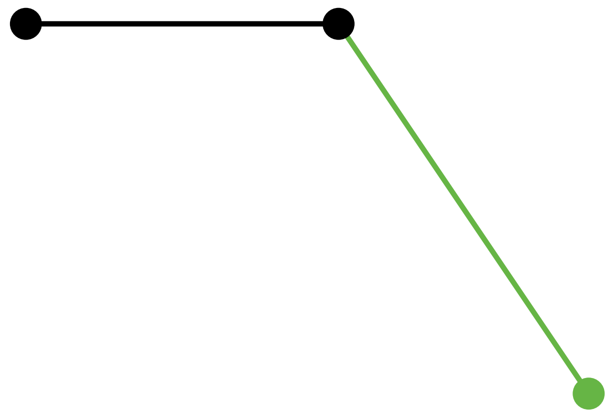

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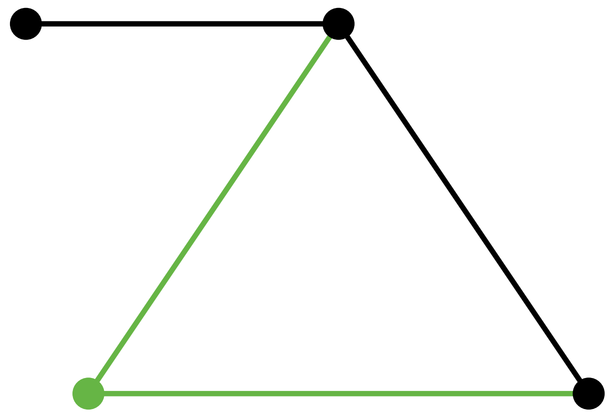

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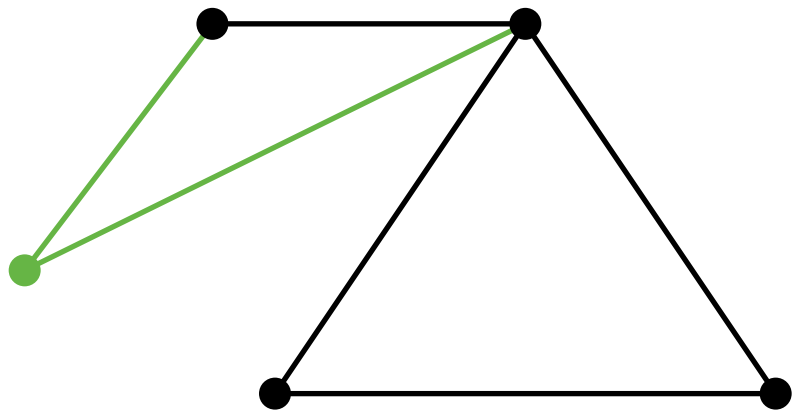

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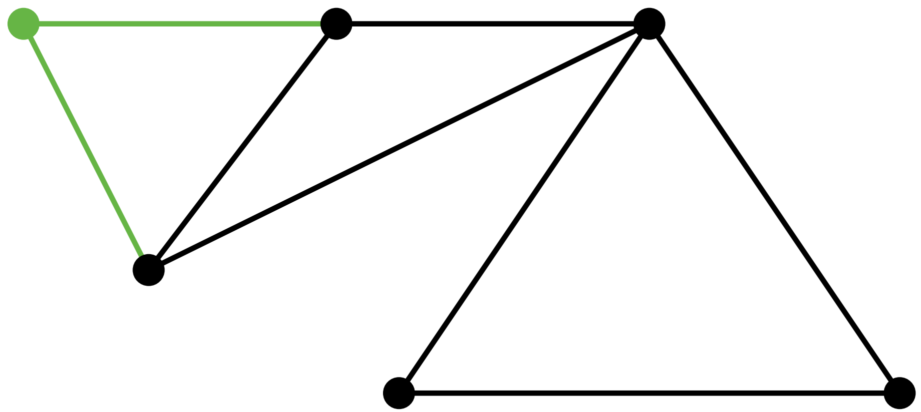

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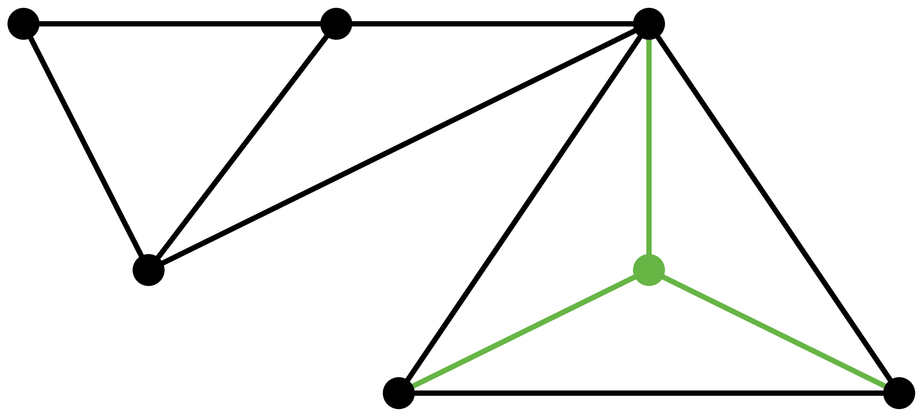

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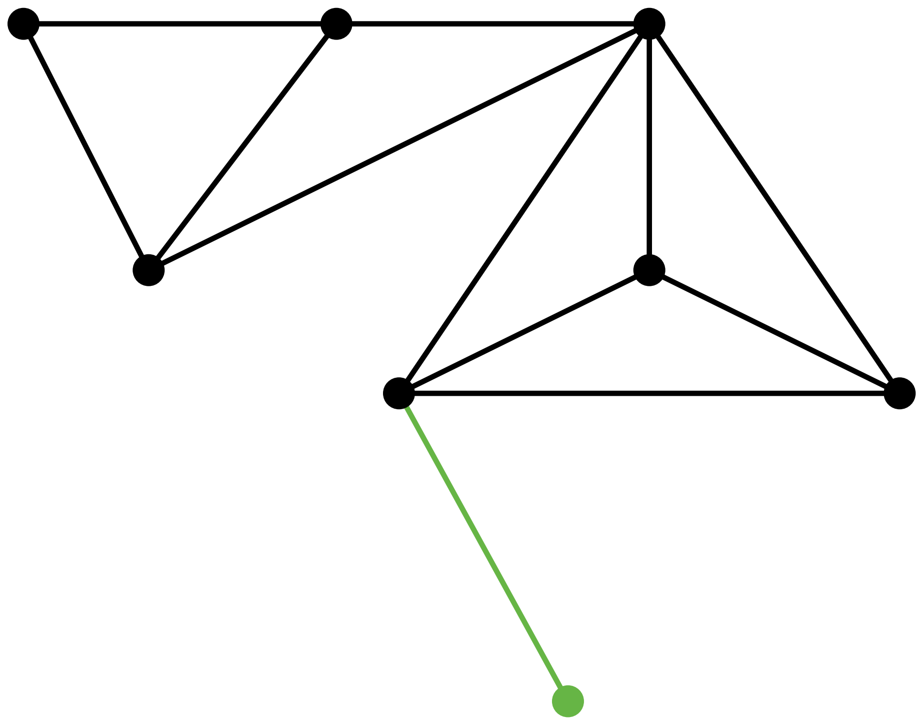

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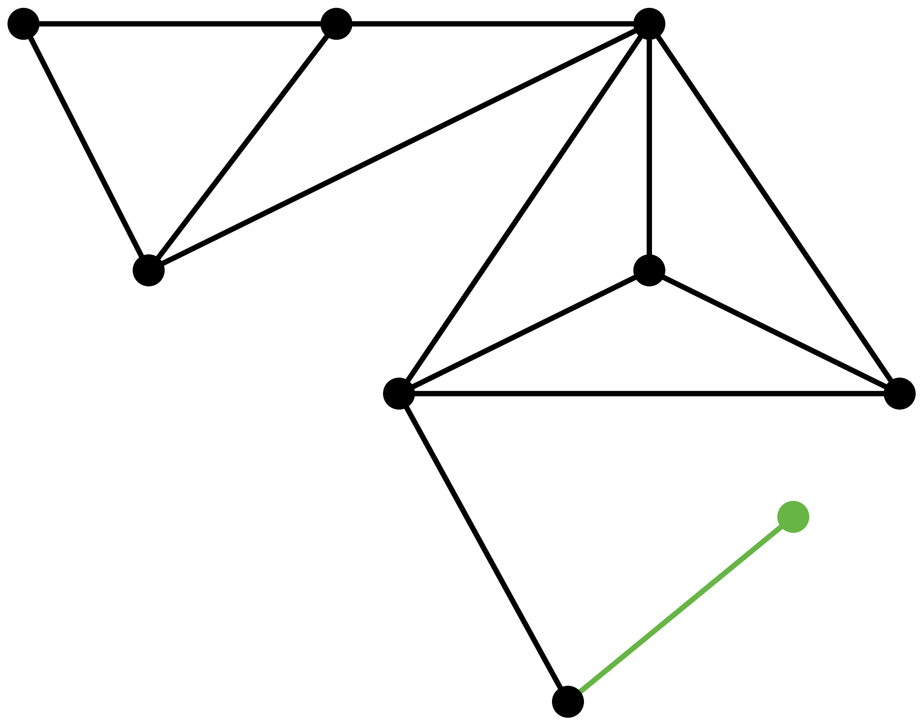

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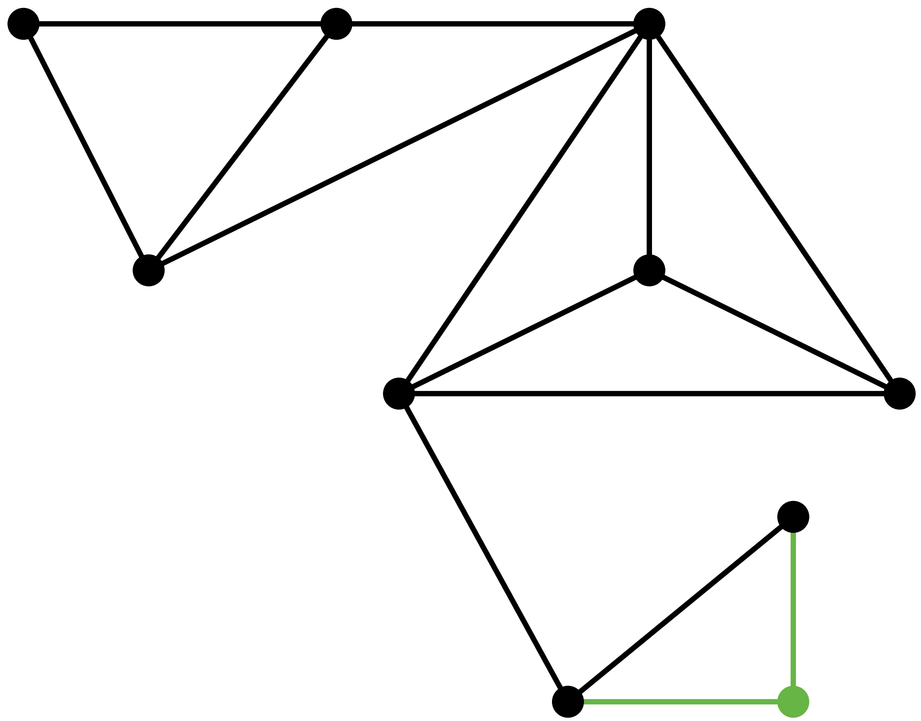

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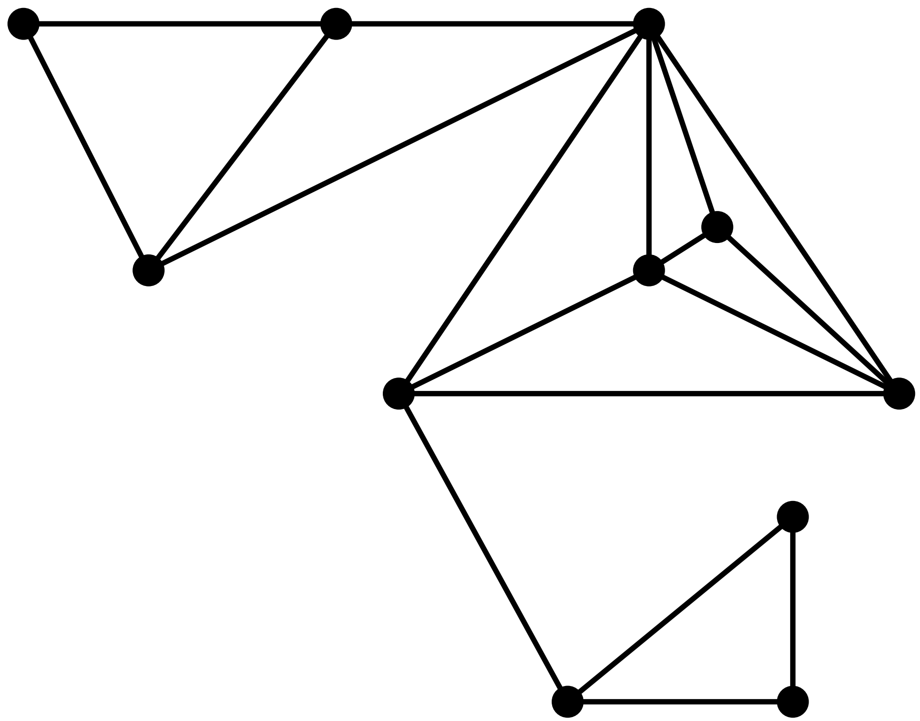

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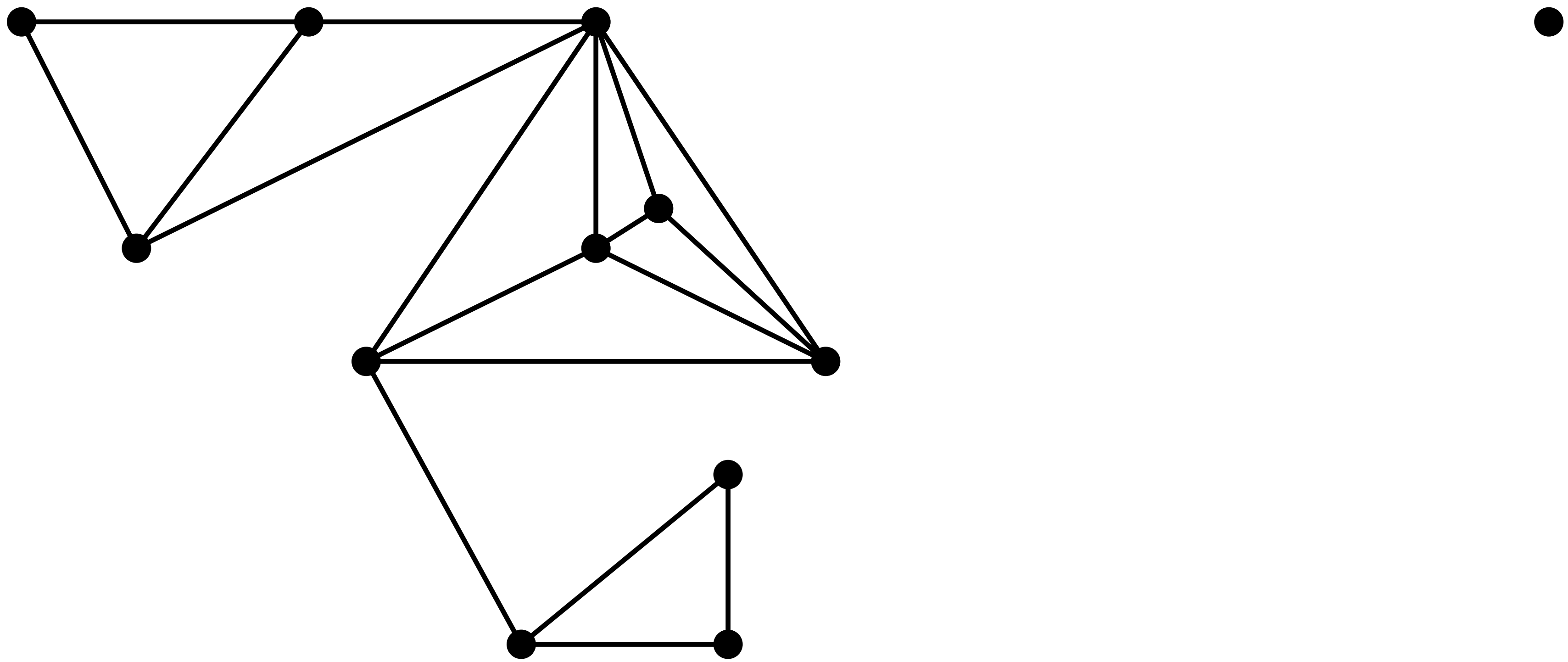

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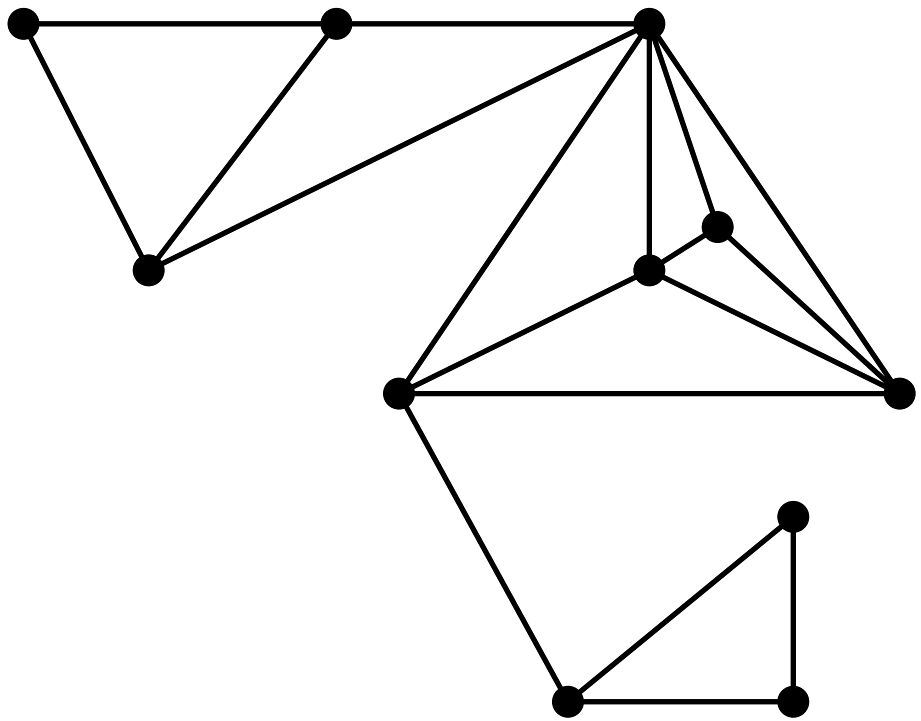

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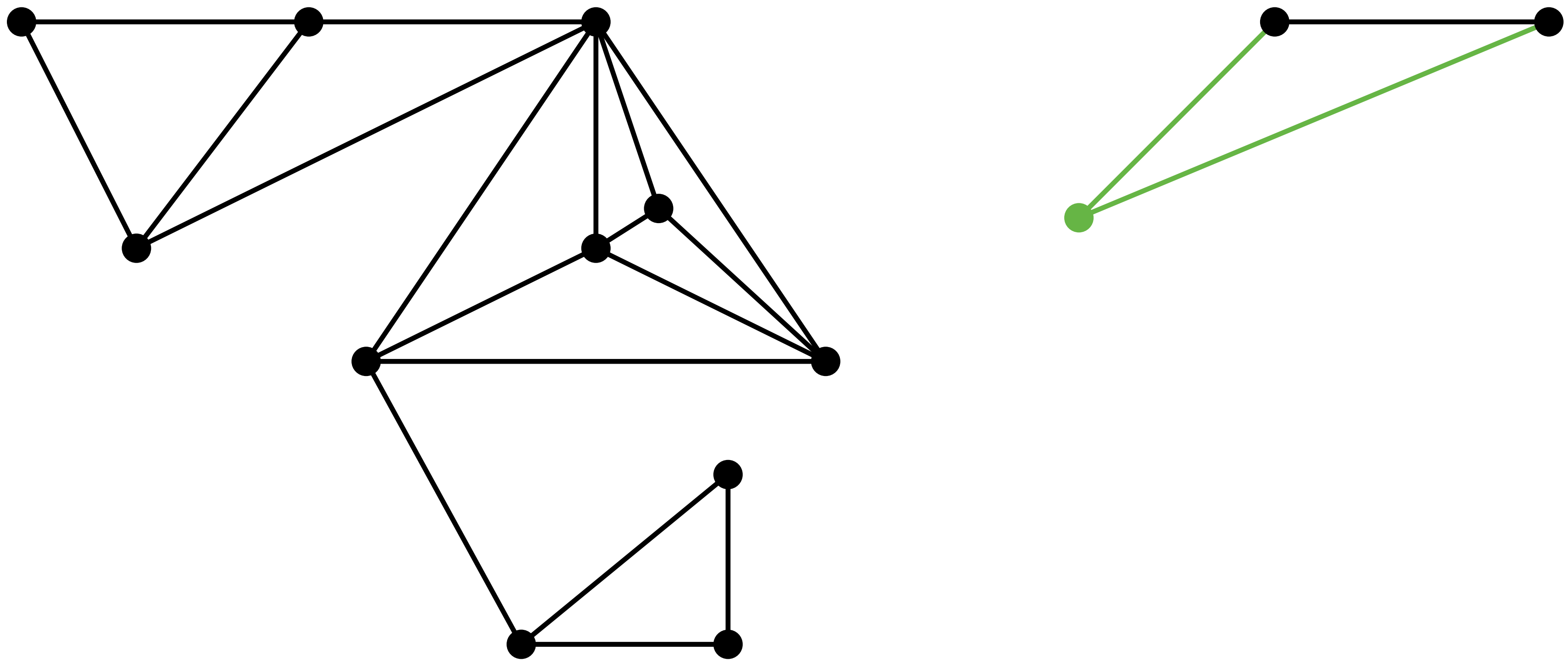

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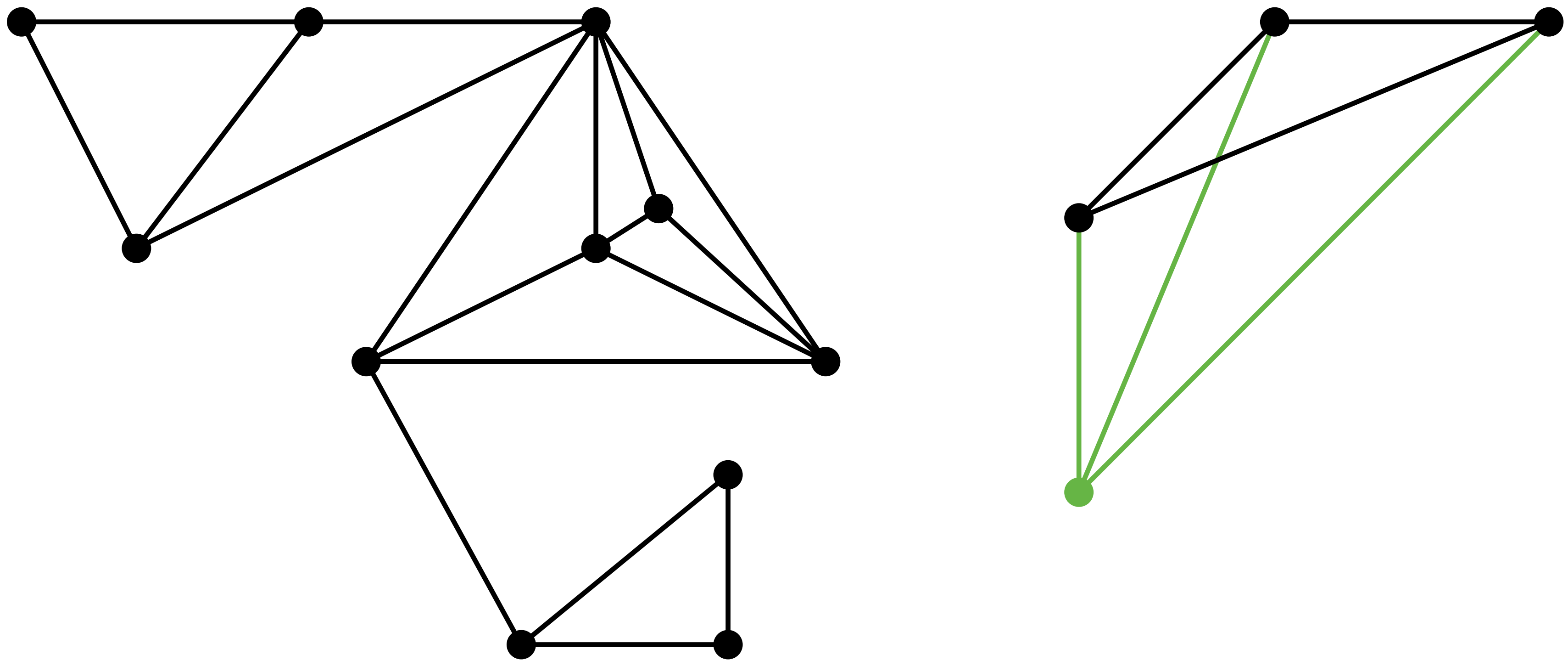

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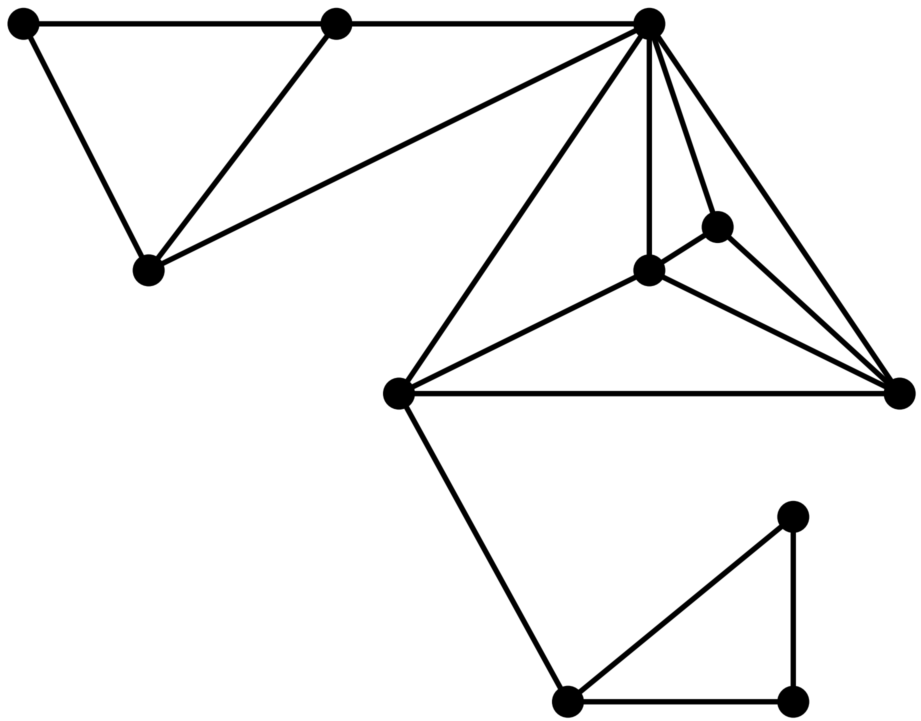

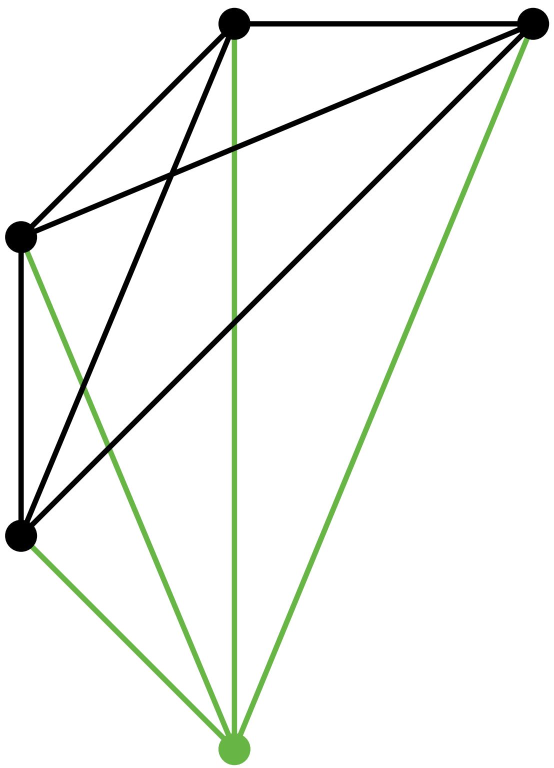

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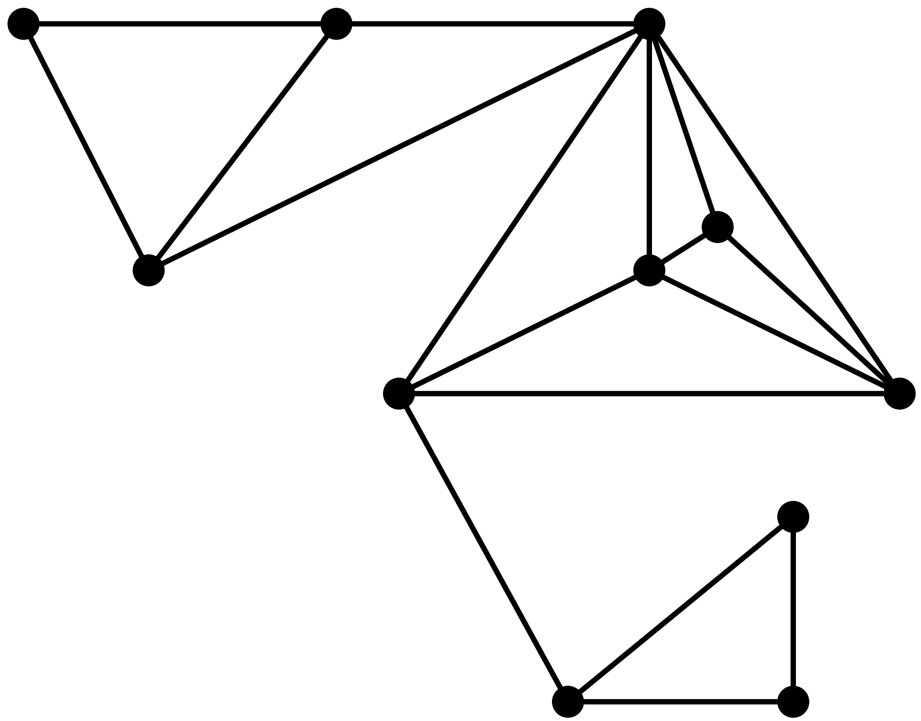

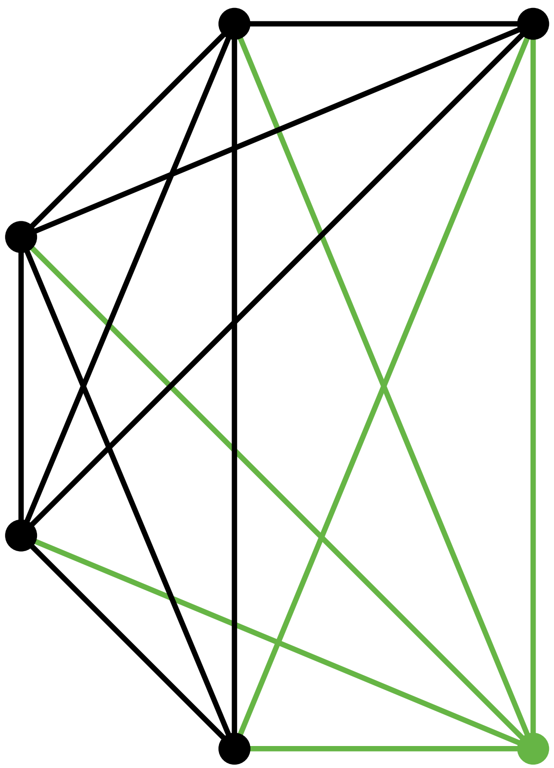

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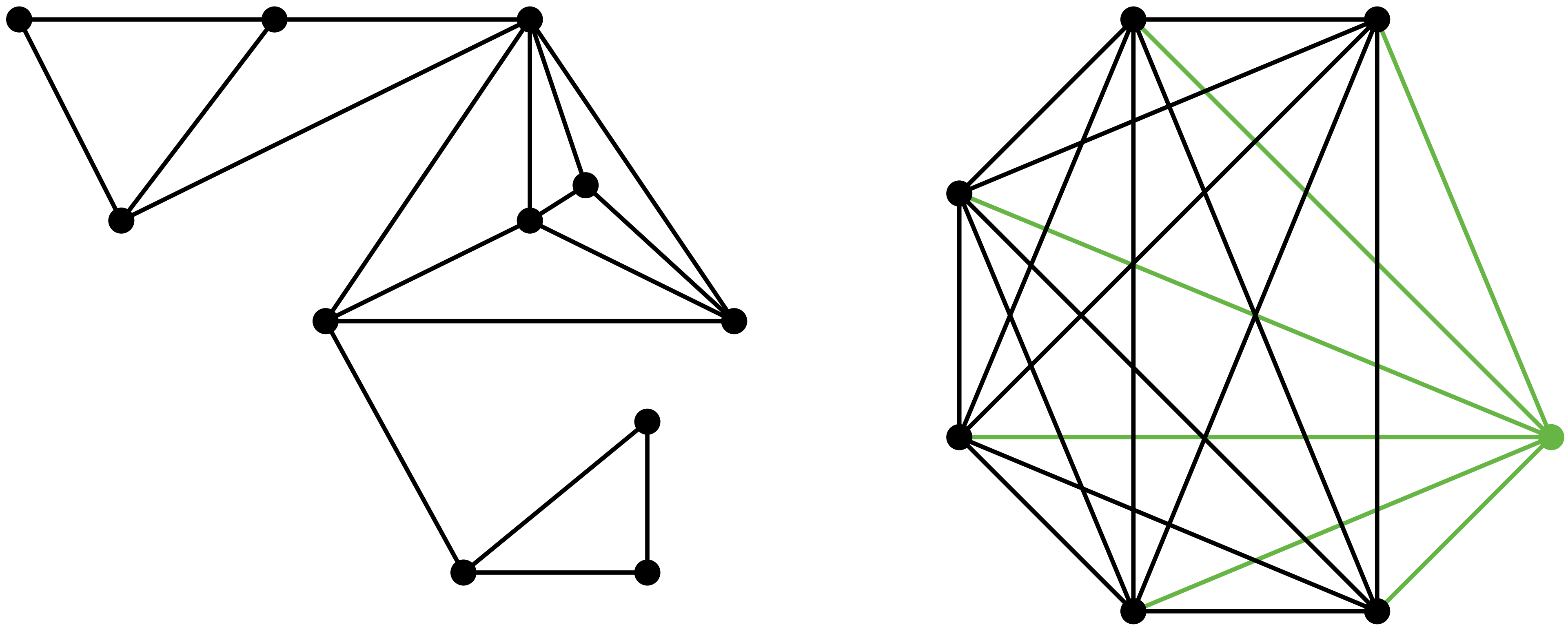

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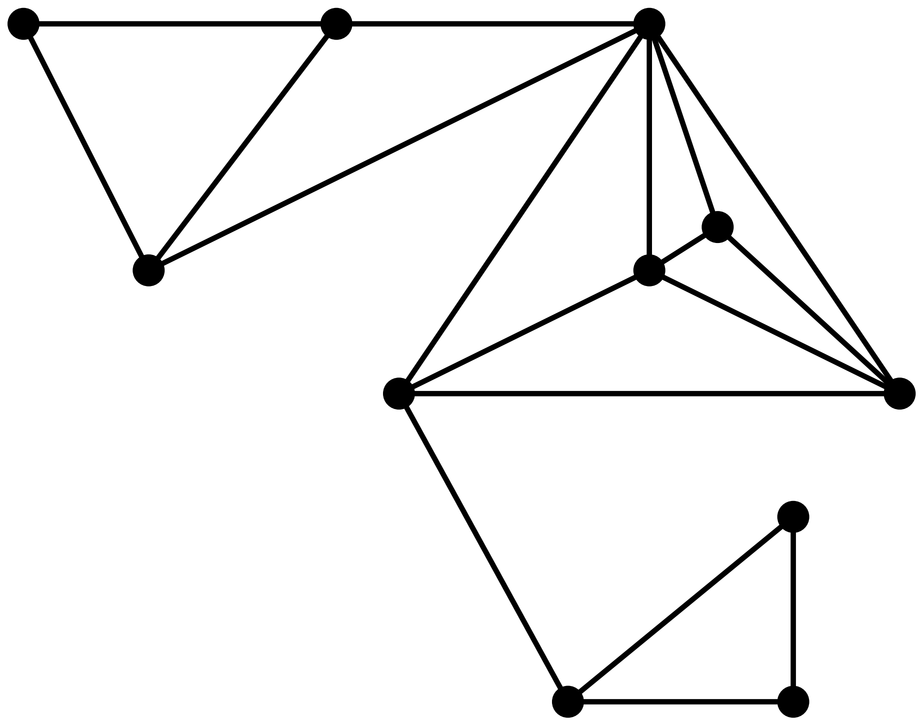

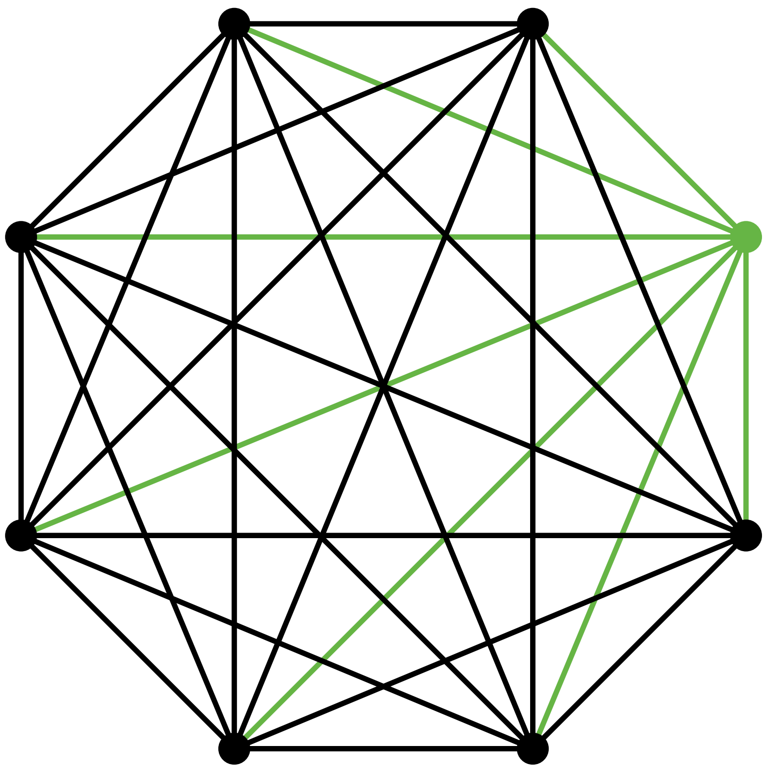

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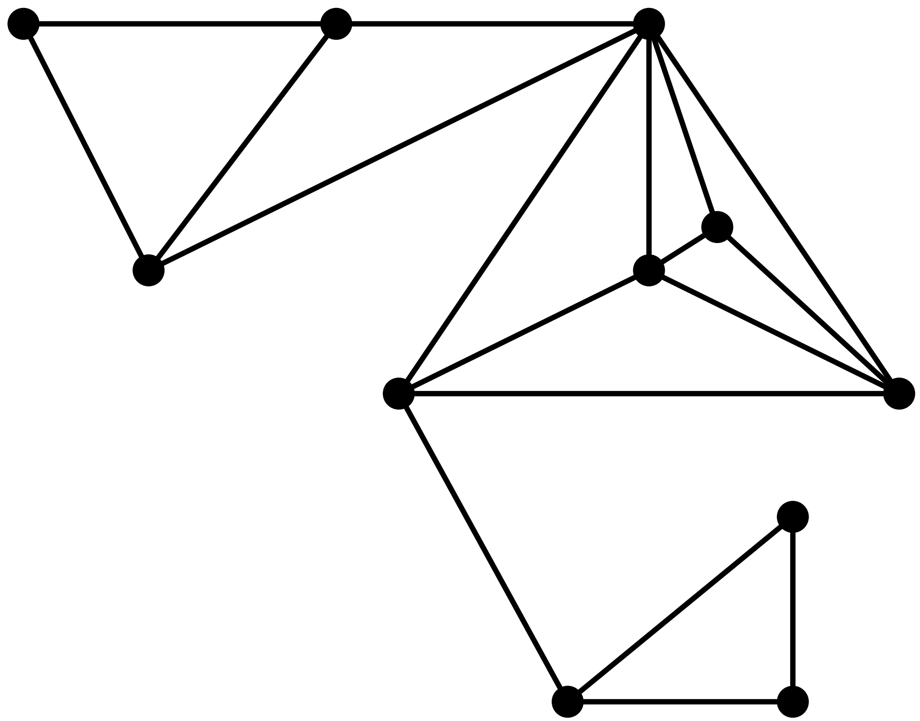

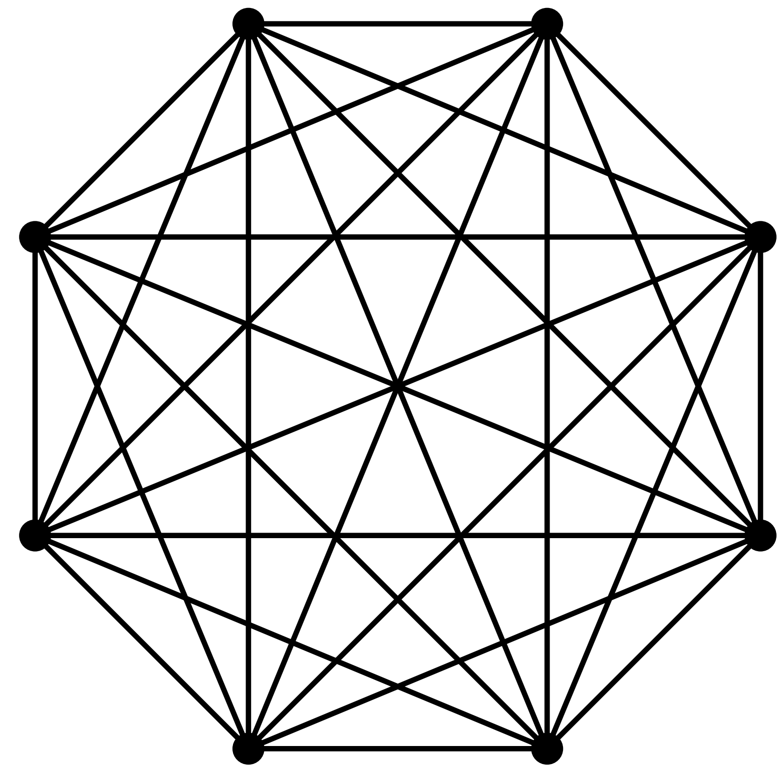

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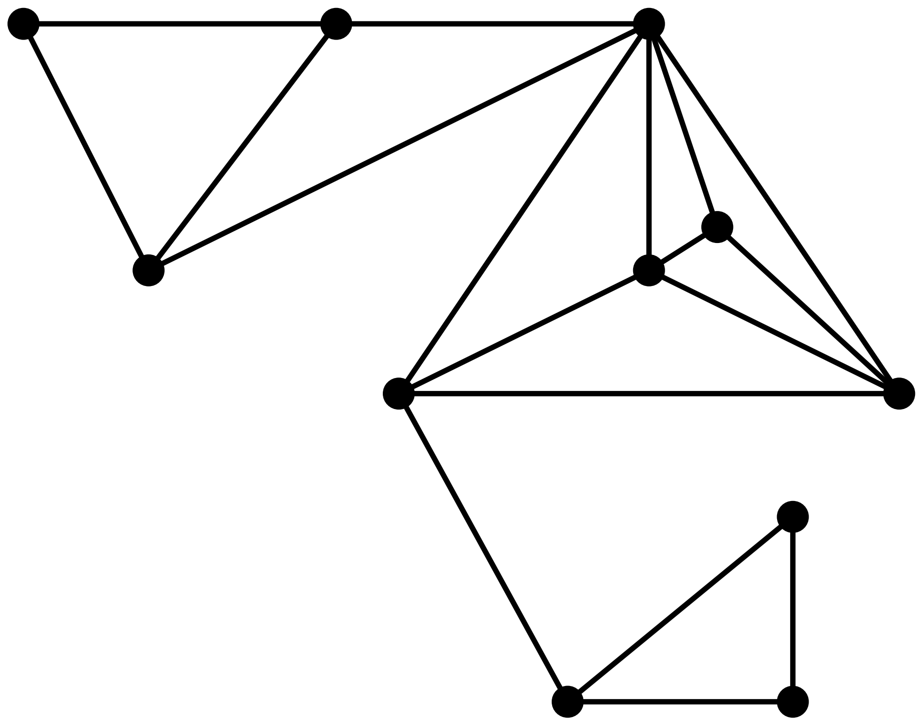

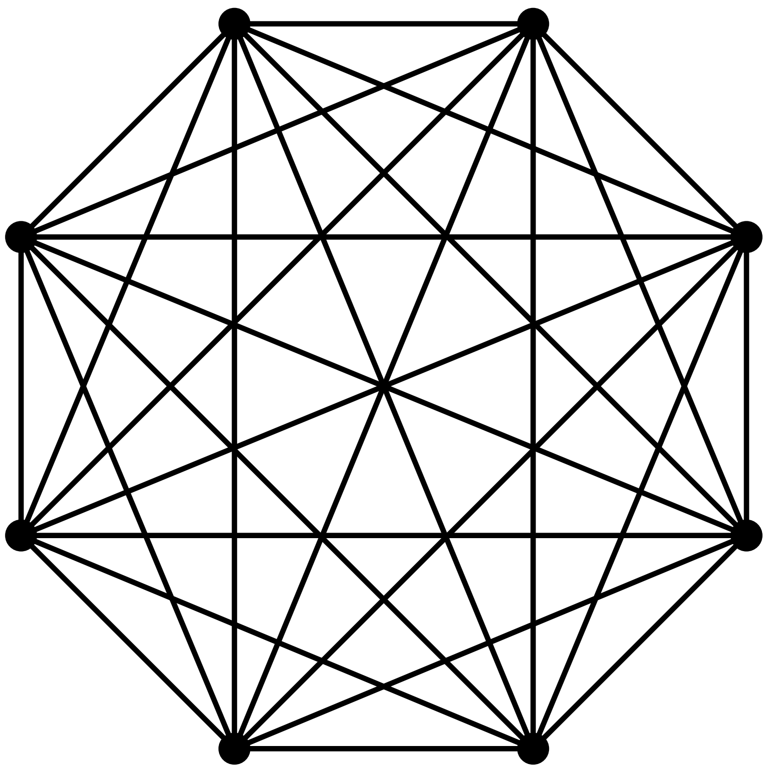

Chordal graphs

Definition. A graph is **chordal** if it has no induced cycle of lengh greater than 3.

Our class of graphs

Iteratively add a new vertex connected to the vertices of an existing **clique of size at most t**.

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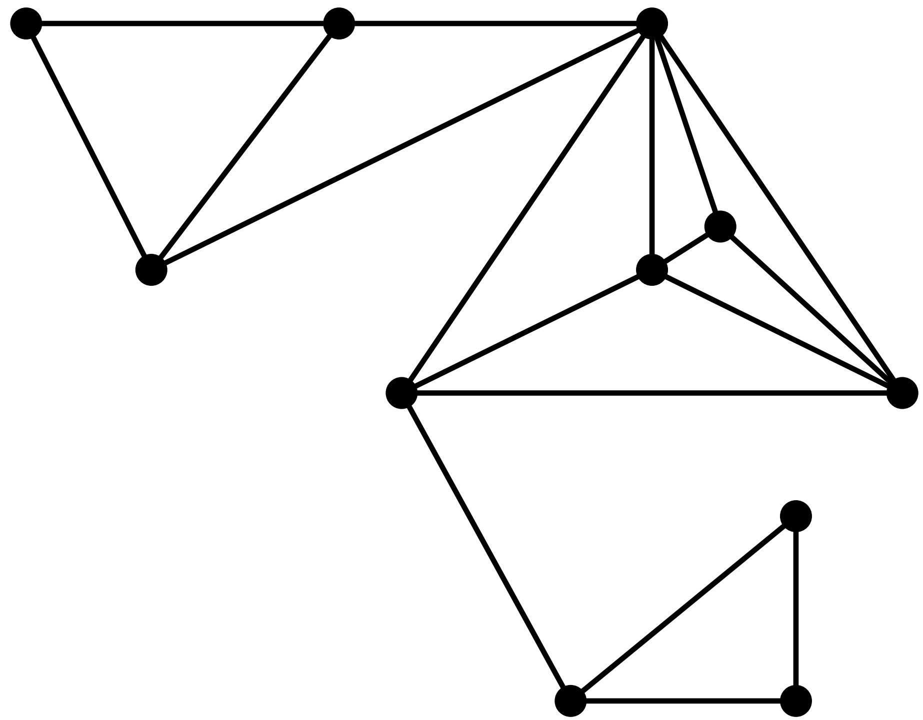

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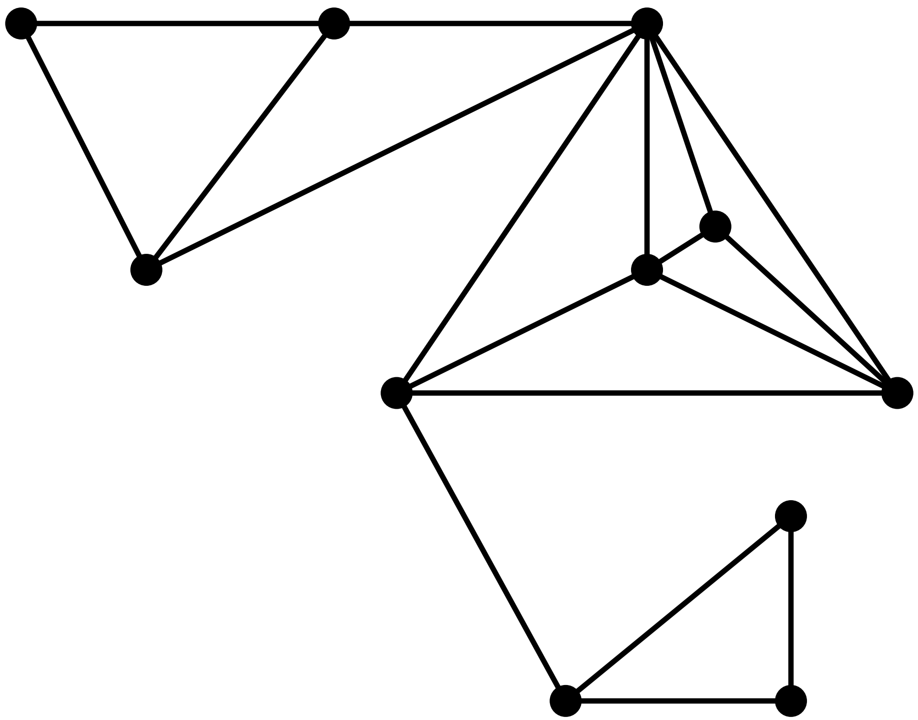

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Chordal graphs with tree-width at most t

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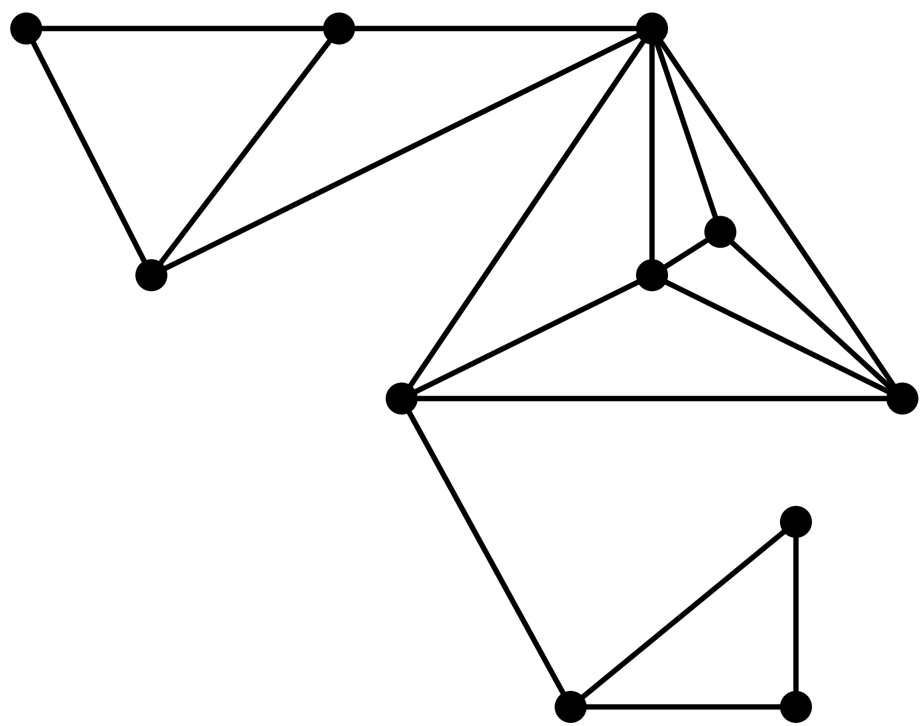

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Chordal graphs with tree-width at most t

Definition. The **tree-width** of a graph G is the minimum k such that G is the subgraph of a k-tree.

Block decomposition

Definition. The **2-connected components** (or blocks) of a connected graph are its maximal 2-connected subgraphs.

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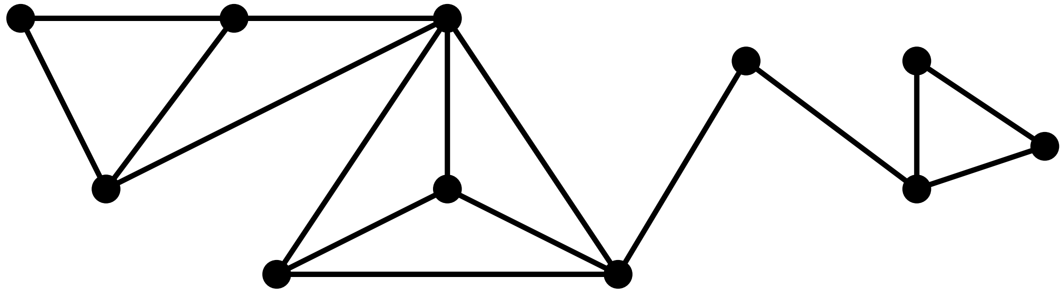

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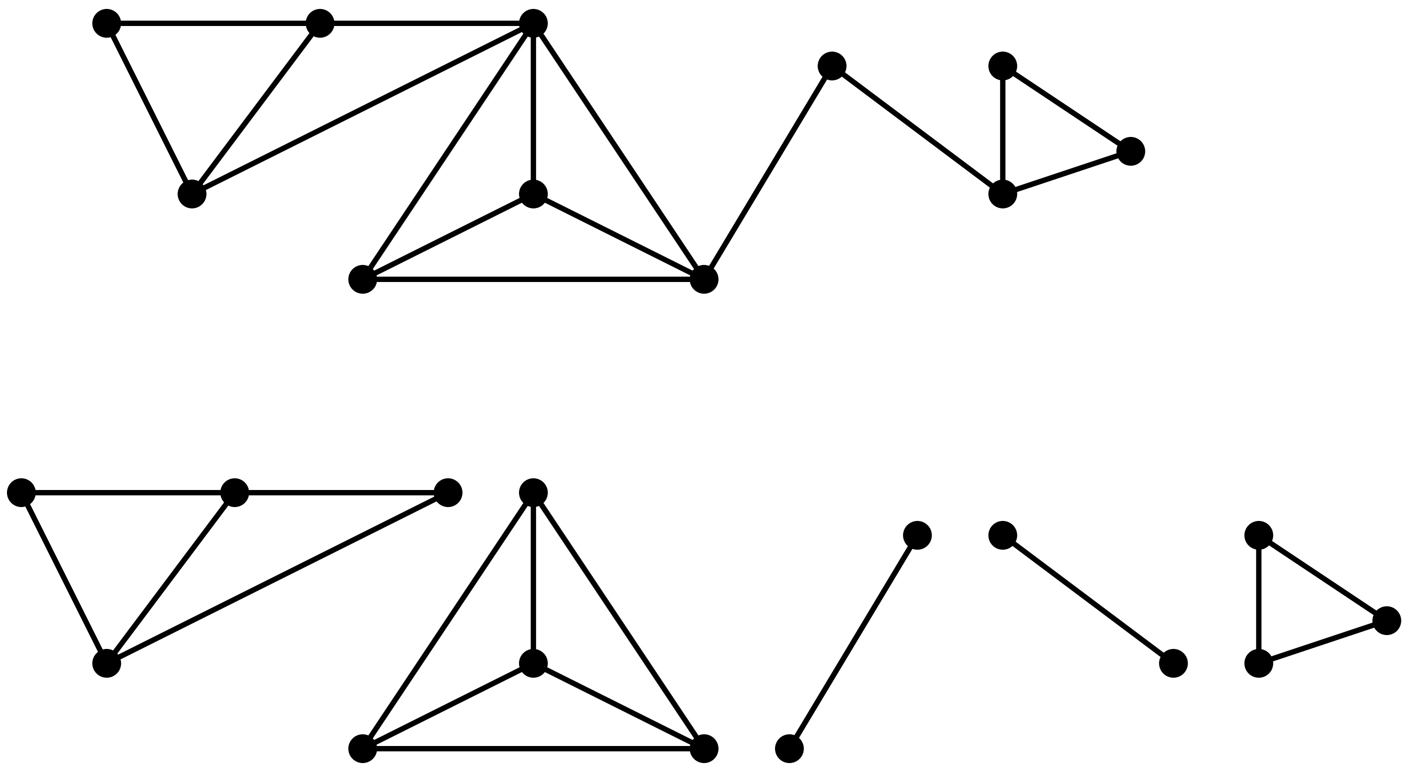

Block decomposition

Let $\mathcal{B} \subset \mathcal{C}$ be the class of the 2-connected members of $\mathcal{G}$. Then,

$$C^\bullet(x) = x\exp(B'(C^\bullet(x))), \quad \text{where } C^\bullet(x) = xC'(x),$$

provided that $\mathcal{G}$ is **block-stable**, i.e., that a graph belongs to $\mathcal{C}$ iff its blocks belong to $\mathcal{B}$.

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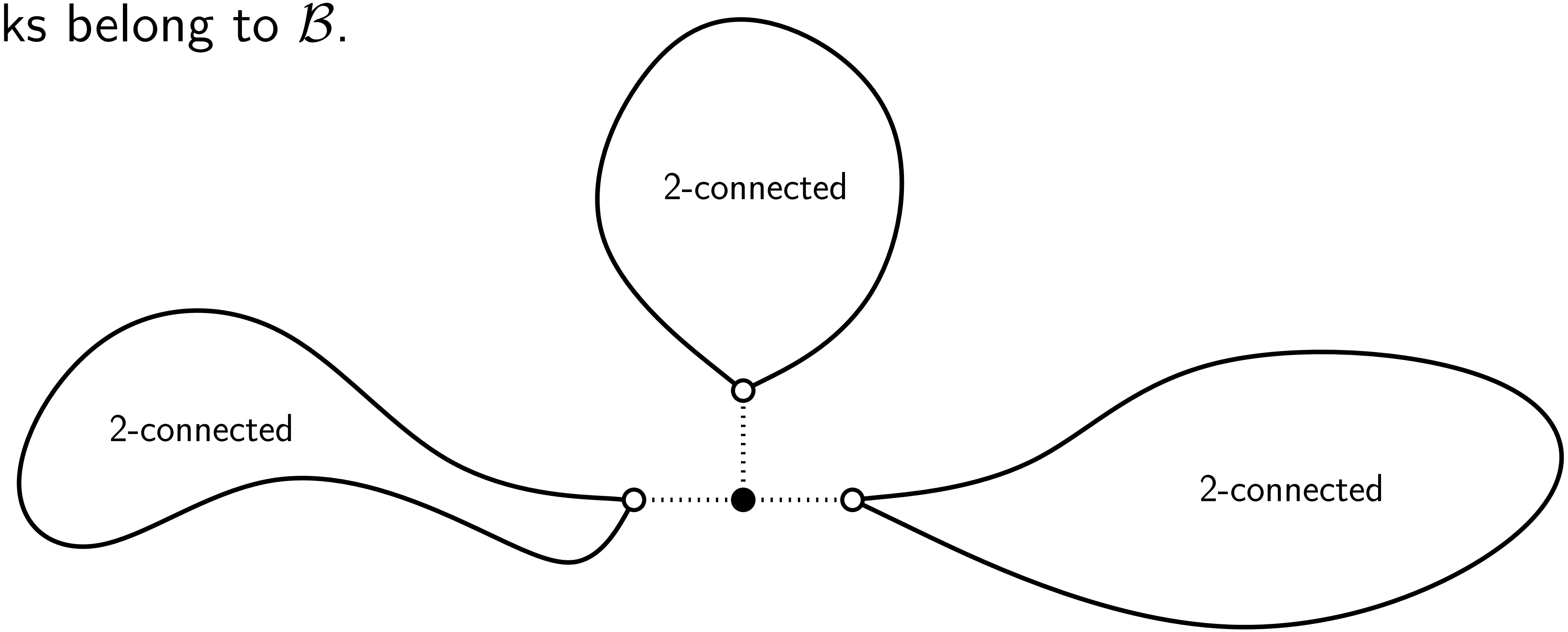

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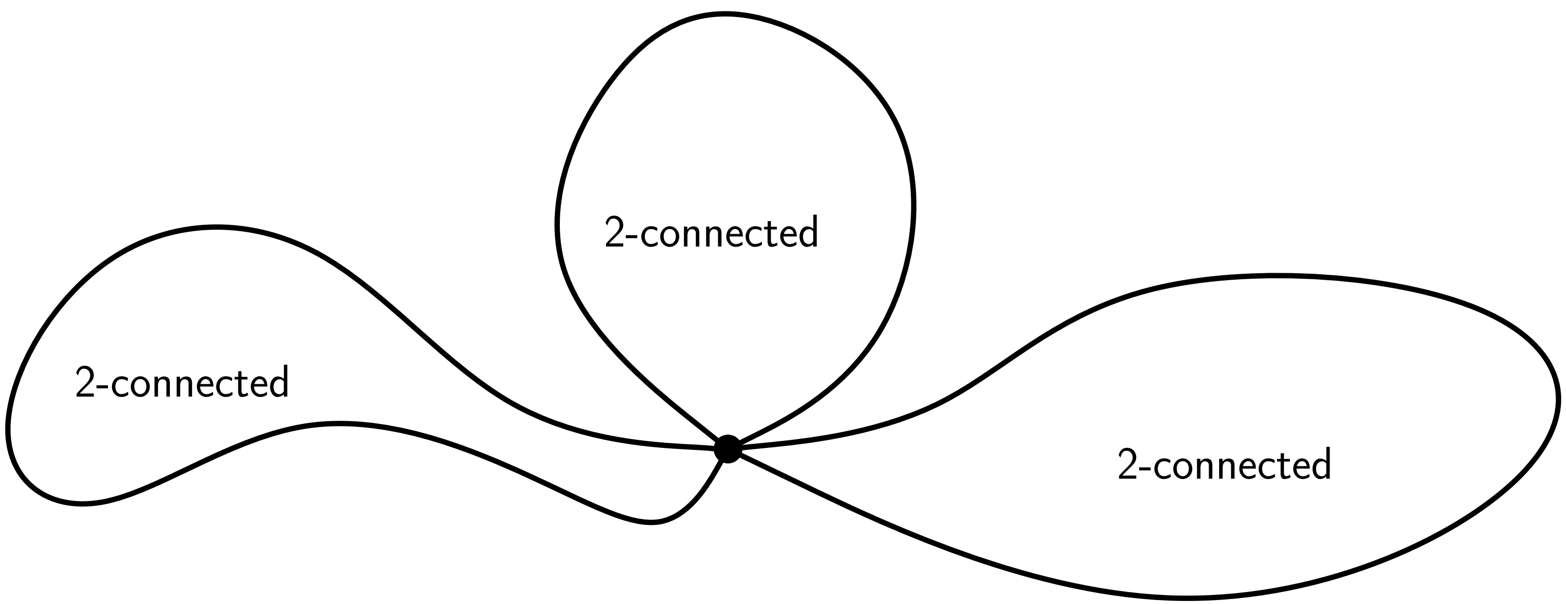

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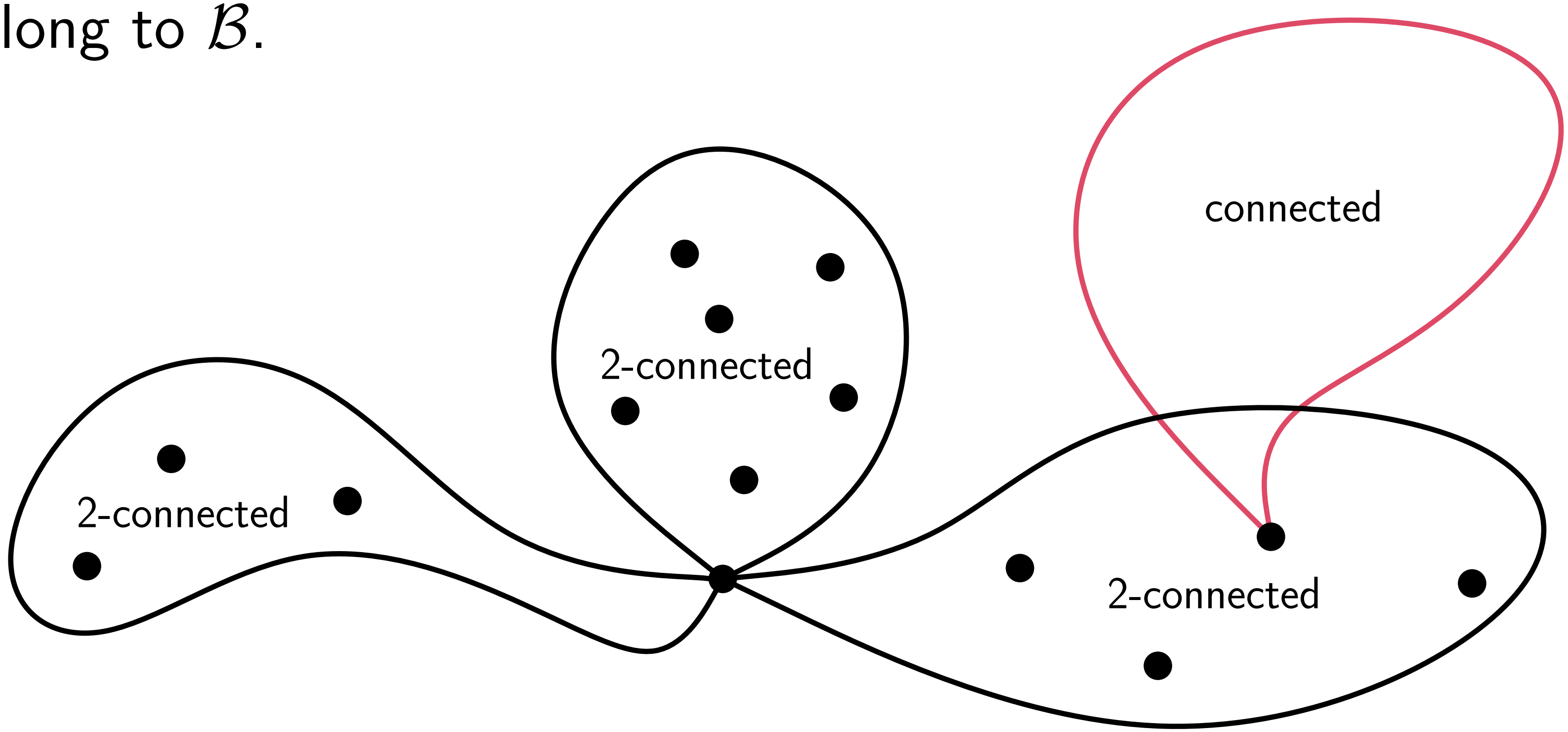

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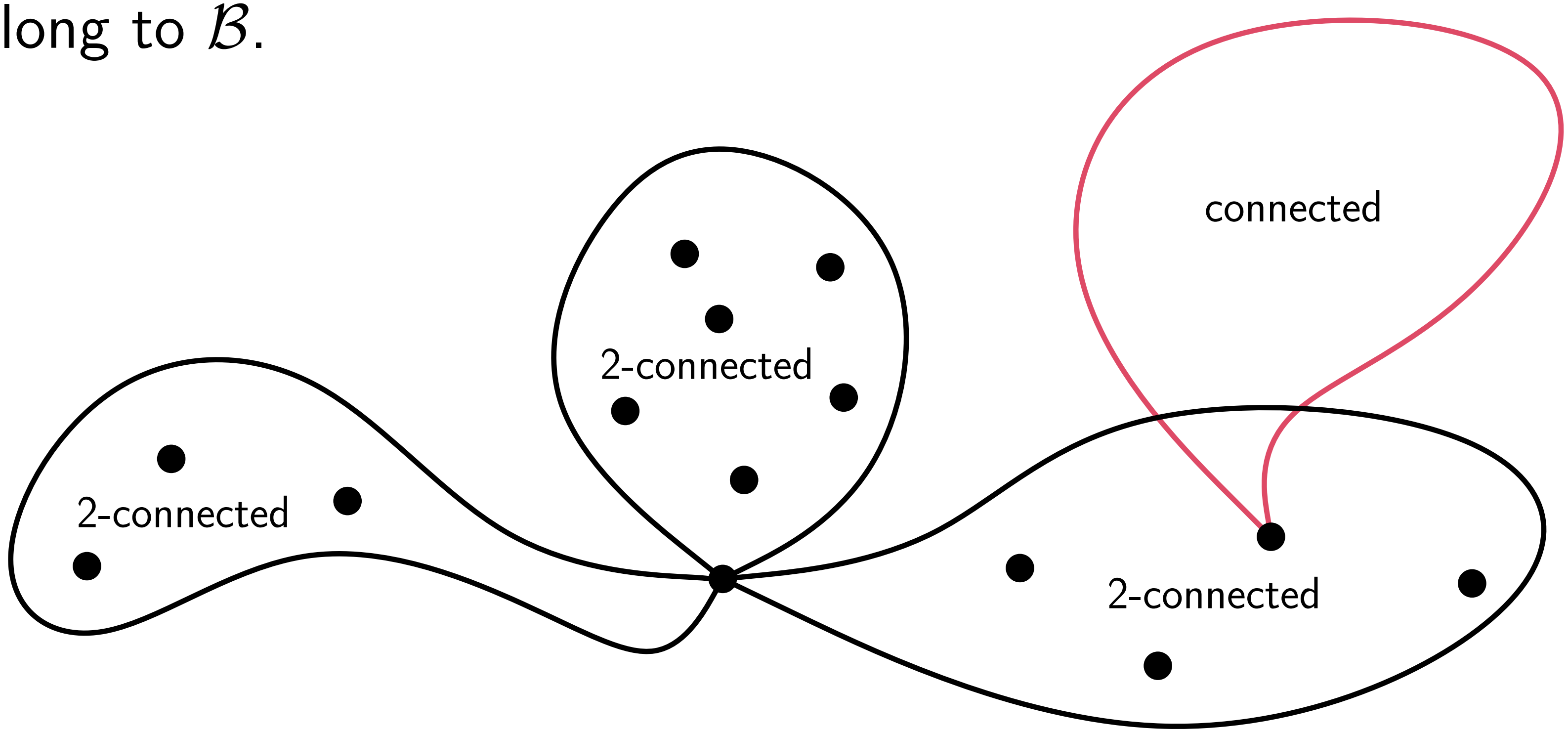
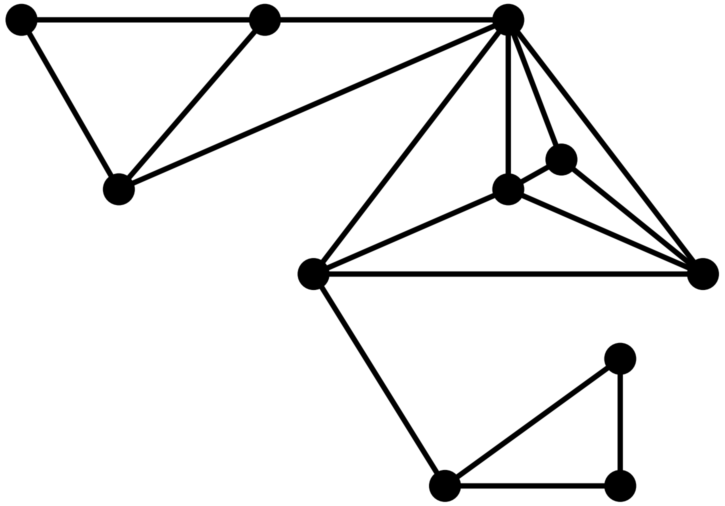


The 3-connected components of a 2-connected graph can also be defined, but the details are a little more involved. However, 4-connected components cannot be defined in general.

Our class of graphs



Chordal graphs with tree-width at most t

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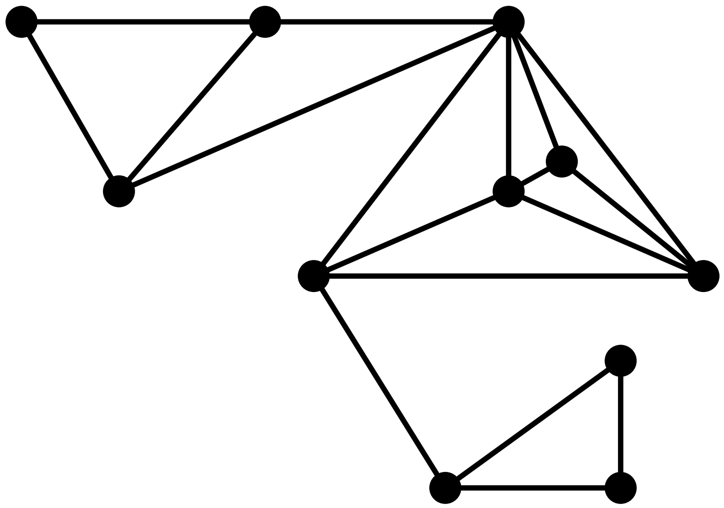

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[Wormald (1985)]: they admit a decomposition into k-connected components.

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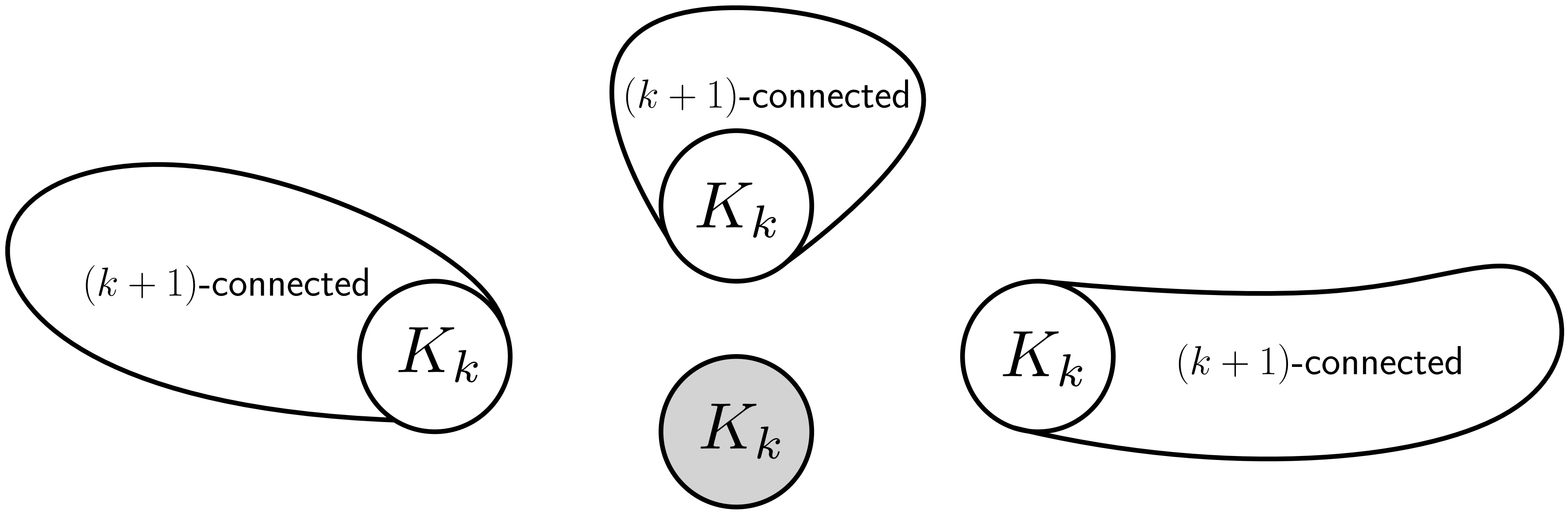

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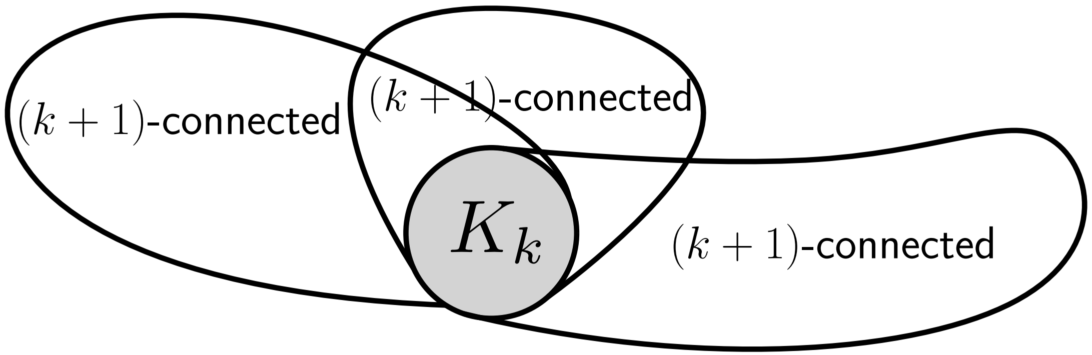

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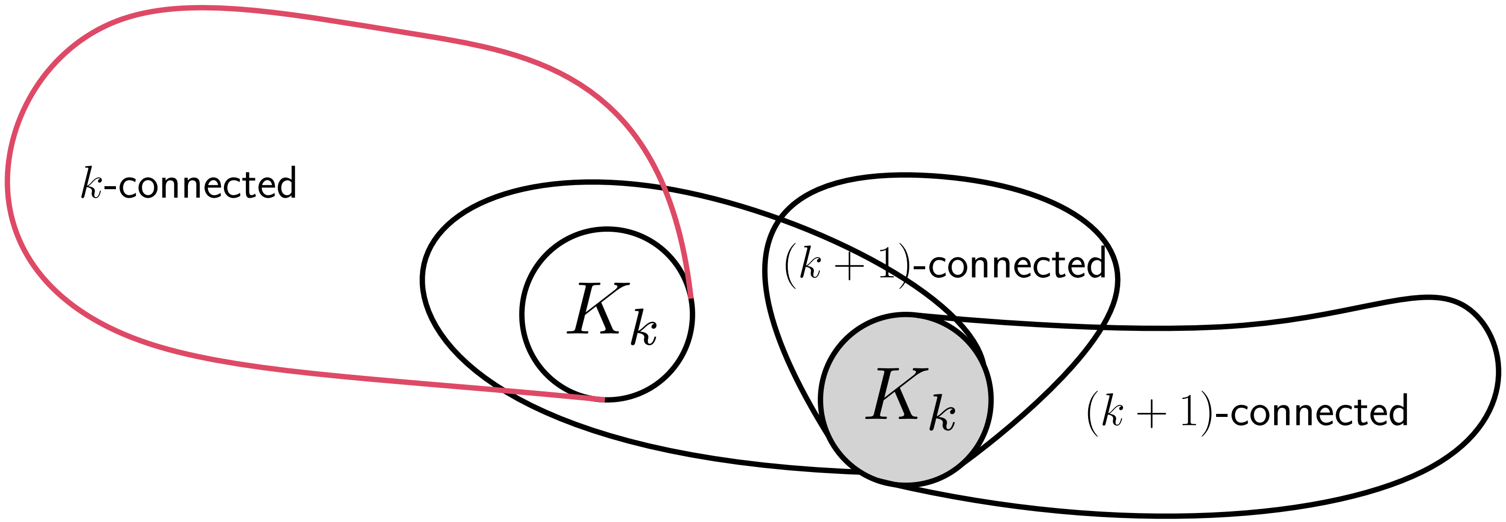

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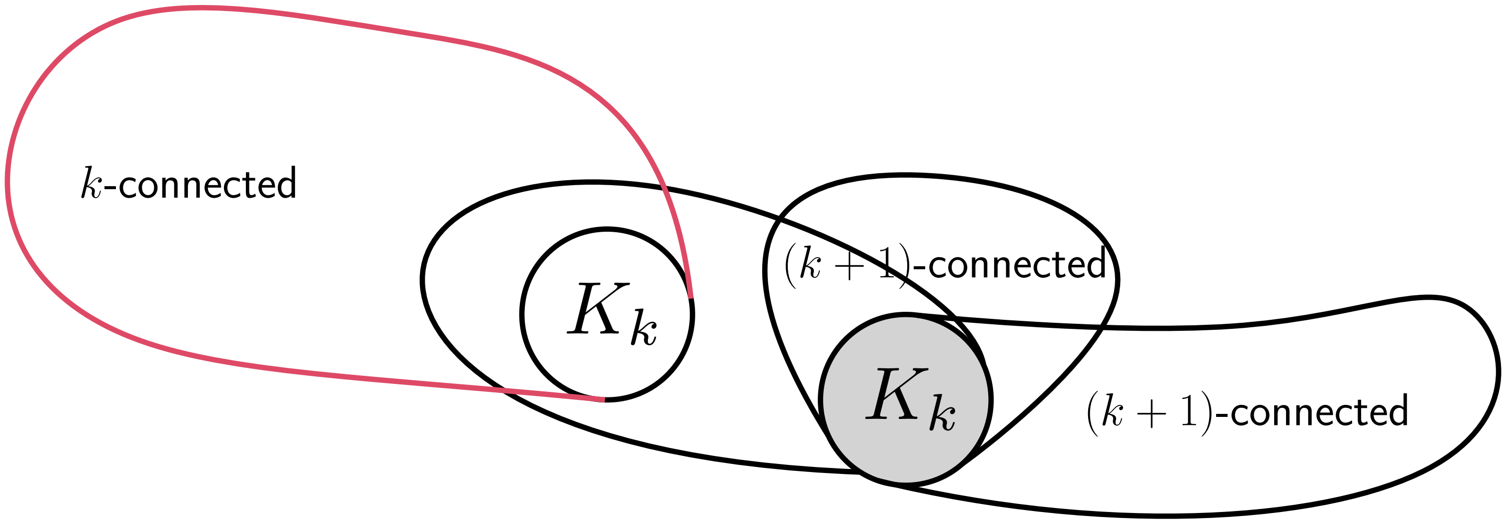

[Wormald (1985)]: they admit a decomposition into k-connected components.

[C., Drmota, Noy & Requilé (2023)]: assyptotic enumeration of the labeled class.

$$|\mathcal{G}_{t,n}| \sim c_t \cdot n^{-5/2} \cdot \gamma_t^n \cdot n! \qquad \text{as } n \to \infty,$$

for some $c_t > 0$ and $\gamma_t > 1$

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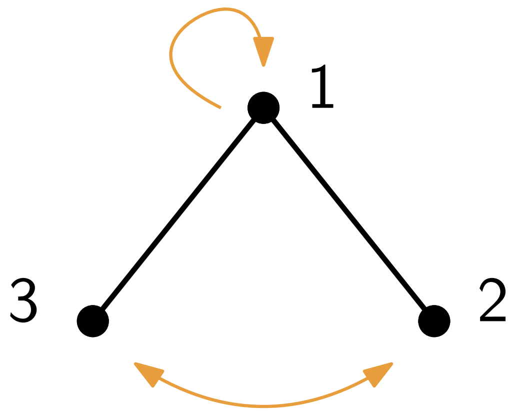

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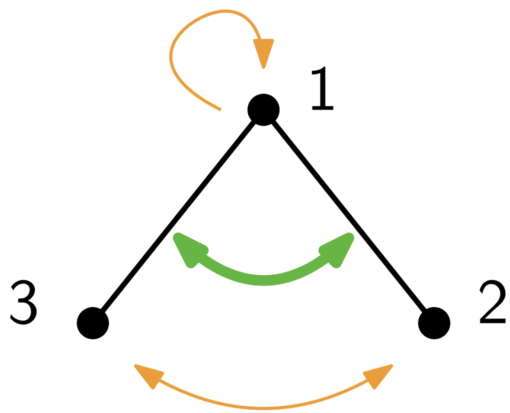

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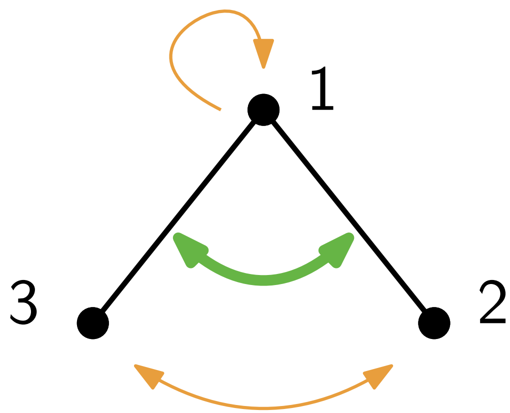

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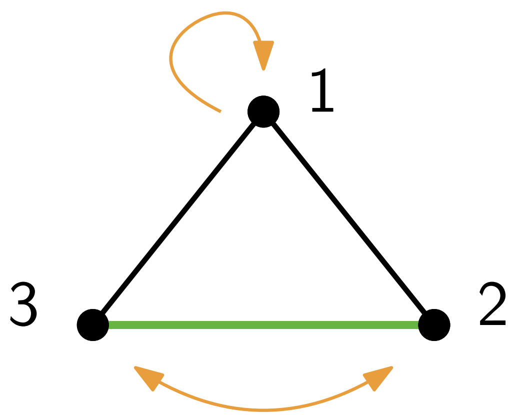

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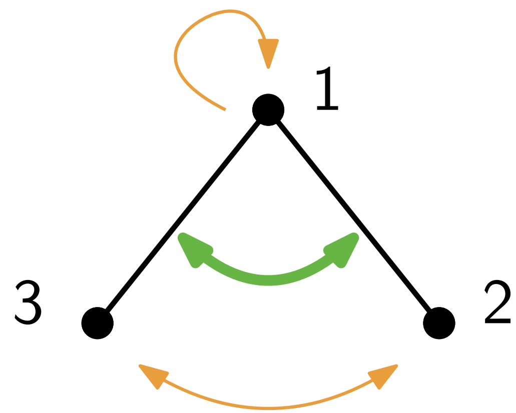

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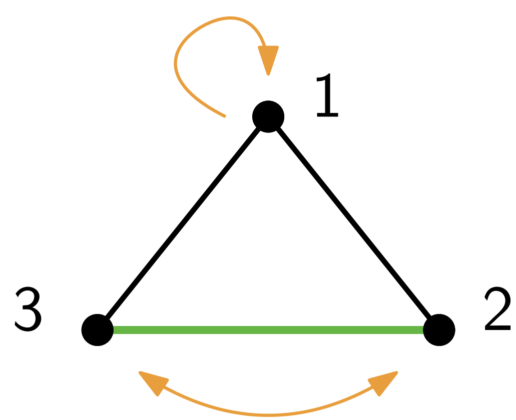

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What we do:

- Refinement of cycle index sums to encode cycles of cliques.
- Extend cycle-pointing to cycles of cliques.

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Future:

- [C.,Drmota & Requilé (soon?)]: asymptotic enumeration of unlabeled chordal graphs with bounded tree-width and unlabeled chordal planar graphs.

The system

$$\begin{cases} X^{\lambda}_{\mathcal{G}^{(k)}_{t,k+1}} = \frac{k!}{\alpha(\lambda)\kappa(\lambda)} \frac{\partial}{\partial s_{\lambda,1}} X_{\mathcal{G}_{t,k+1}}, \\ X^{\lambda}_{\mathcal{G}^{(k)}_{t,k}} = Z_{\mathrm{SET}}(s_j \to (X^{\lambda^j}_{\mathcal{G}^{(k)}_{t,k+1} \circ_k \mathcal{G}^{(k)}_{t,k}})^{[j]})_{j\geq 1}, \\ X^{\lambda}_{\mathcal{G}^{(k)}_{t,k+1} \circ_k \mathcal{G}^{(k)}_{t,k}} = X^{\lambda}_{\mathcal{G}^{(k)}_{t,k+1}} (s_{\mu,j} \to (X^{\mu}_{\mathcal{G}^{(k)}_{t,k}})^{[j]})_{\mu\vdash k, j\geq 1}, \\ X_{\mathcal{G}^{\bullet_k}_{t,k}} = \sum_{\mu\vdash k} \frac{\alpha(\mu)\kappa(\mu)}{k!} t_{\mu,1} X^{\mu}_{\mathcal{G}^{(k)}_{t,k}} + X_{(\mathcal{G}_{t,k})^{\bullet_k}_{\geq 2}}, \\ X_{(\mathcal{G}_{t,k})^{\bullet_k}_{\geq 2}} = X_{(\mathcal{G}_{t,k+1})^{\bullet_k}_{\geq 2}} (s_{\mu,j} \to (X^{\mu}_{\mathcal{G}^{(k)}_{t,k}})^{[j]}, t_{\mu,j} \to (X^{\mu}_{(\mathcal{G}^{(k)}_{t,k})^{\bullet_k}})^{[j]})_{\mu\vdash k, j\geq 1} \\ \qquad + \sum_{\mu\vdash k} \frac{\alpha(\mu)\kappa(\mu)}{k!} s_{\mu,1} Z_{\mathrm{SET}^{\bullet}_{\geq 2}}(s_j \to (X^{\mu^j}_{\mathcal{G}^{(k)}_{t,k+1} \circ_k \mathcal{G}^{(k)}_{t,k}})^{[j]}, \\ \qquad\qquad t_j \to (X^{\mu^j}_{(\mathcal{G}^{(k)}_{t,k+1} \circ_k \mathcal{G}^{(k)}_{t,k})^{\bullet_k}})^{[j]})_{j\geq 1}, \\ X_{\mathcal{G}_{t,k}} = \sum_{\lambda\vdash k} \sum_{1\leq j\leq \binom{t+1}{k}} \int \frac{1}{j t_{\lambda,j}} X_{\mathcal{G}^{\bullet_k}_{t,k}} (S(\lambda, j) \to 0, T(\lambda, j) \to 0) ds_{\lambda,j} \end{cases}$$