Enumeration of unlabelled chordal graphs with bounded tree-width

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Work in collaboration with Clément Requilé

CENTRE DE RECERCA MATEMÀTICA

EXCELENCIA MARÍA DE MAEZTU 2022 - 2025

DMD 2024 - Alcalá de Henares

Introduction

How to build a tree?

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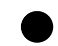
Iteratively add a new vertex connected to an existing vertex.



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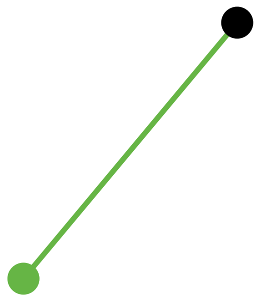

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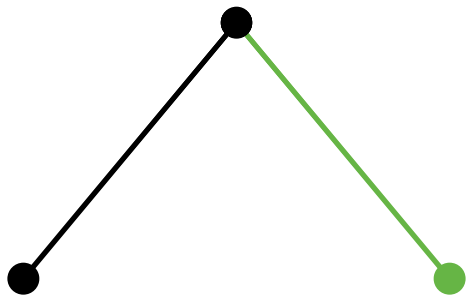

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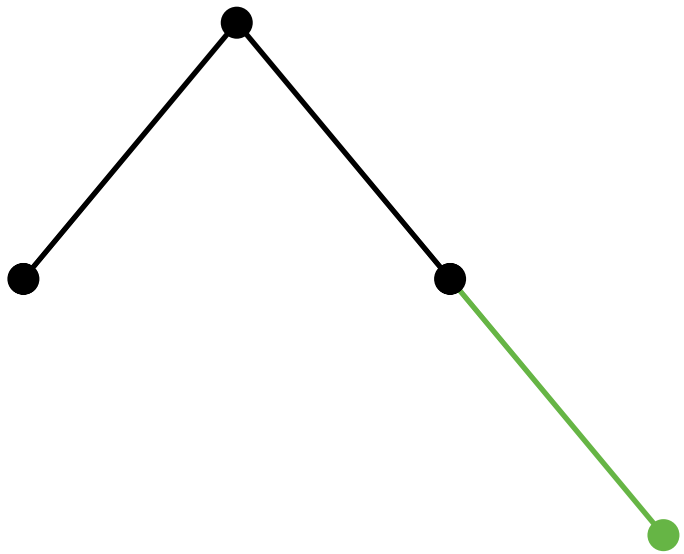

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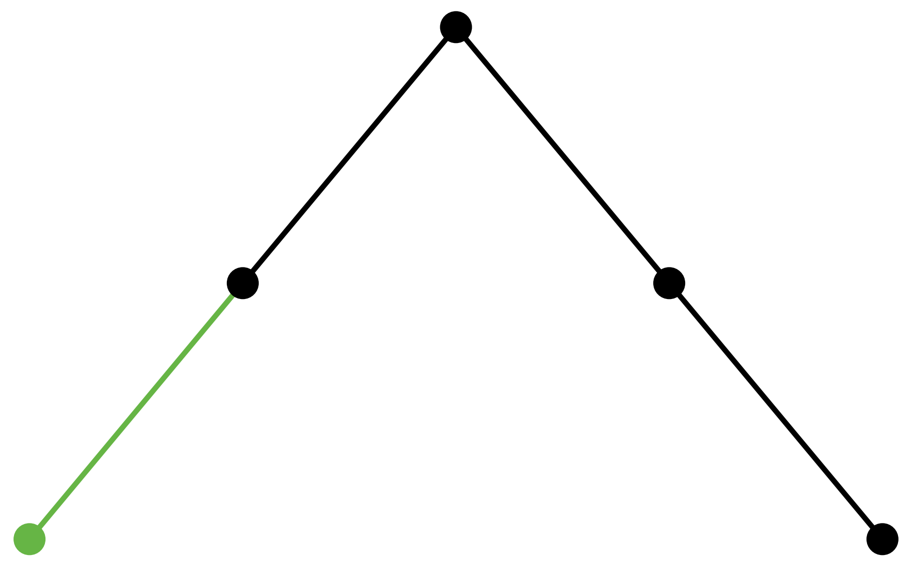

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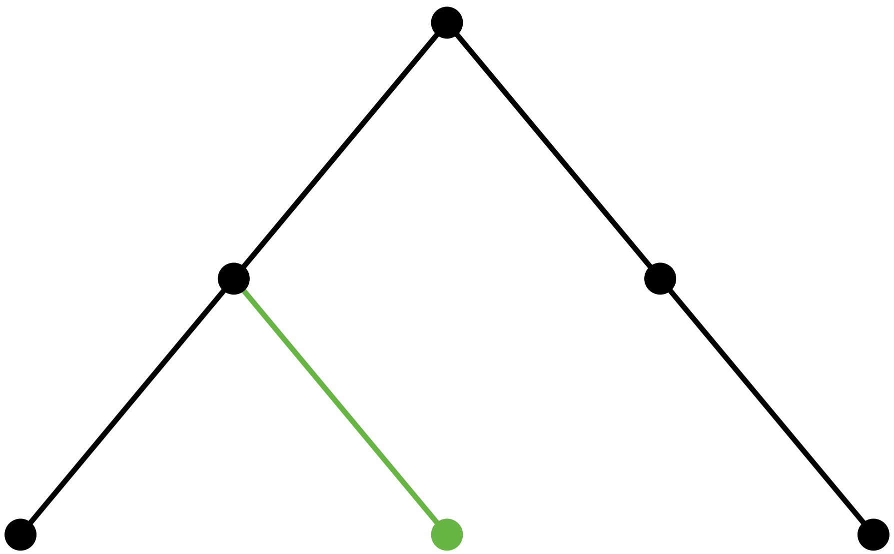

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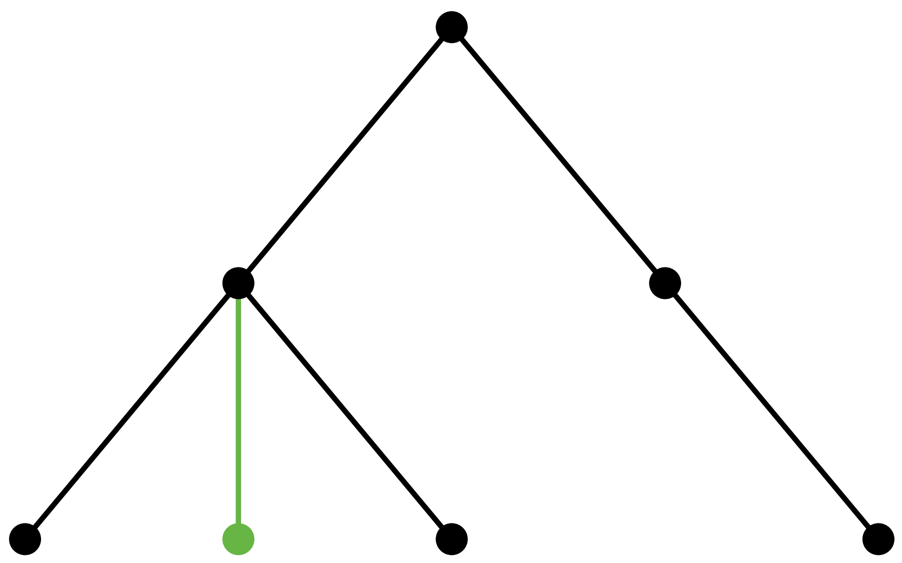

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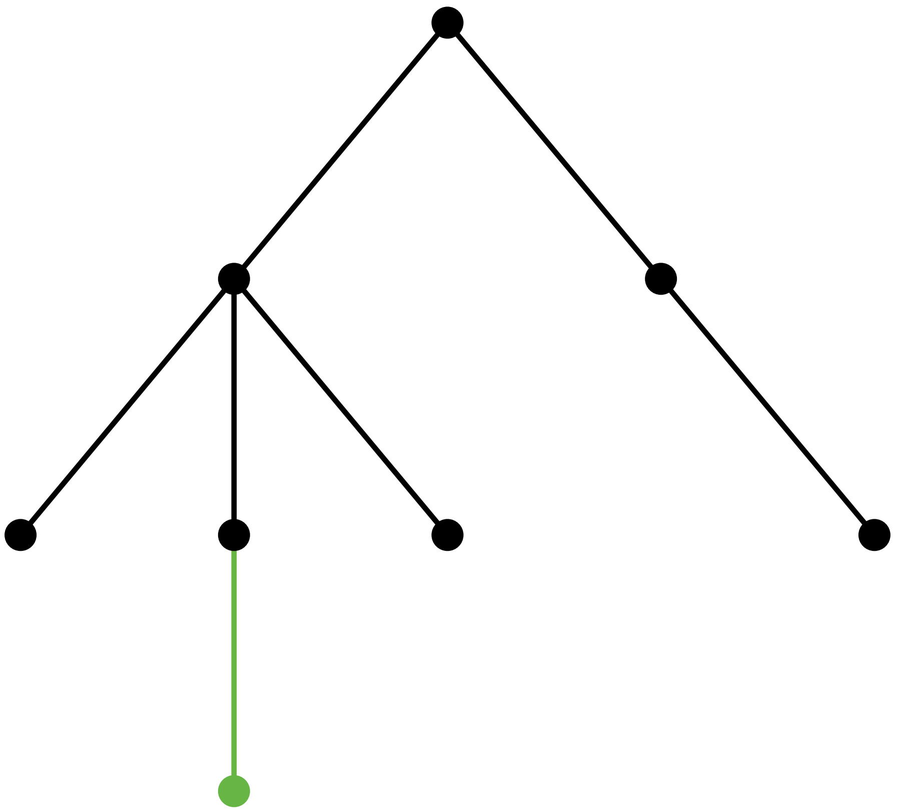

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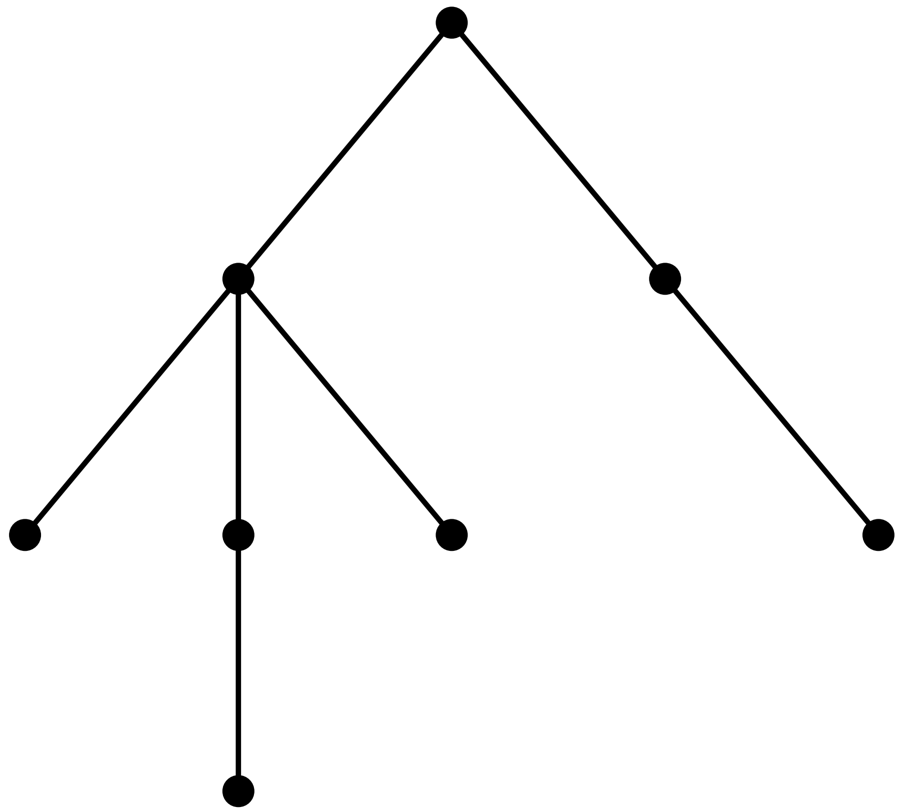

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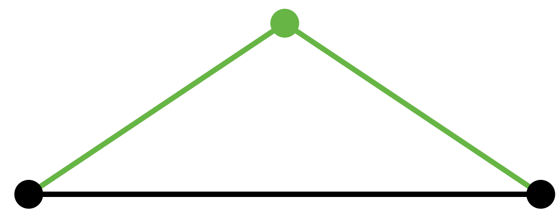

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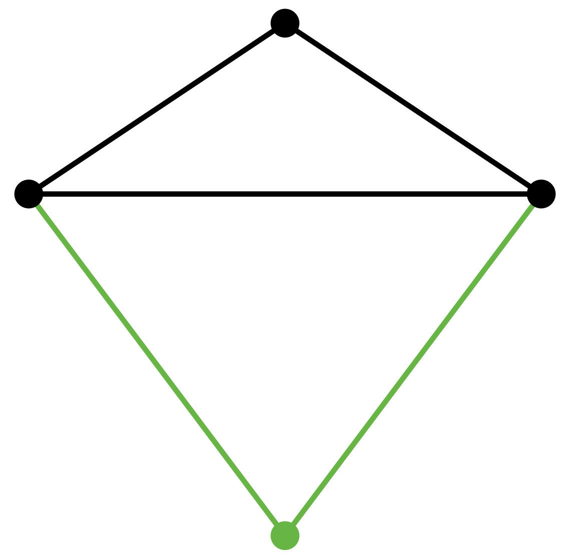

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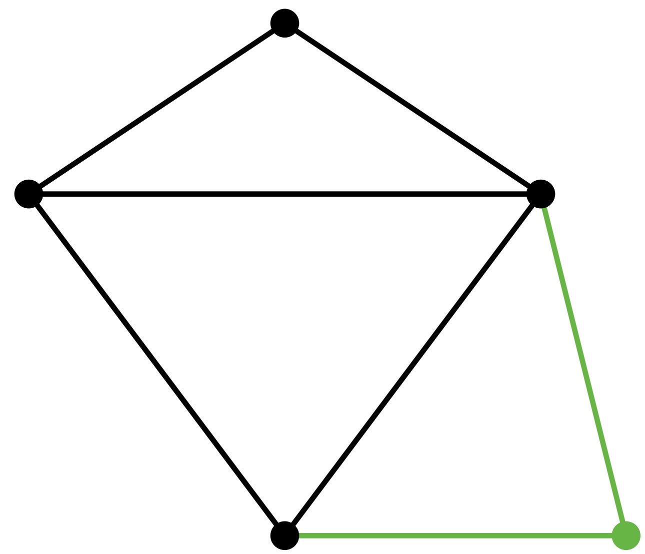

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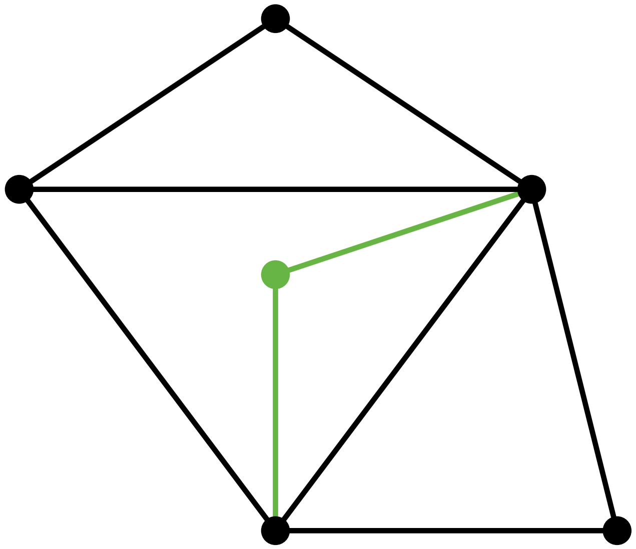

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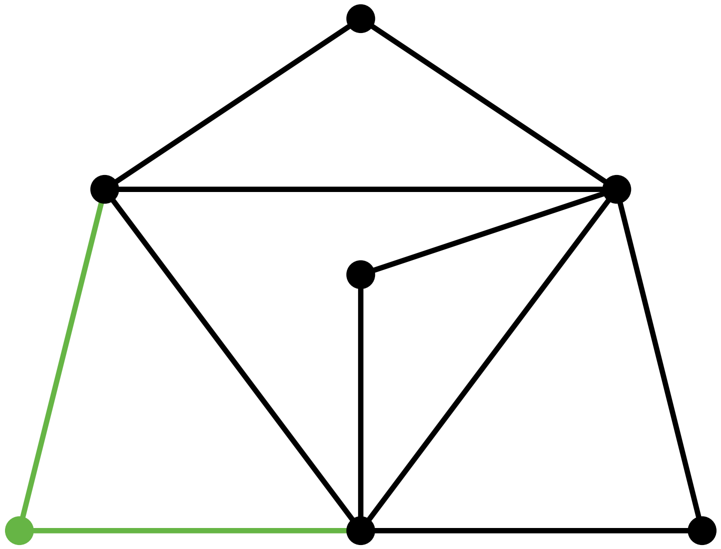

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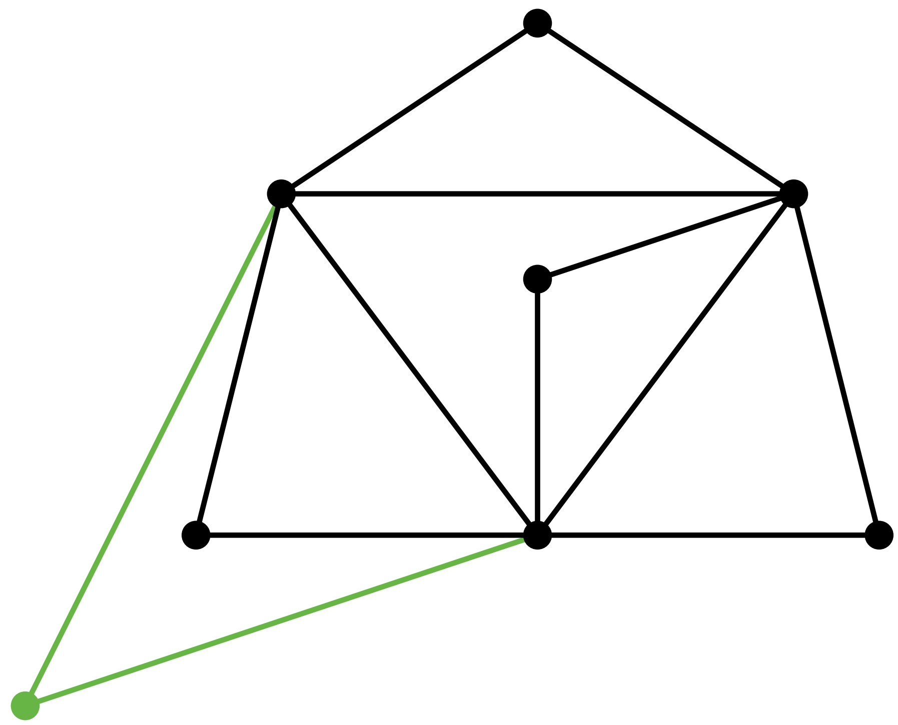

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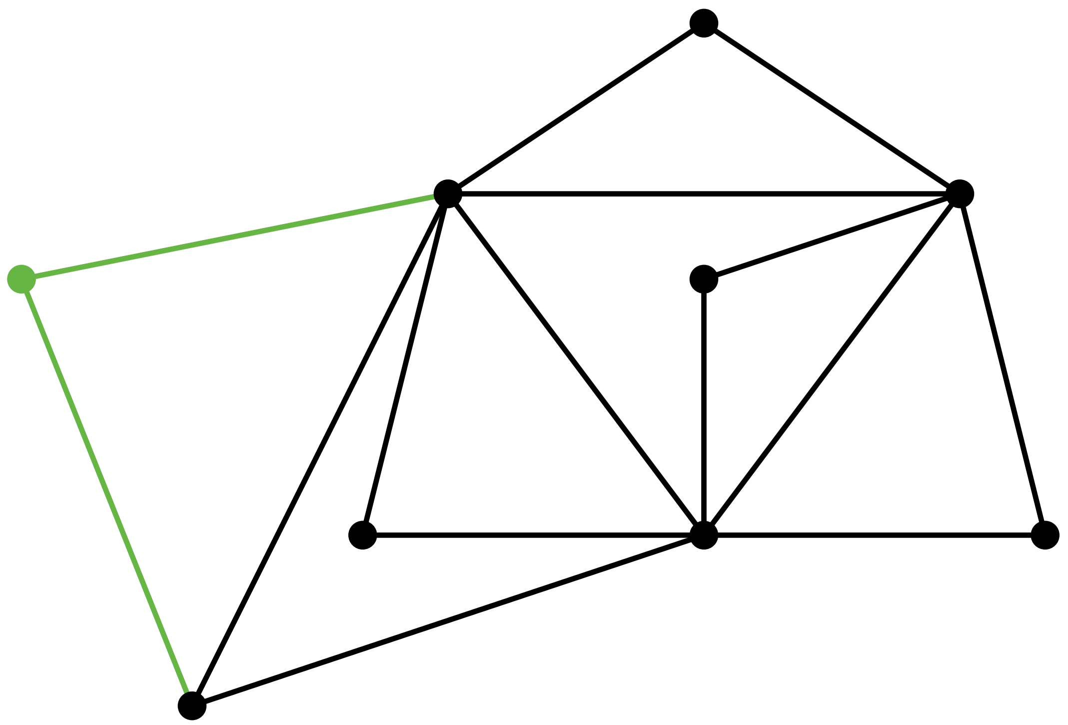

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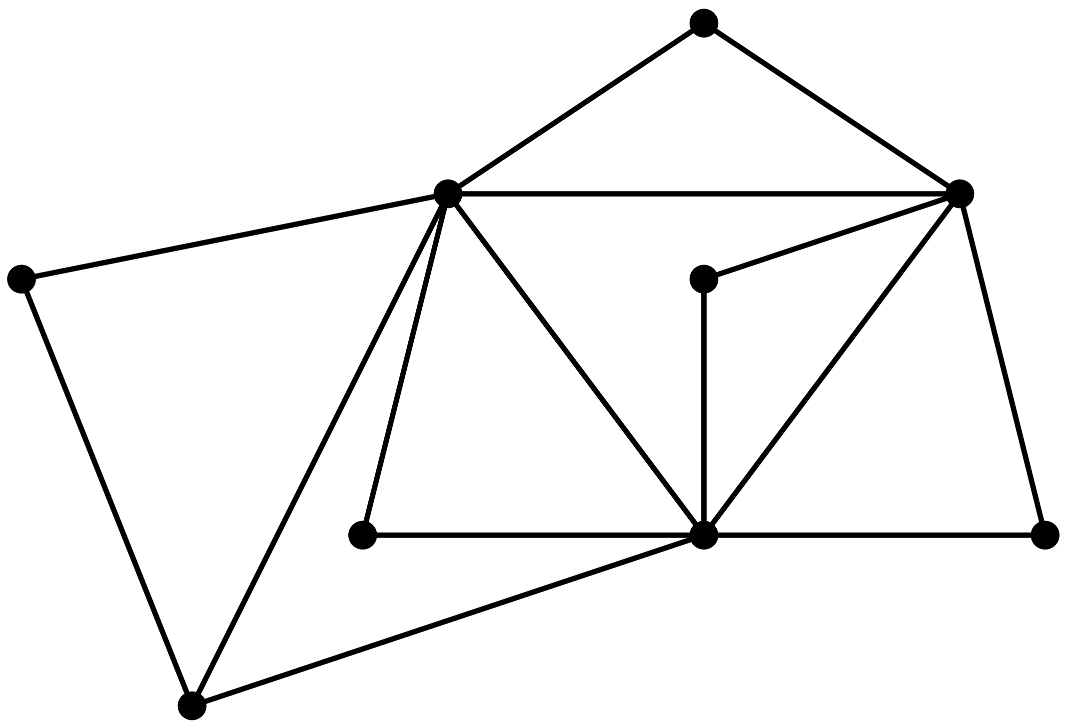

2-trees

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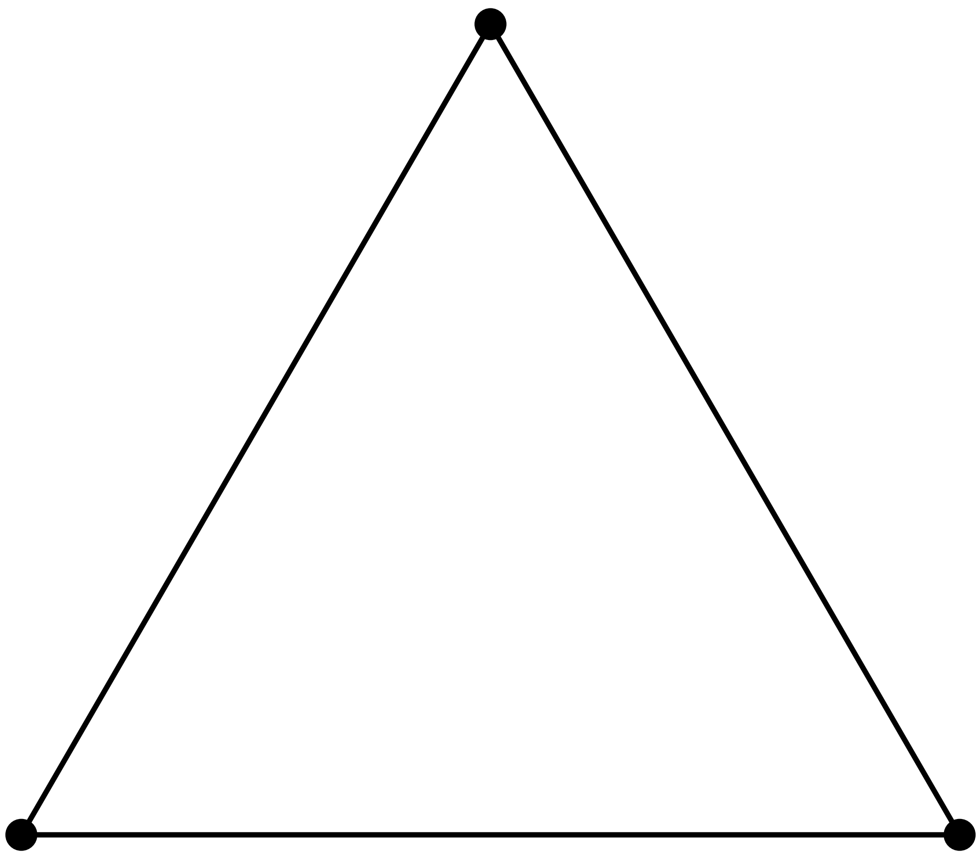

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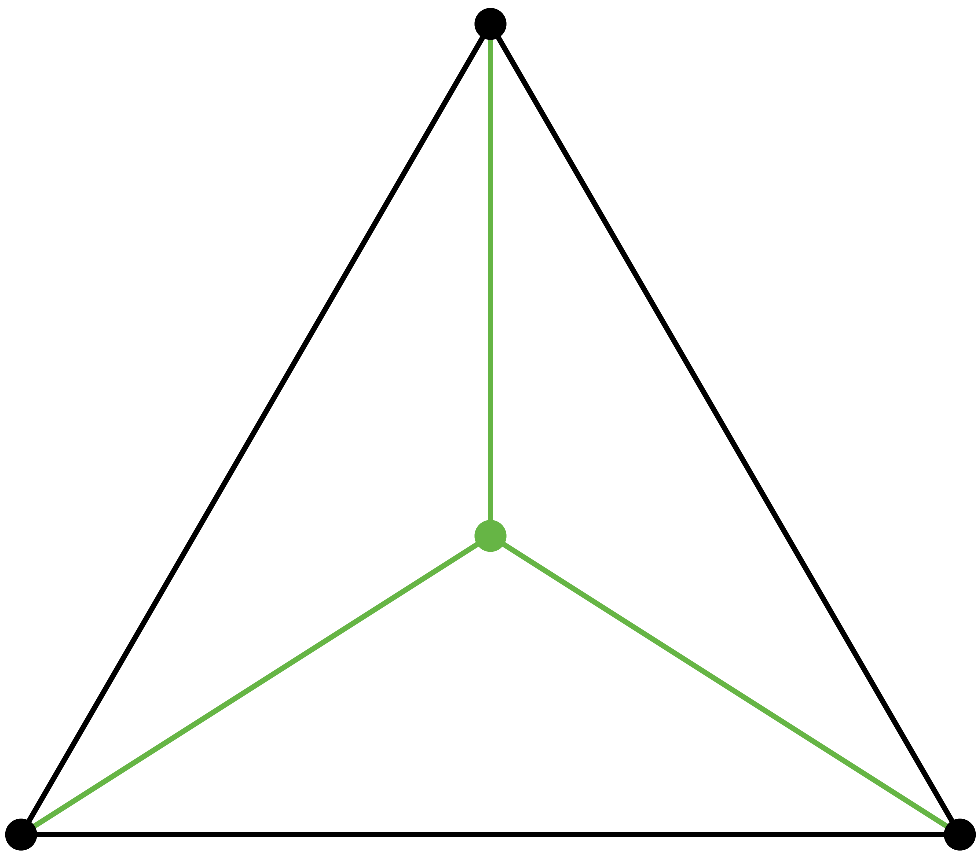

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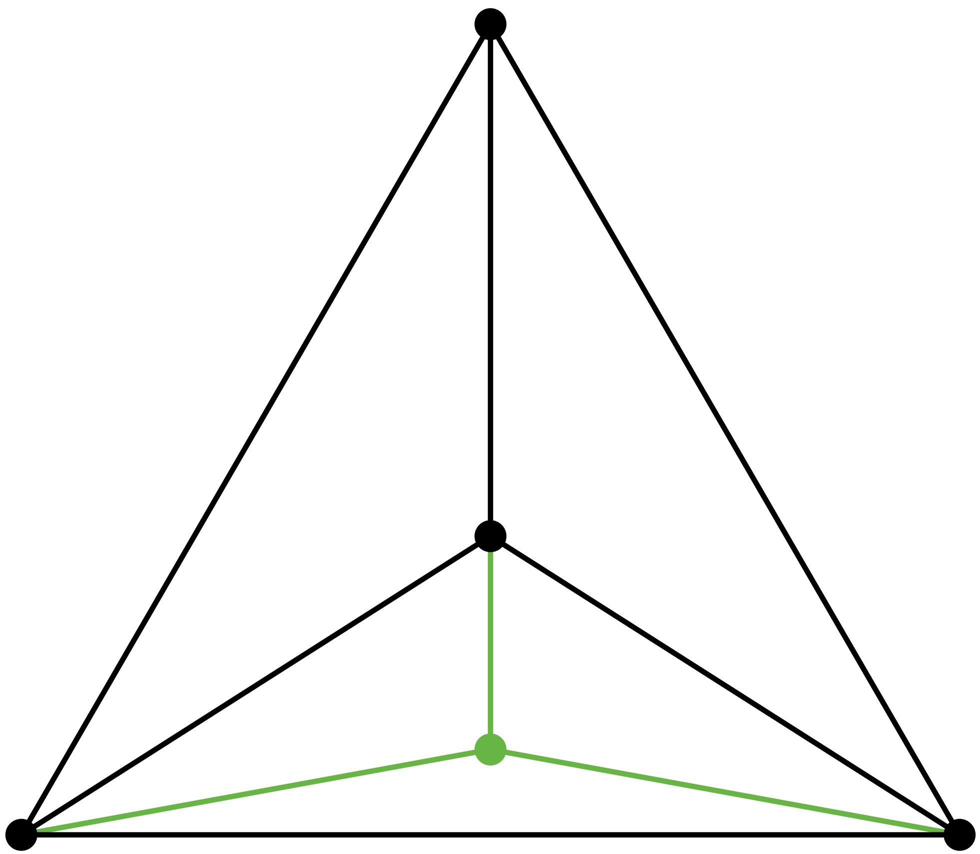

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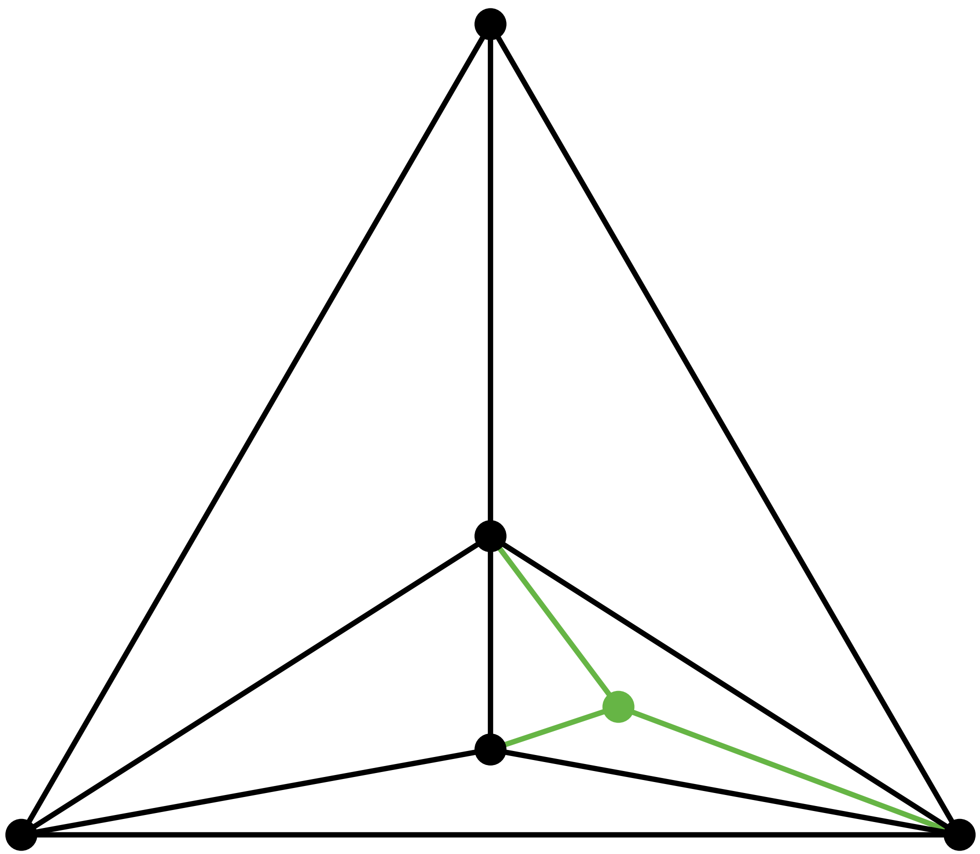

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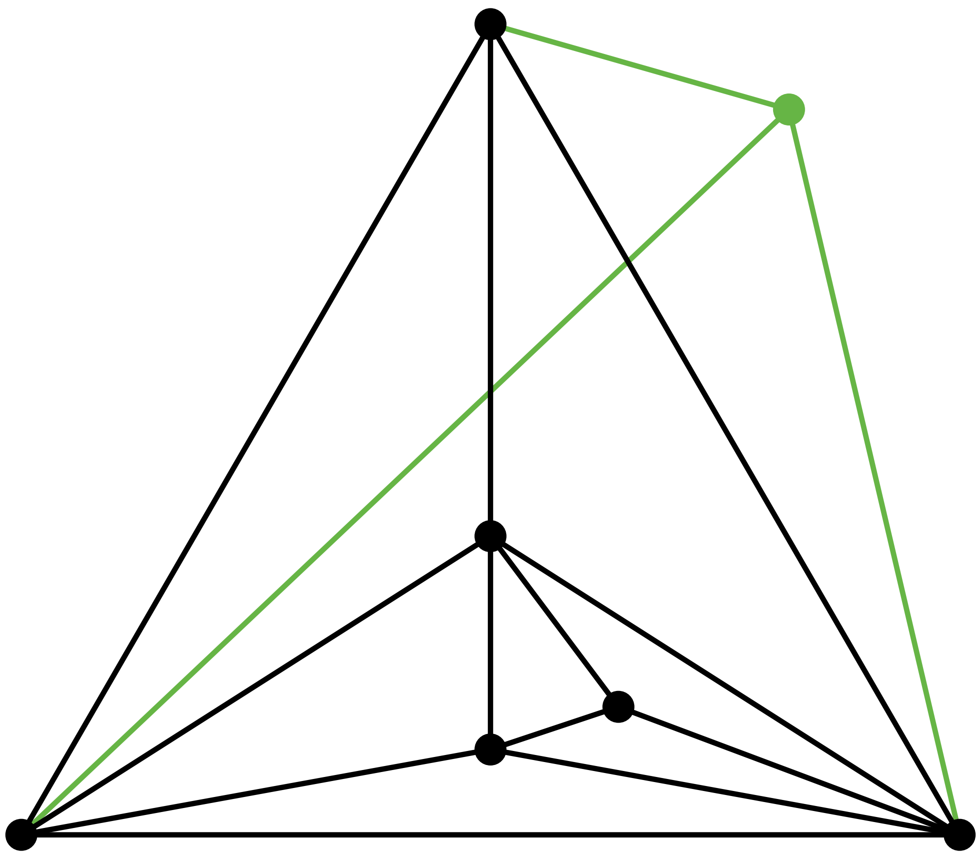

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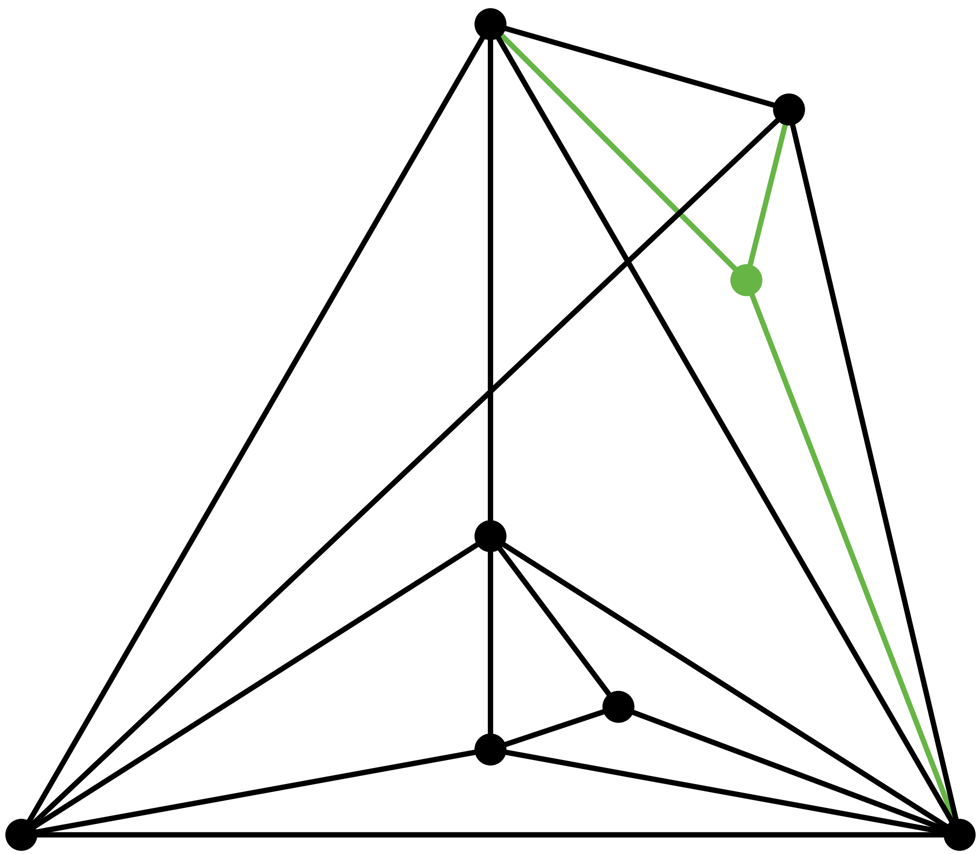

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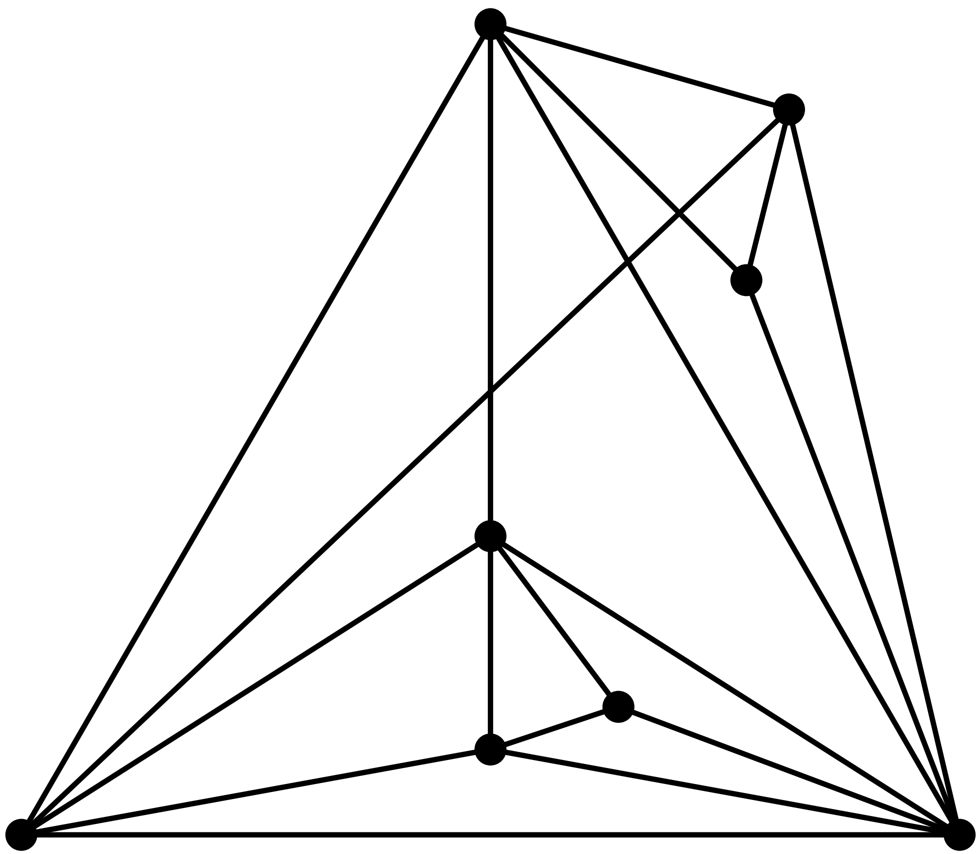

3-trees

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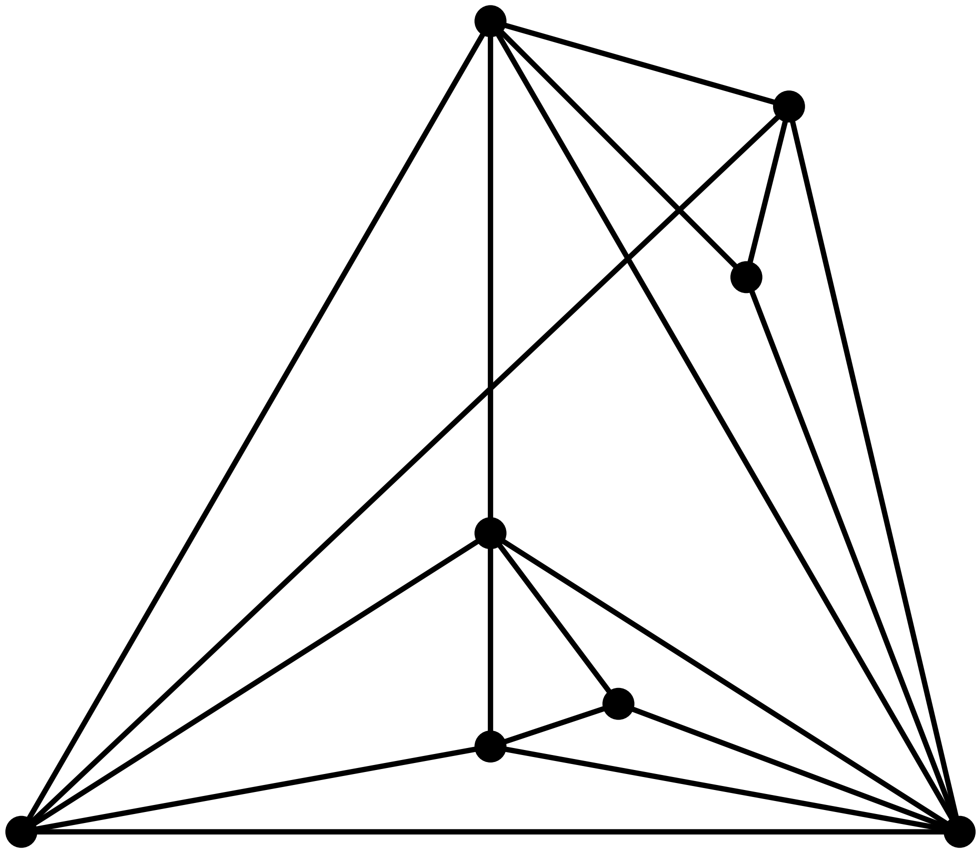

3-trees

Definition. A k-**tree** is a graph obtained from a $(k+1)$-clique by successively adding a new vertex connected to all vertices of an existing k-clique.

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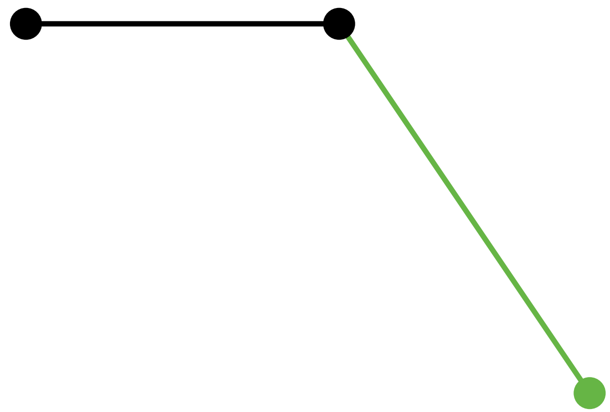

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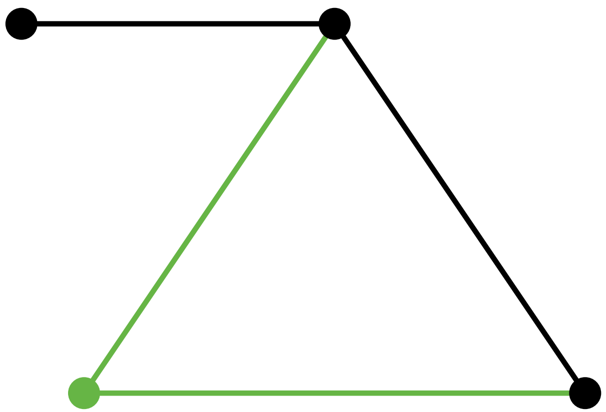

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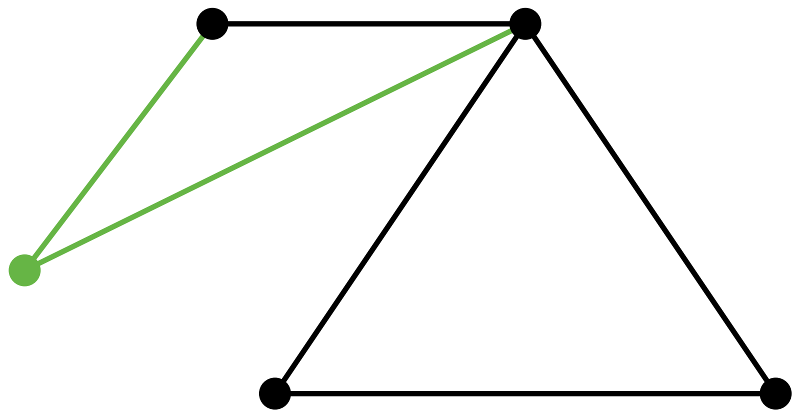

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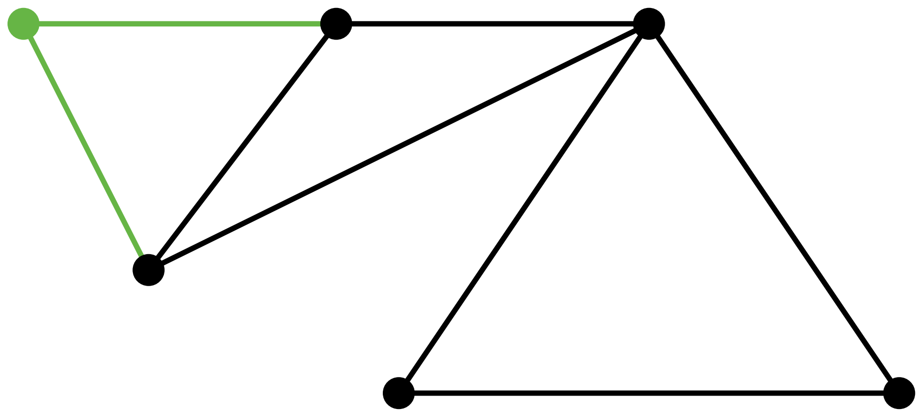

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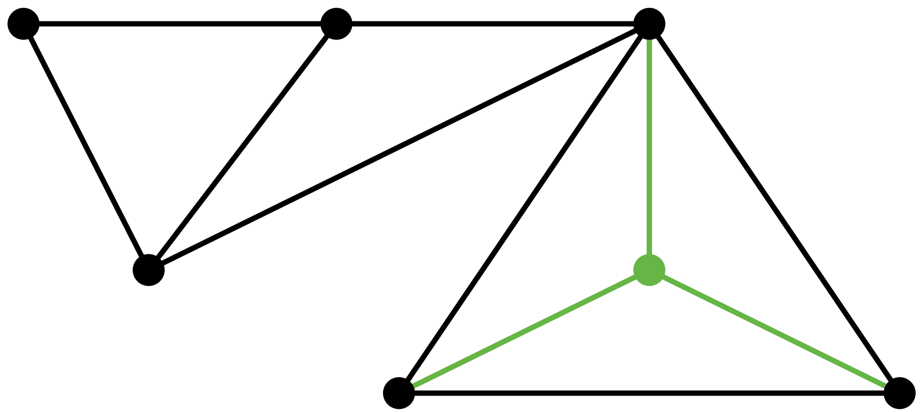

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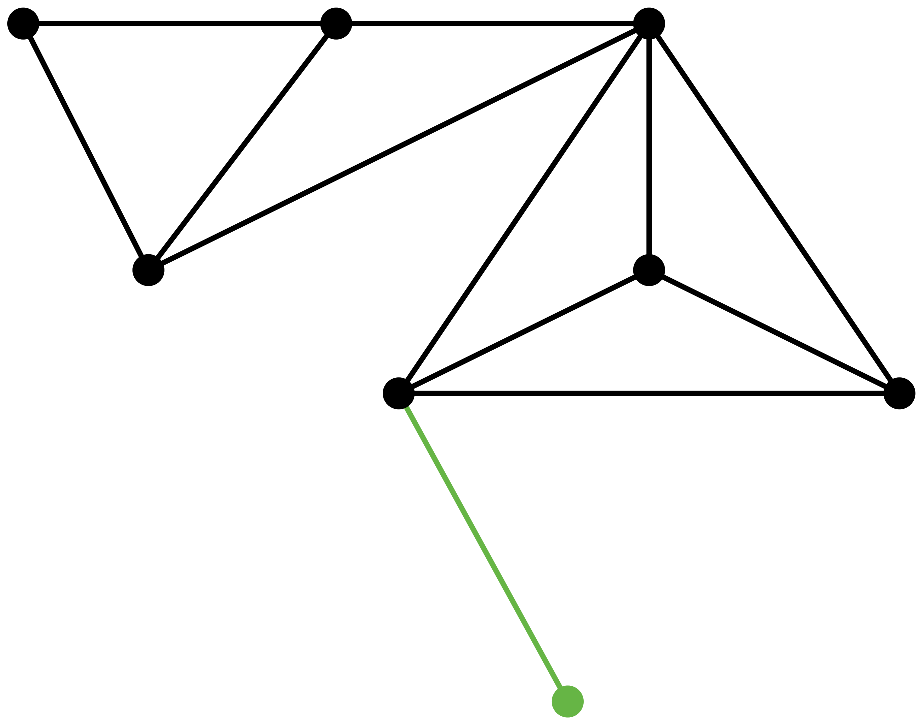

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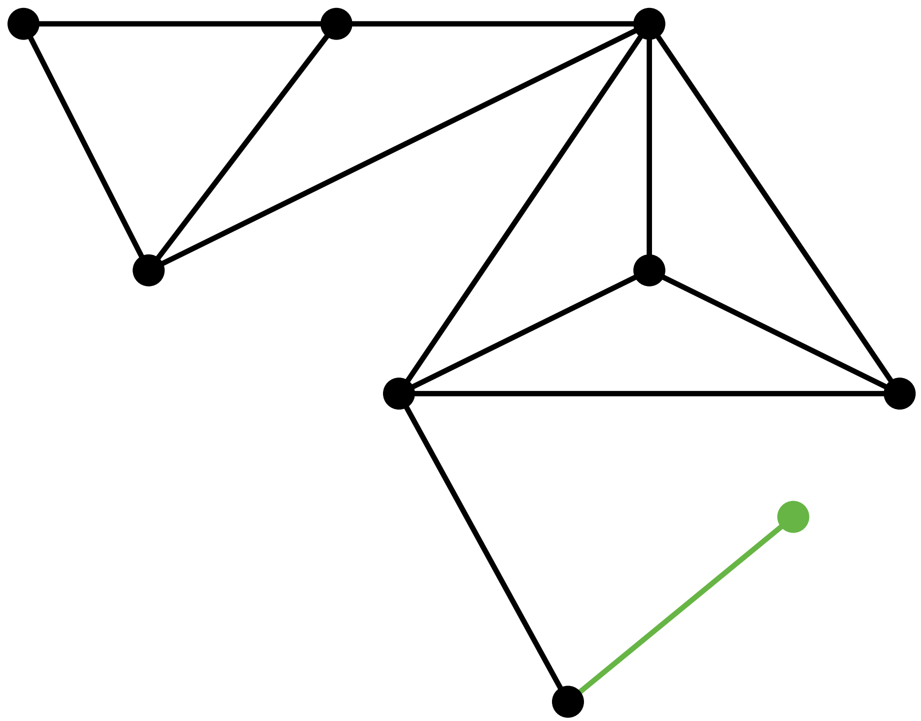

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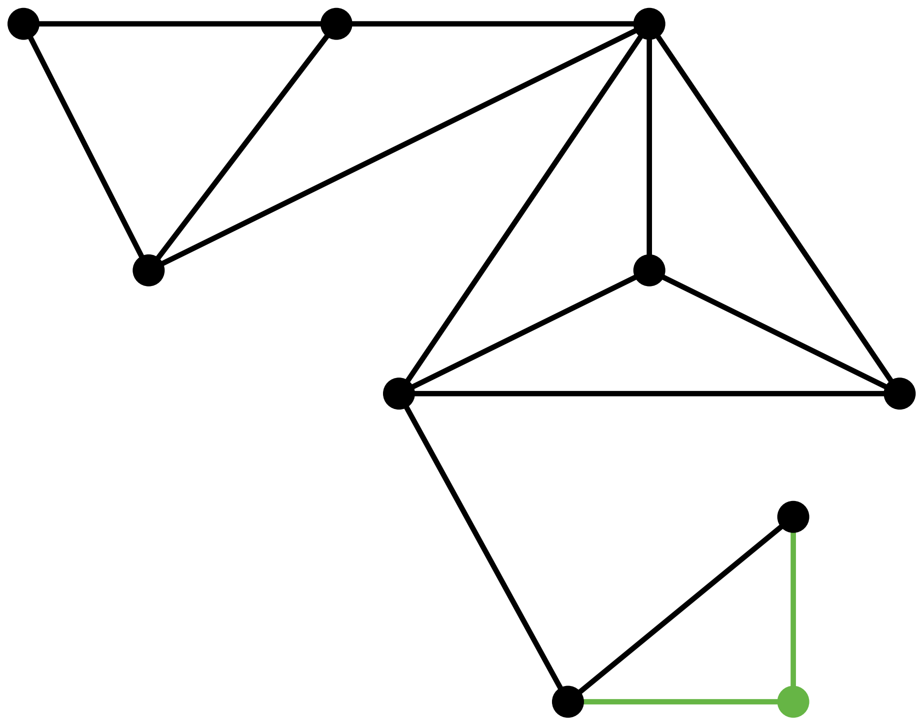

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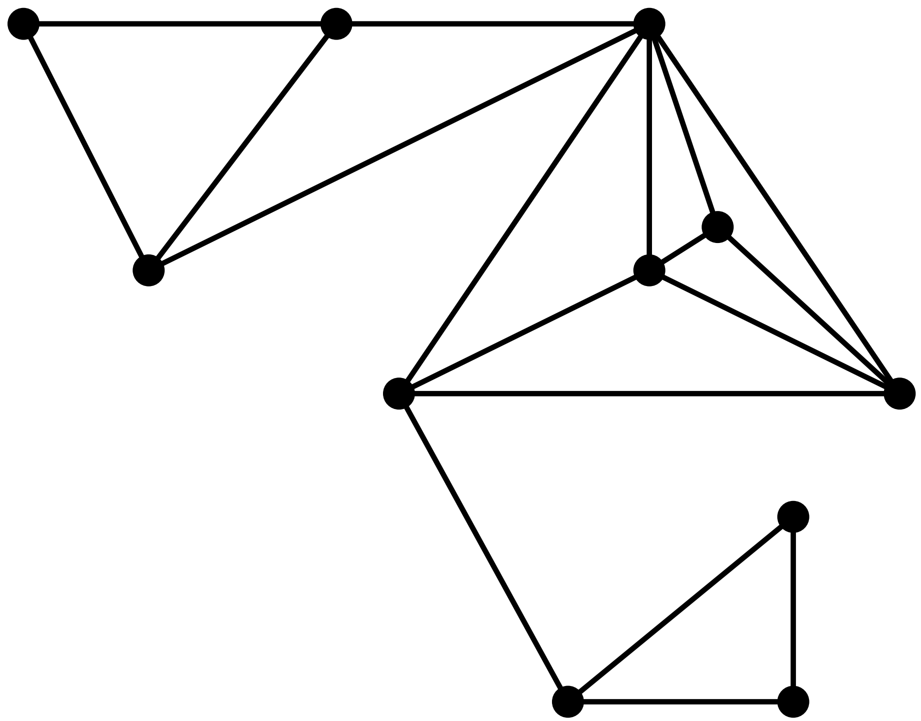

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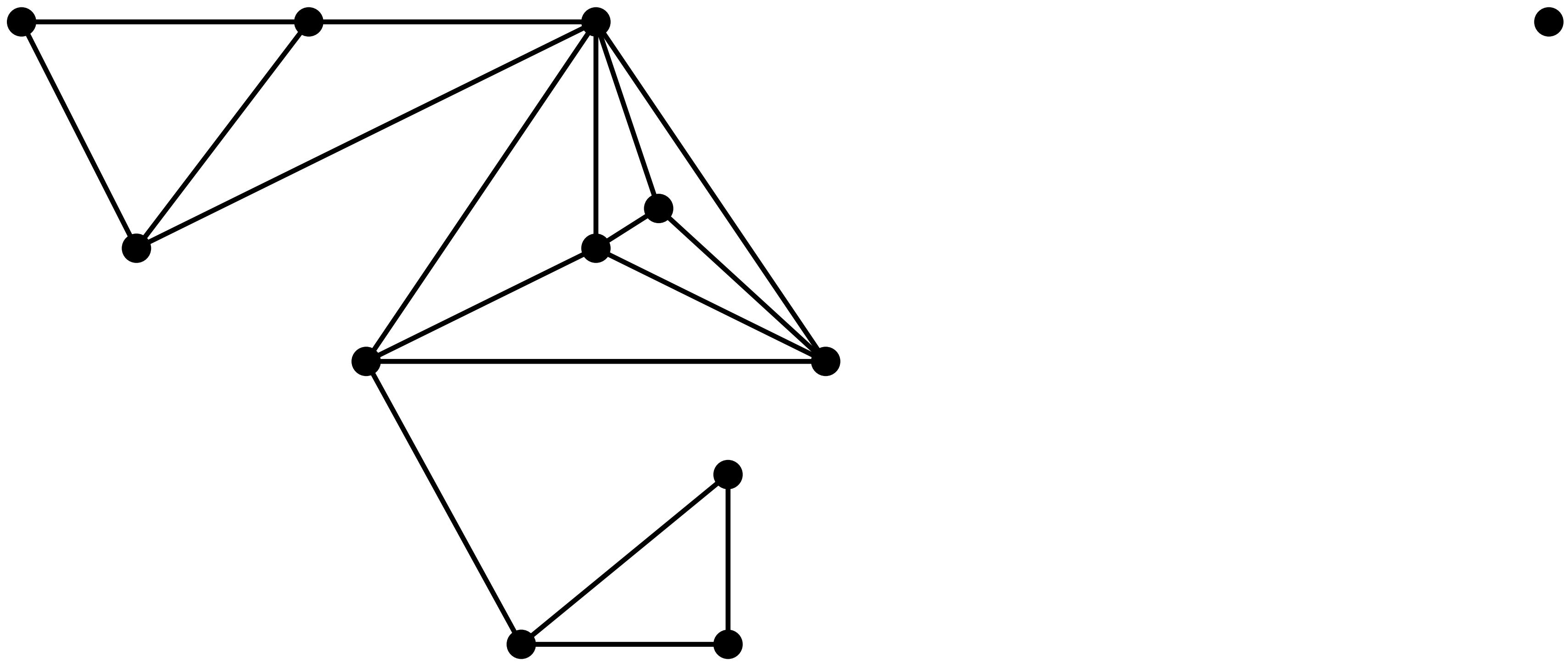

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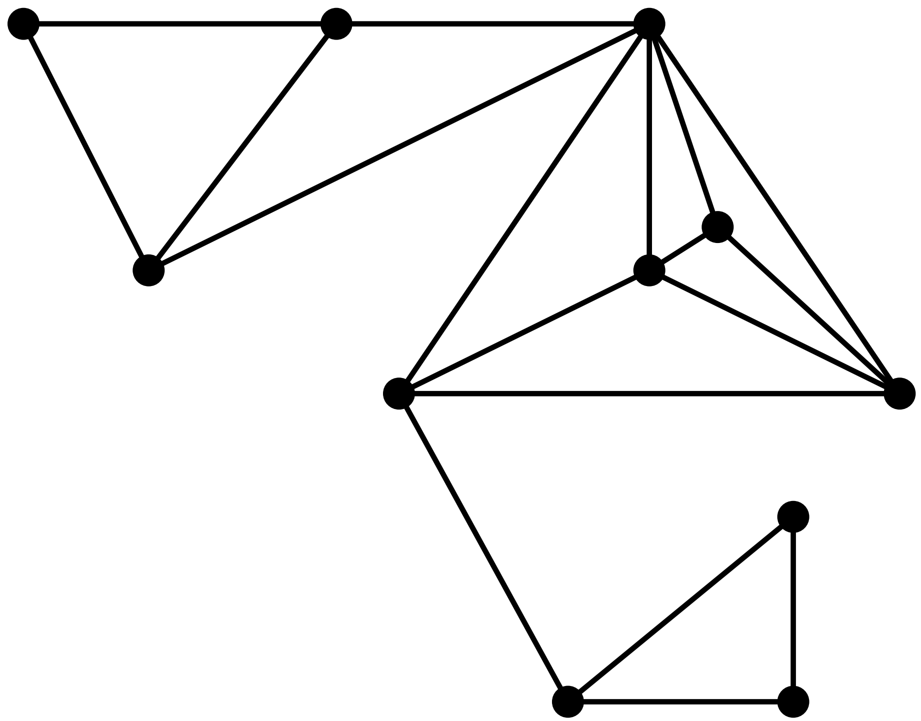

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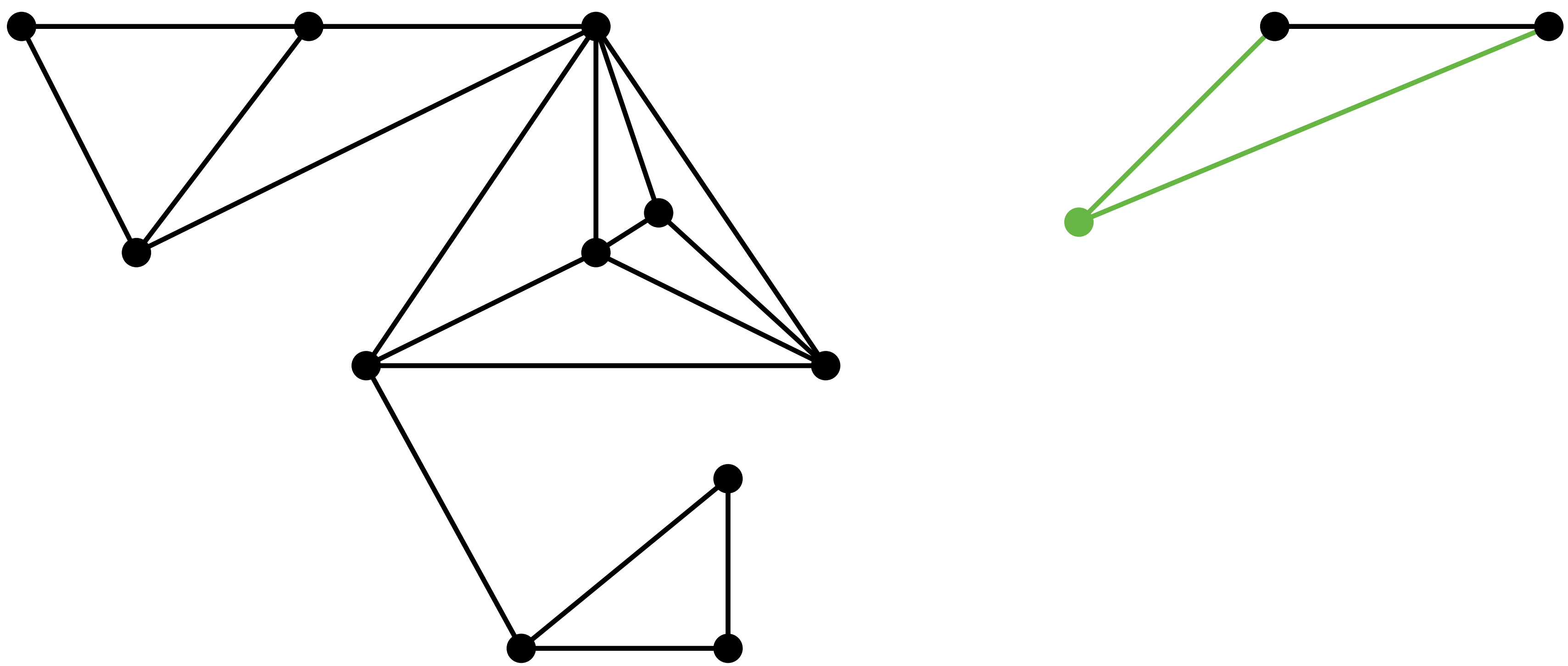

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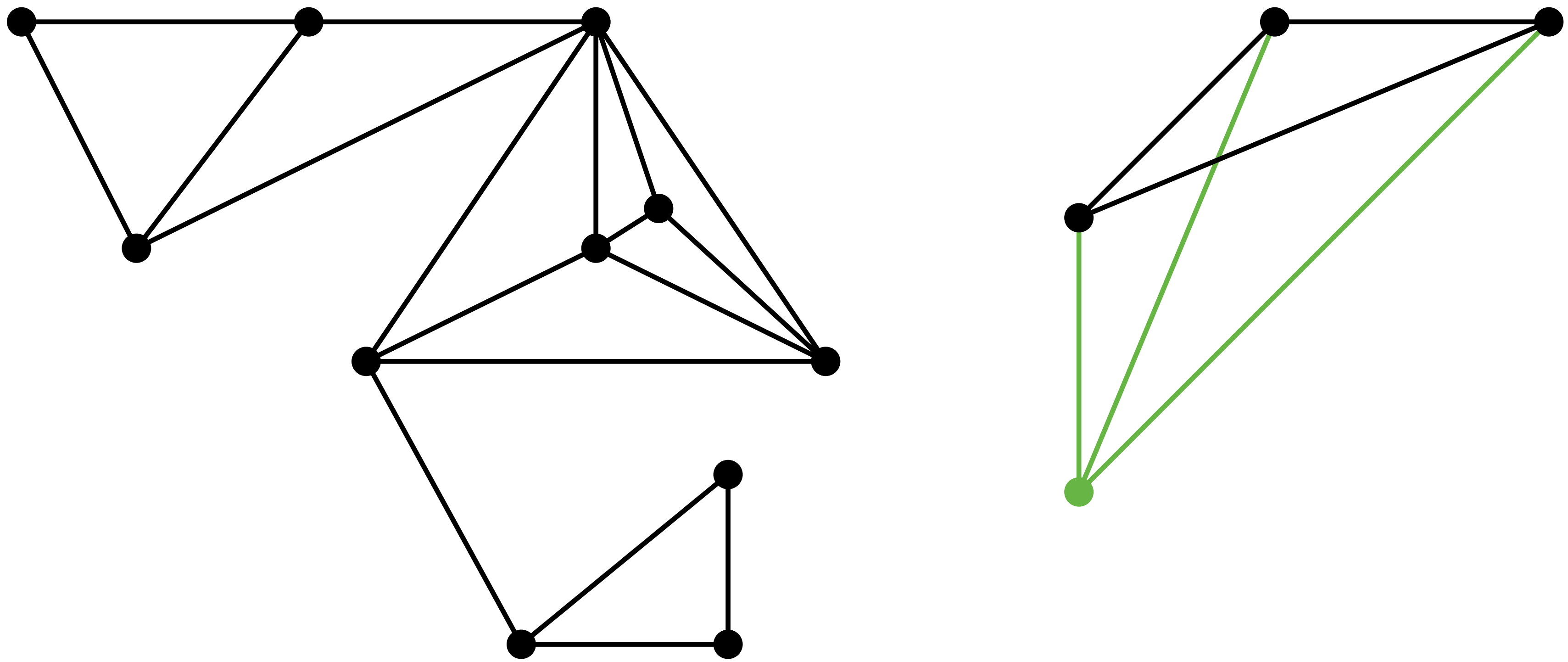

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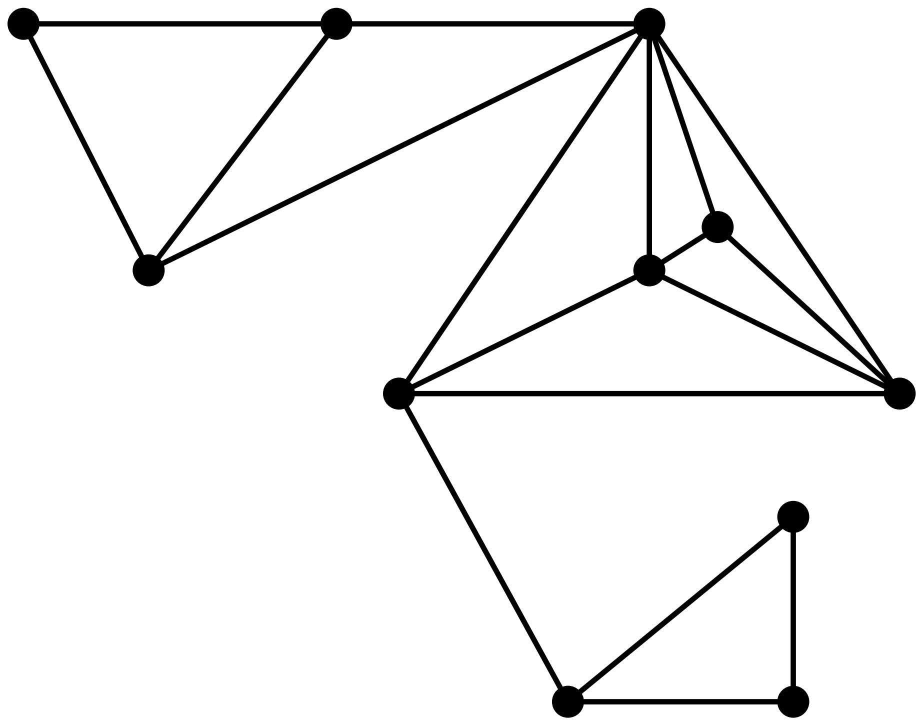

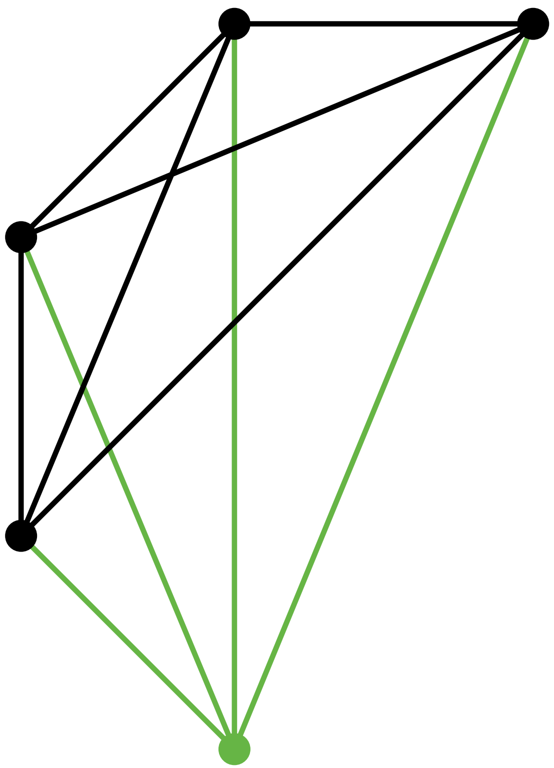

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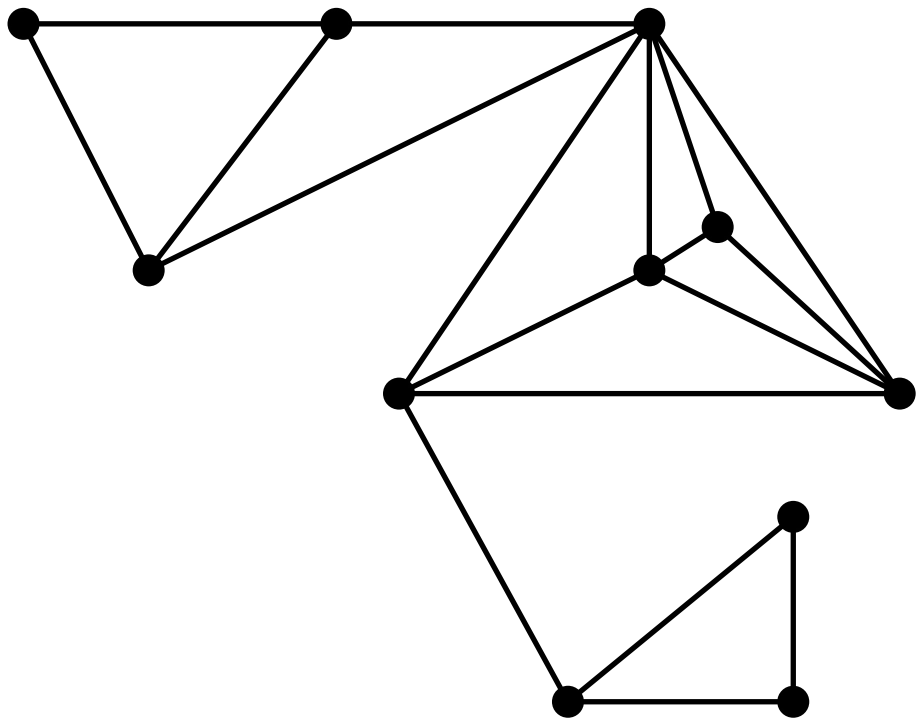

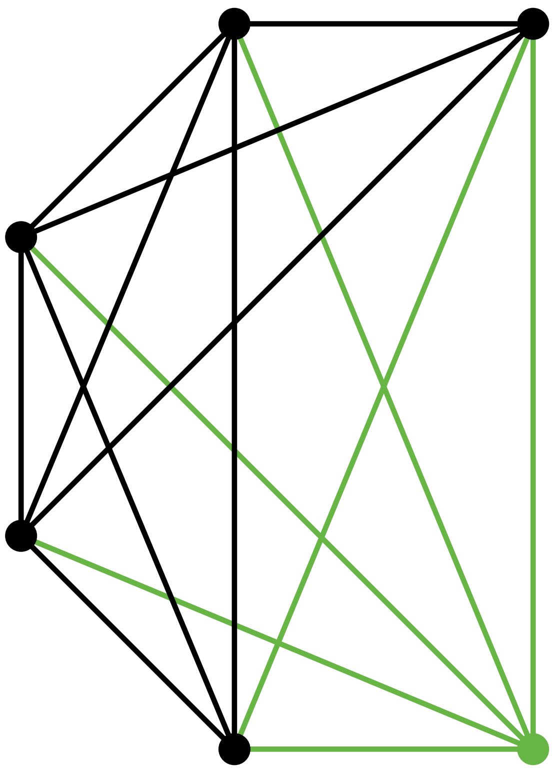

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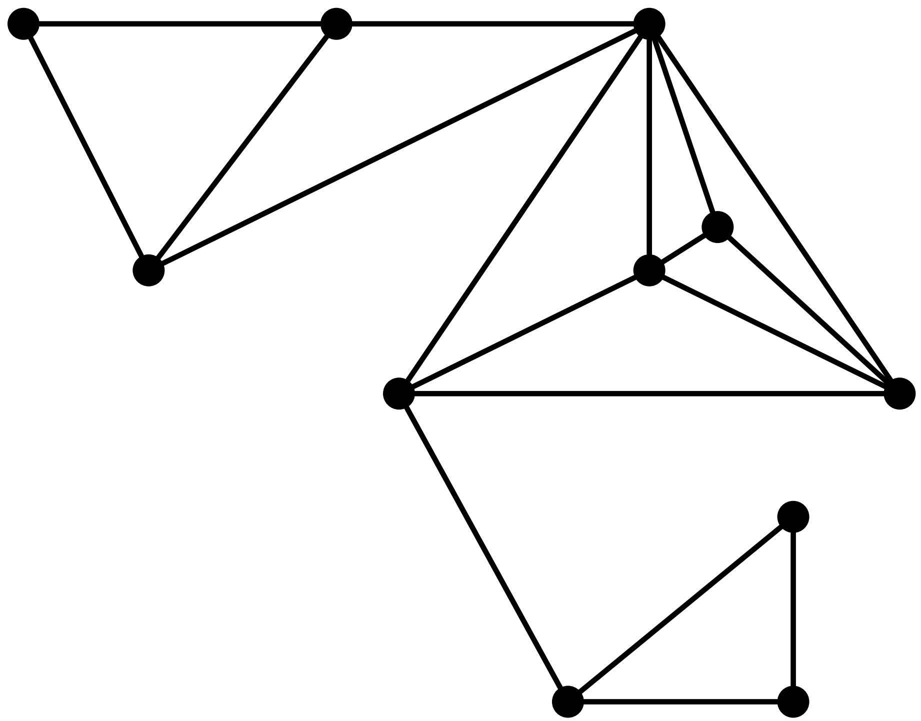

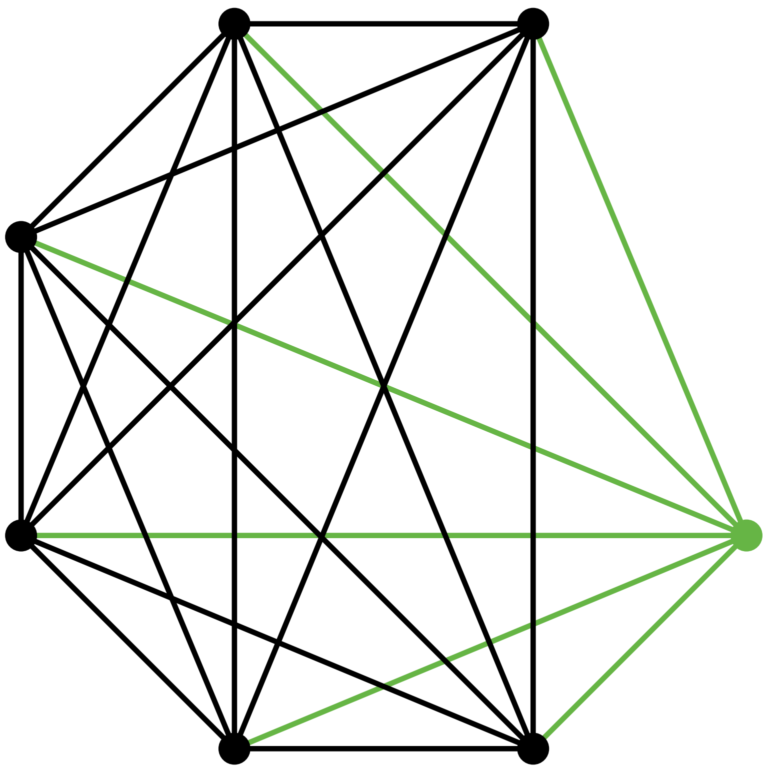

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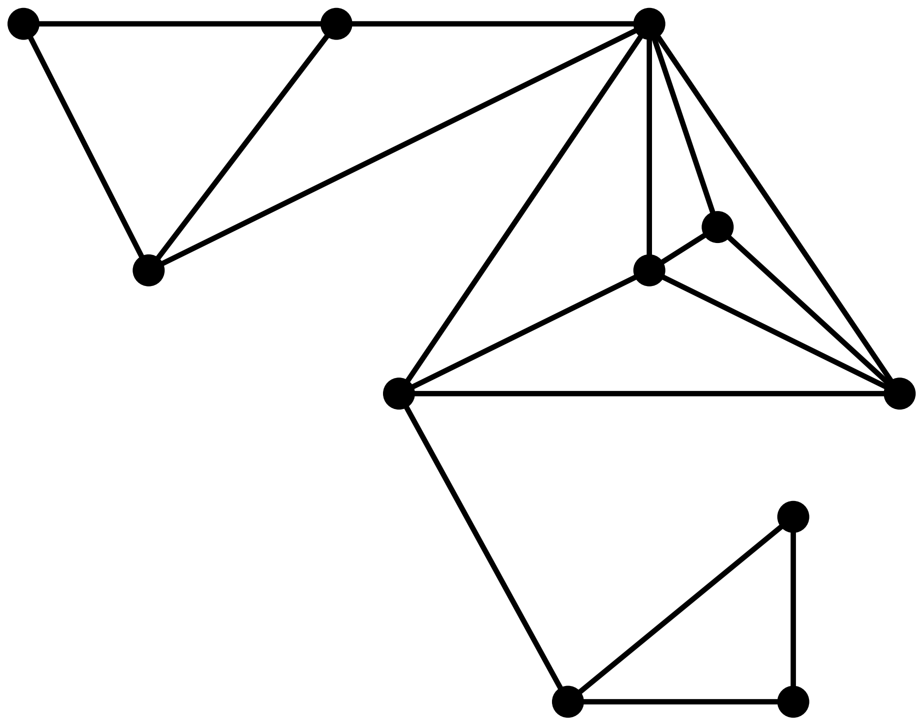

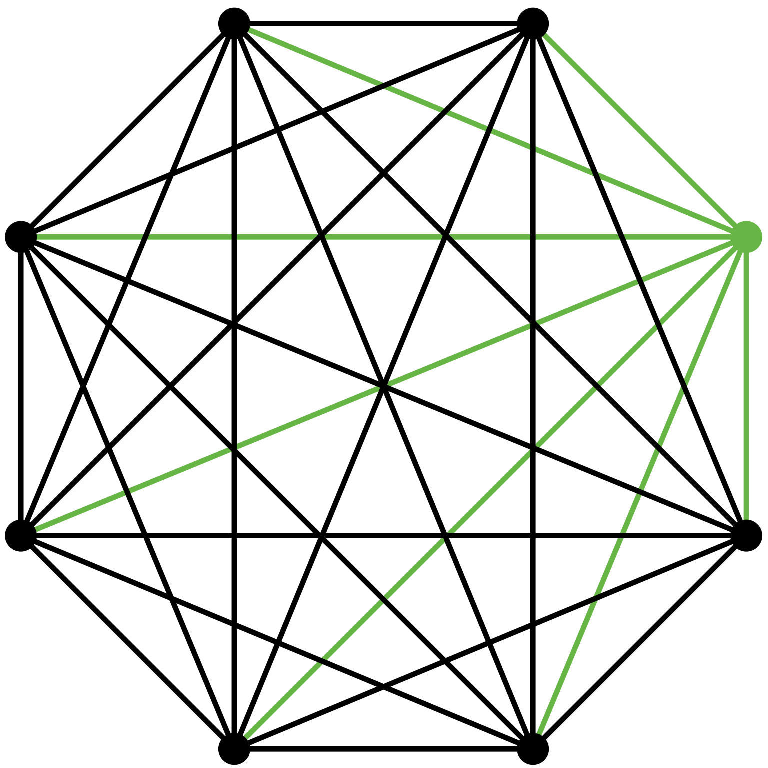

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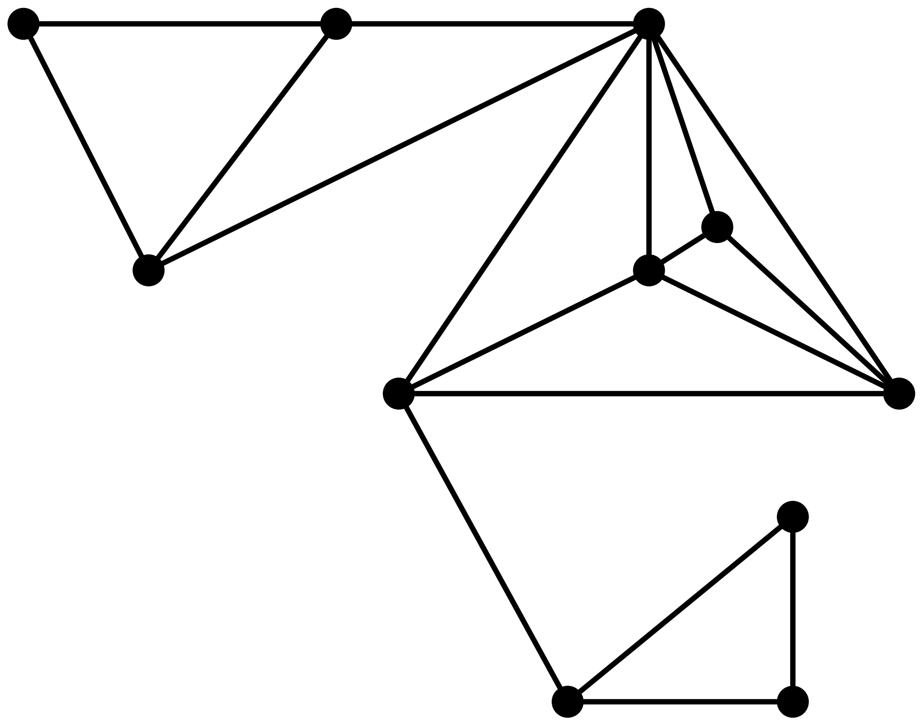

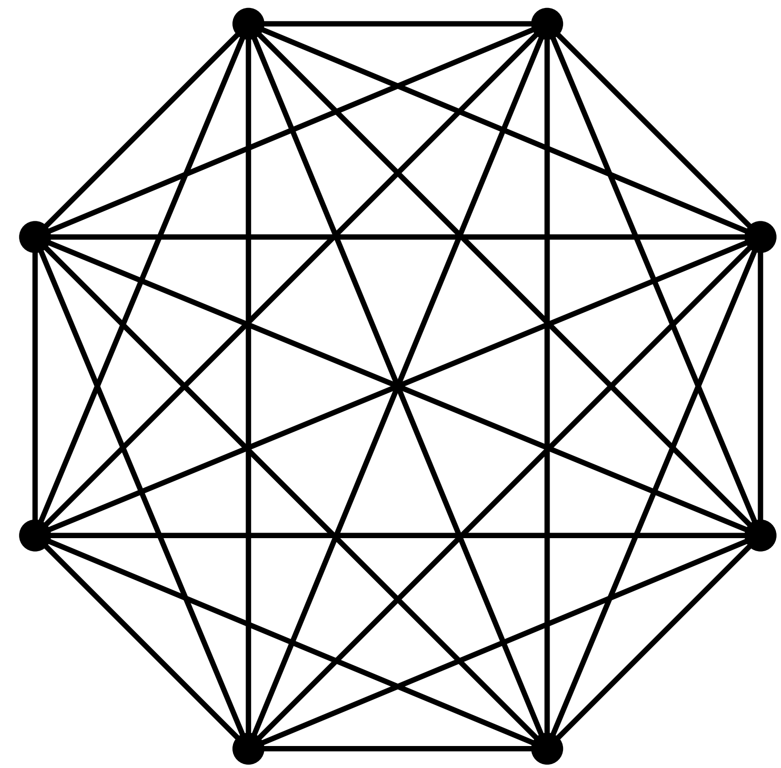

Chordal graphs

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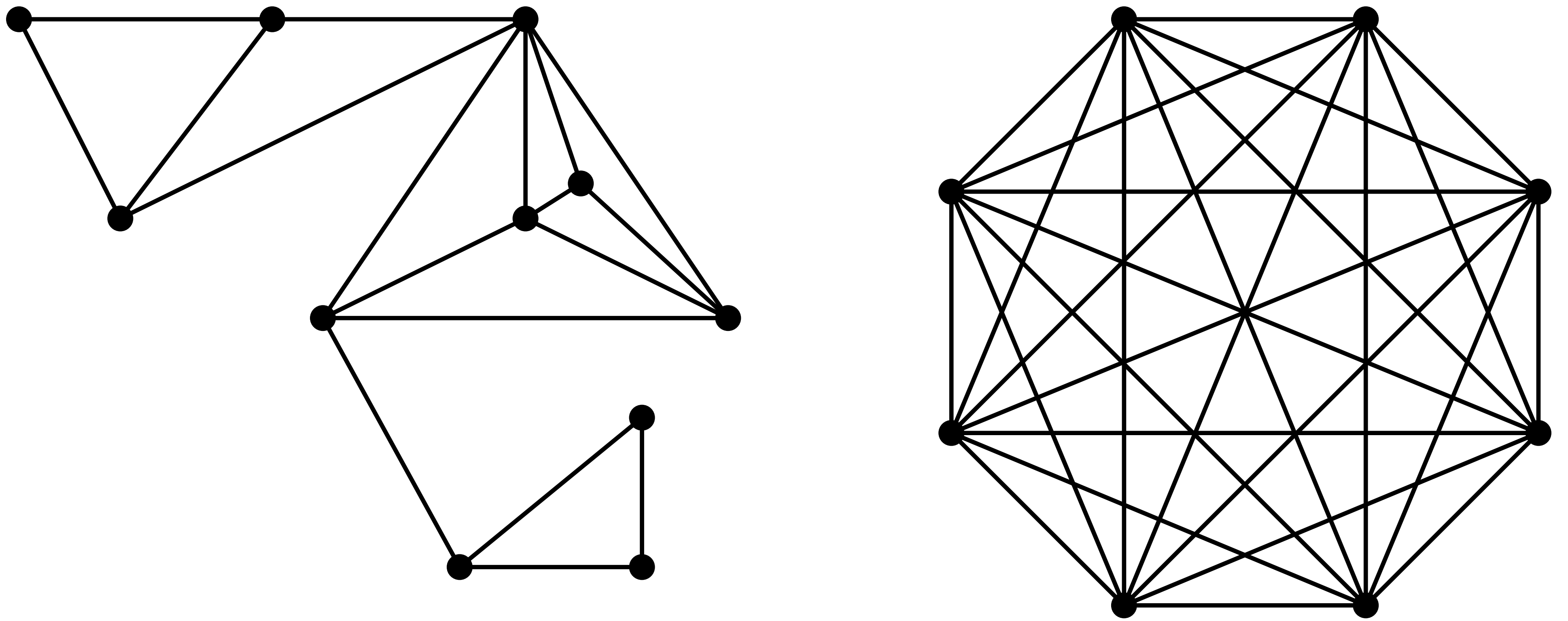

Chordal graphs

Definition. A graph is **chordal** if it has no induced cycle of lengh greater than 3.

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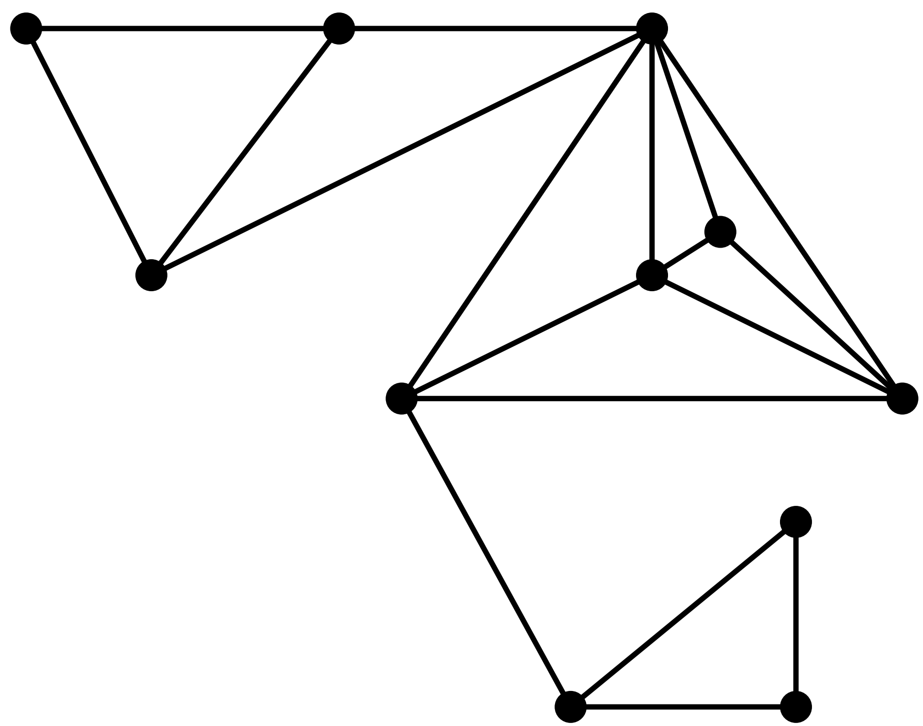

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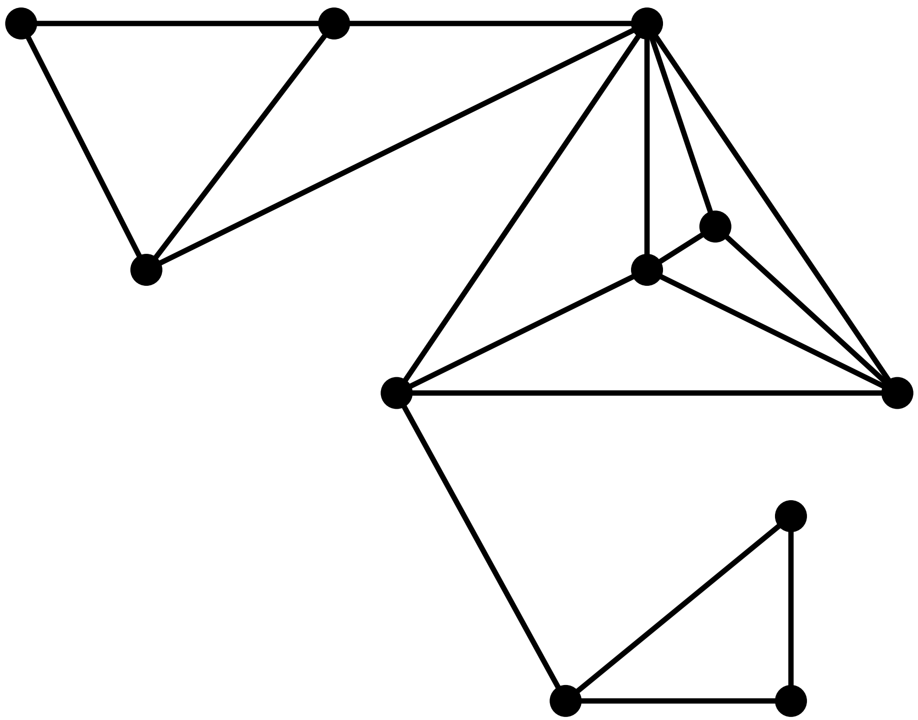

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Chordal graphs with tree-width at most t

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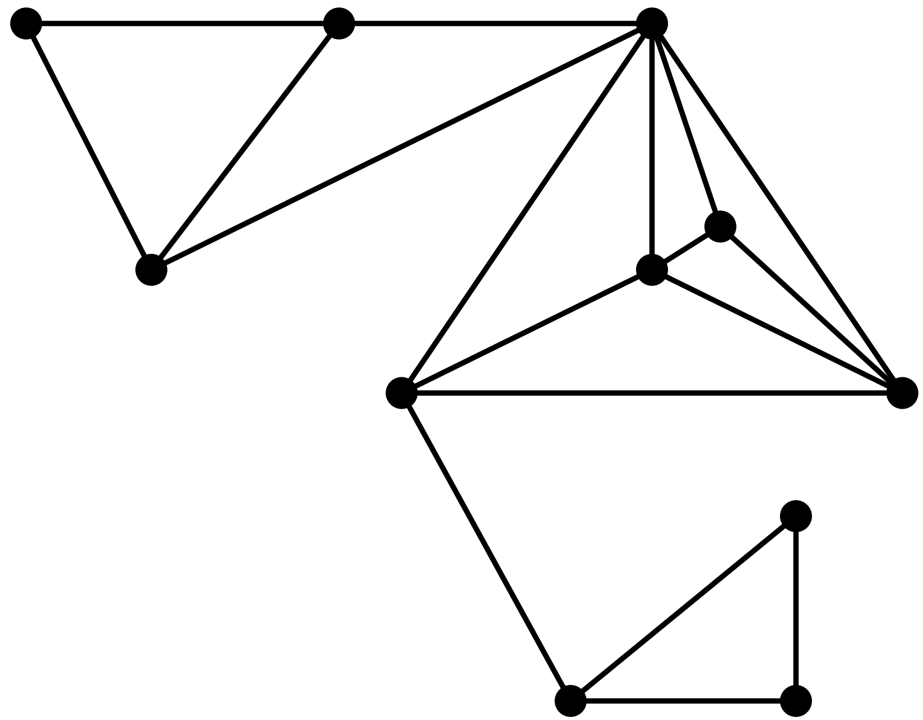

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Chordal graphs with tree-width at most t

Definition. The **tree-width** of a graph G is the minimum k such that G is the subgraph of a k-tree.

Labelled vs unlabelled

A graph with n vertices is **labelled** if each vertex carries a different label in $\{1, 2, \ldots, n\}$.

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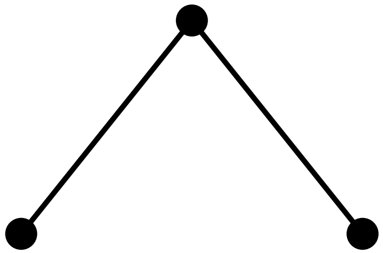

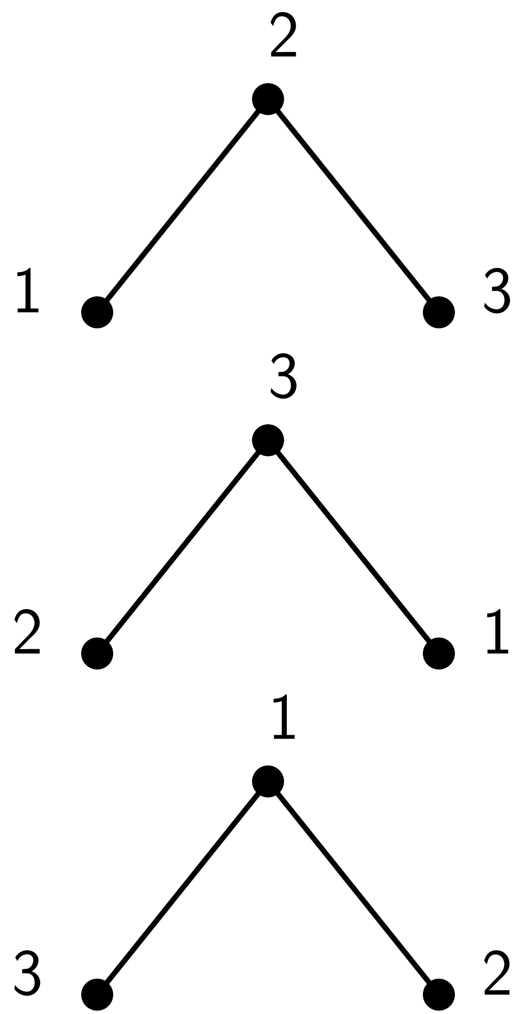

The symbolic method

Our goal is to determine the number of graphs in the family with size n.

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Definition. A **combinatorial class** is a pair $(\mathcal{A}, |\cdot|)$ where

- $\mathcal{A}$ is a family of combinatorial objects,
- $|\cdot| : \mathcal{A} \to \mathbb{N}$ is a size function,
- The number of objects with size n is $a_n < \infty$.

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$$A(x) = \sum_{n \geq 0} a_n x^n.$$

Suitable for unlabelled classes.

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Operations between **classes** translate into relations involving their **generating functions**. **The goal** is to obtain (a system of) equations that determine the GF of our class.

Labelled trees

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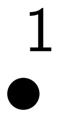

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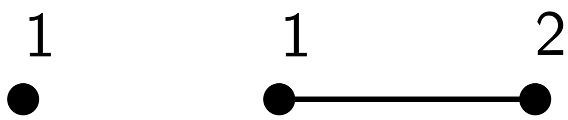

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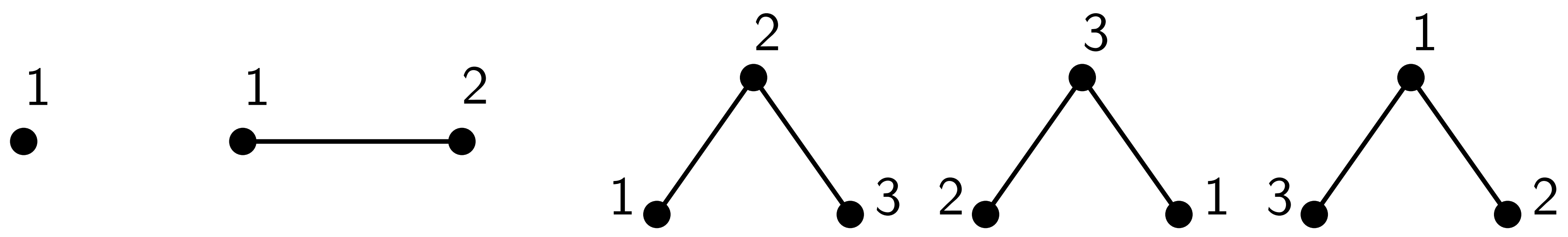

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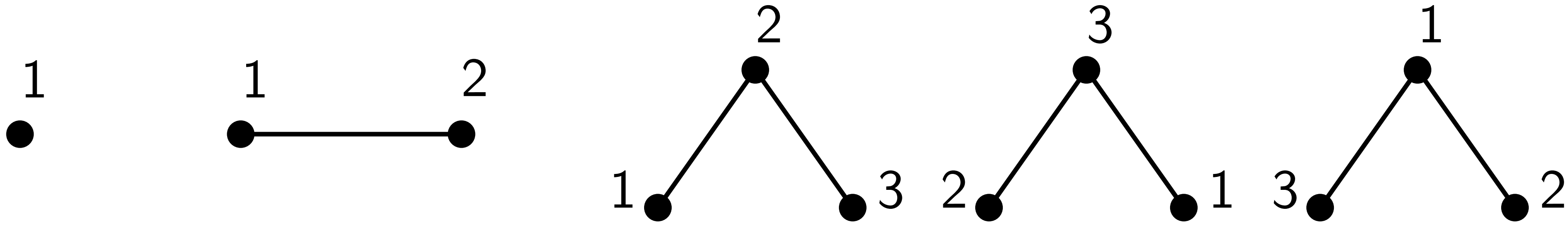

Rooting. Let $\mathcal{T}^\bullet$ be the class of rooted labelled trees.
Since all vertices are distinguishable, there are n ways to root a tree with n vertices. Thus,

$$T^\bullet(x) = \sum_{n\geq 0} n \frac{t_n}{n!} x^n = xT'(x).$$

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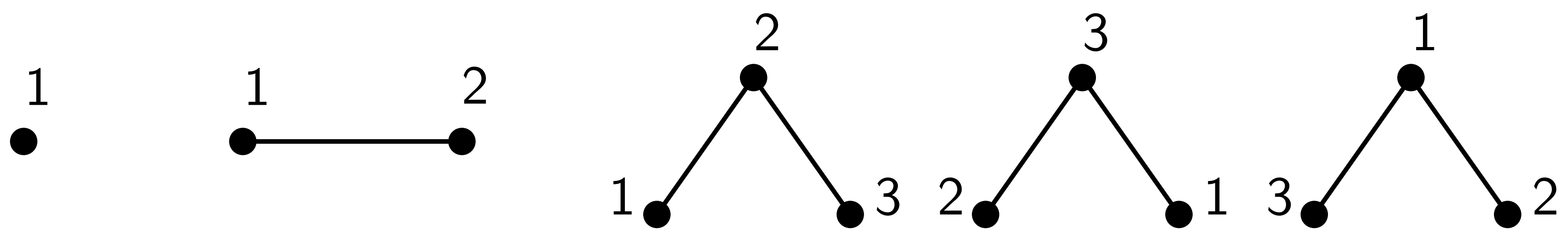

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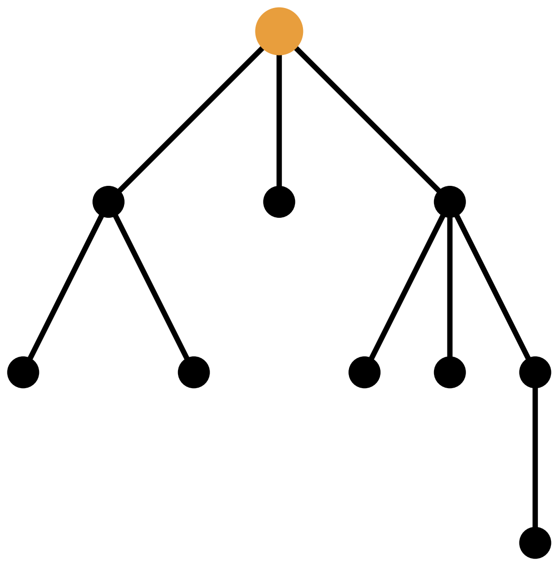
Unrooting. To do the inverse operation, we can simply integrate:

$$T(x) = \int T^\bullet(x)/x\; dx.$$

Labelled trees



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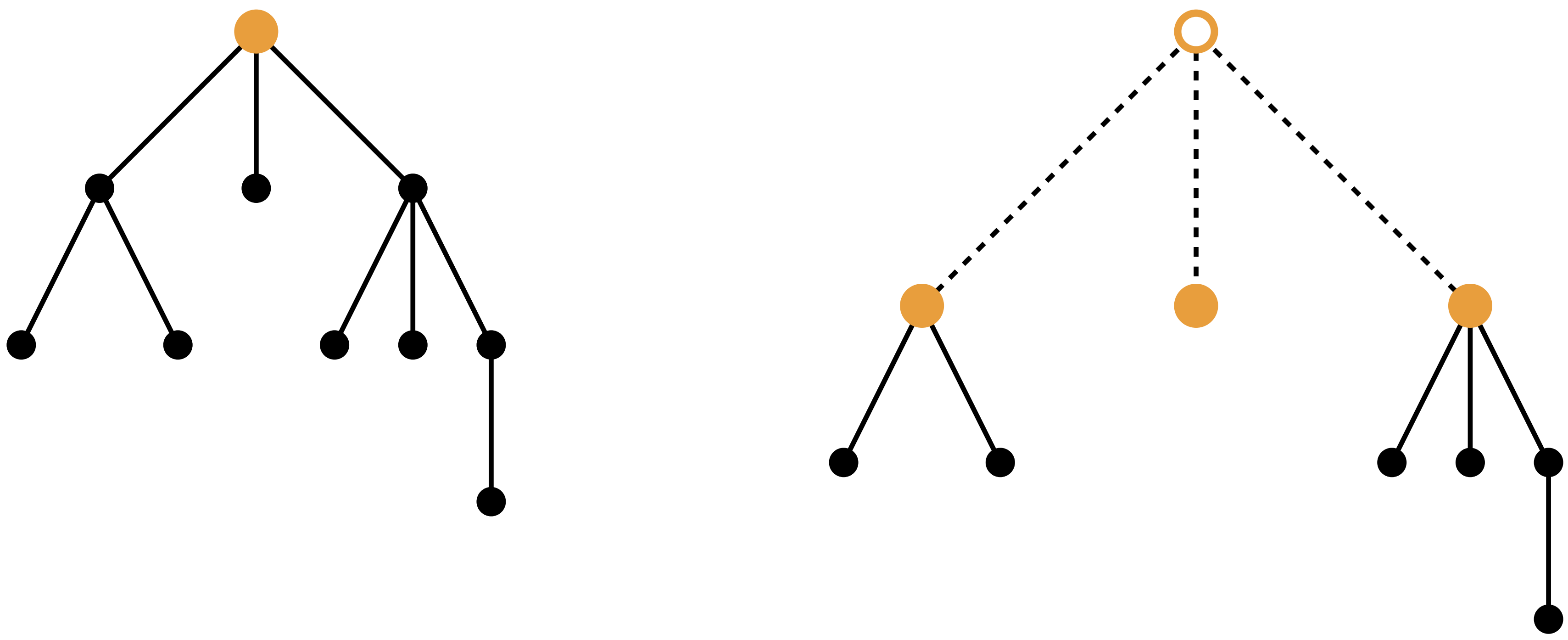

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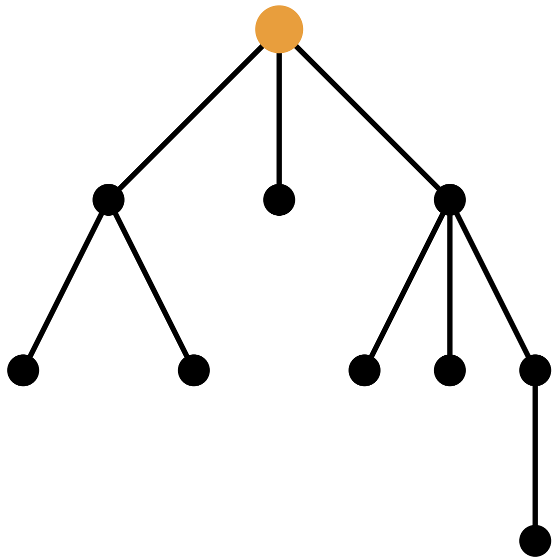

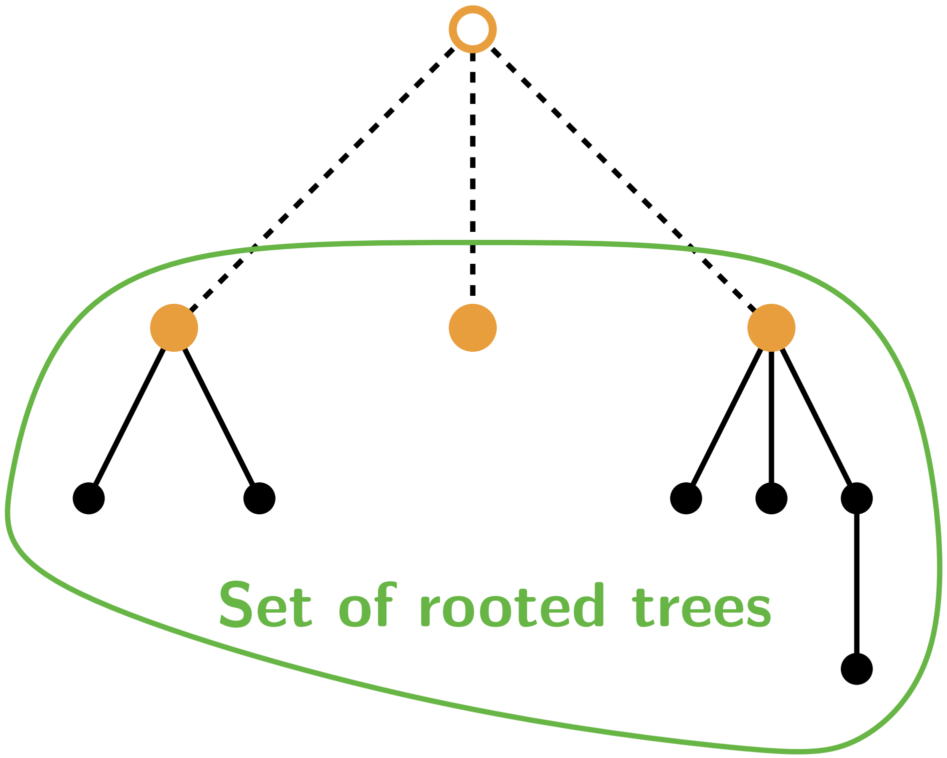

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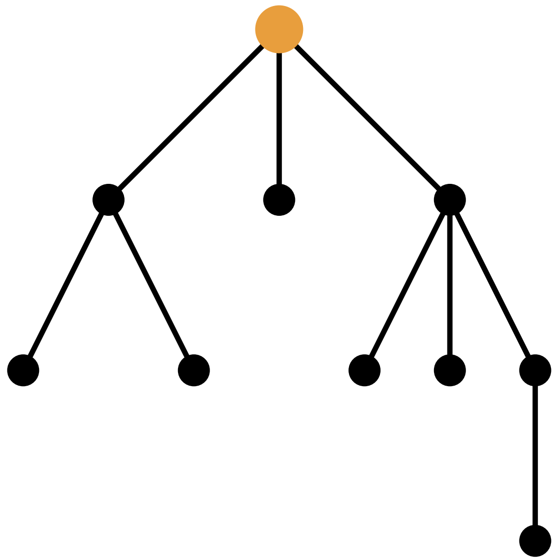

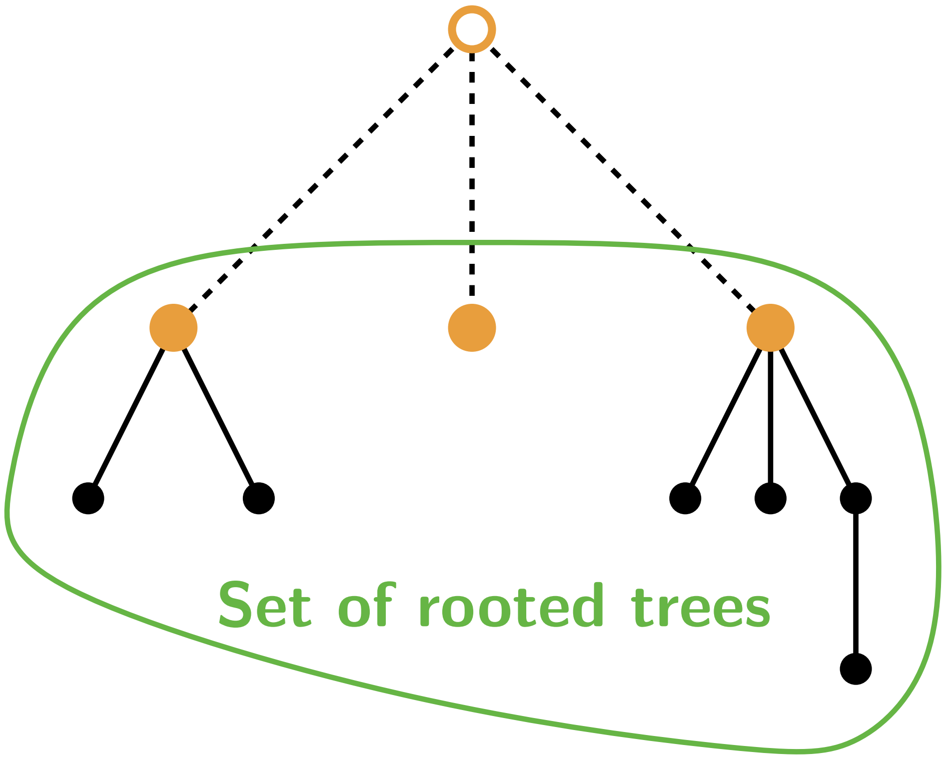

Implicit equation:

$$T^{\bullet}(x) = x \exp\left(T^{\bullet}(x)\right) = x + x^2 + \frac{3x^3}{2} + \cdots . \tag{2}$$

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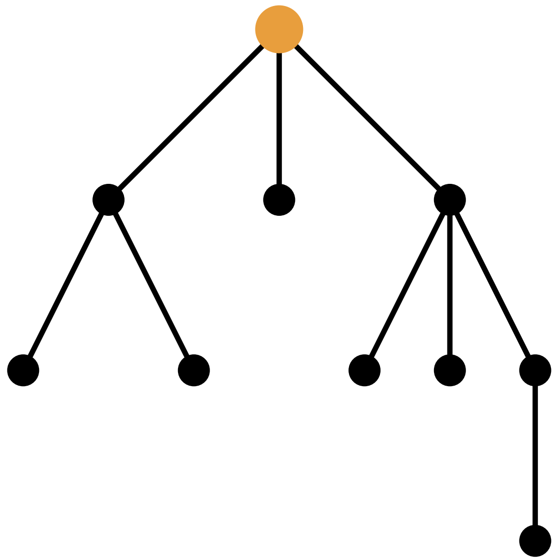

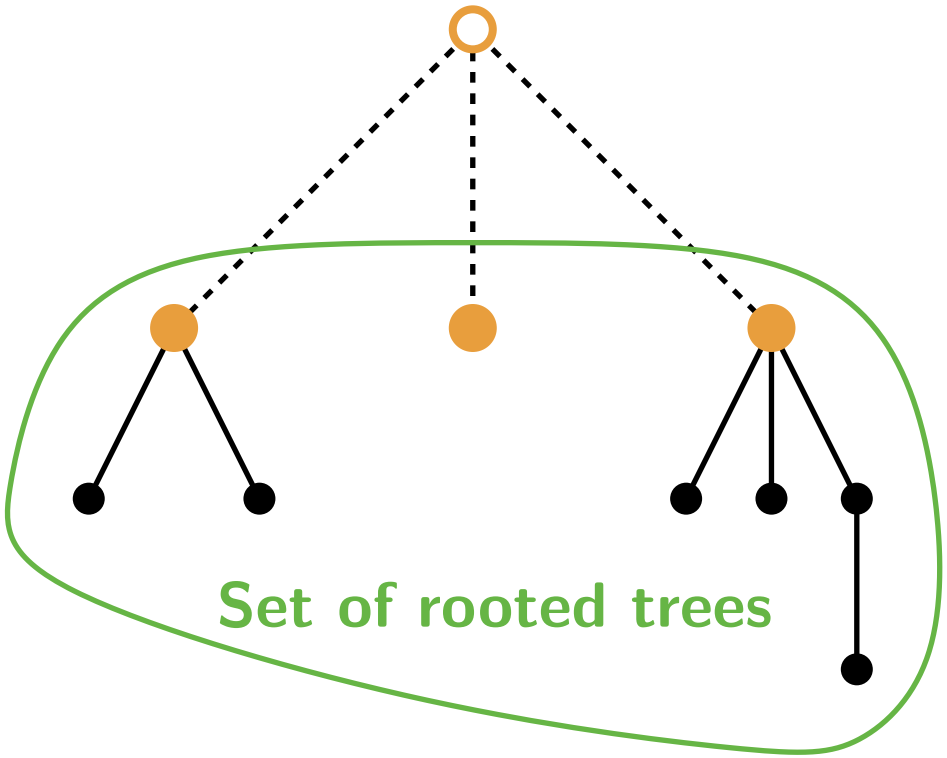

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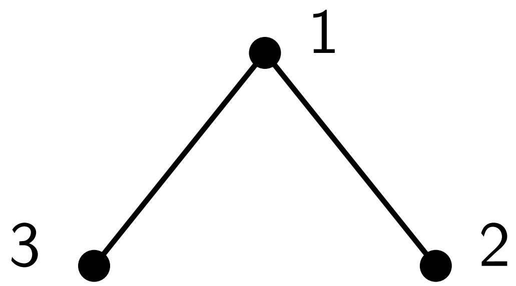
$$T^\bullet(x) = x\exp\left(T^\bullet(x)\right) = x + x^2 + \frac{3x^3}{2} + \cdots . \qquad (2)$$

By using the **Lagrange inversion formula** we obtain:

$$|\mathcal{T}_n^\bullet| = n![x^n]T^\bullet(x) = n^{n-1} \implies |\mathcal{T}_n| = |\mathcal{T}_n^\bullet|/n = n^{n-2}.$$

Pólya theory



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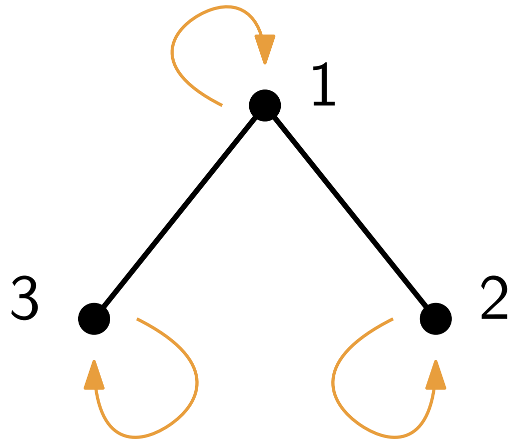

$$(1)(2)(3) \longrightarrow s_1^3$$

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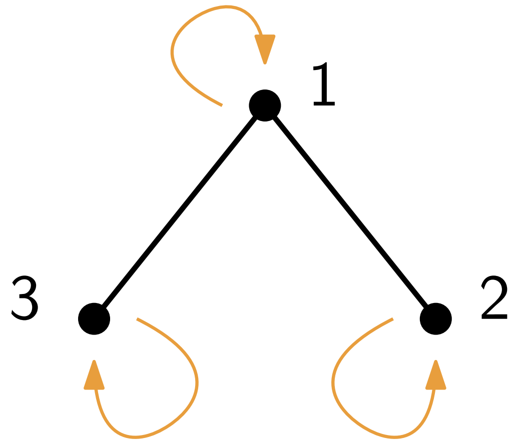

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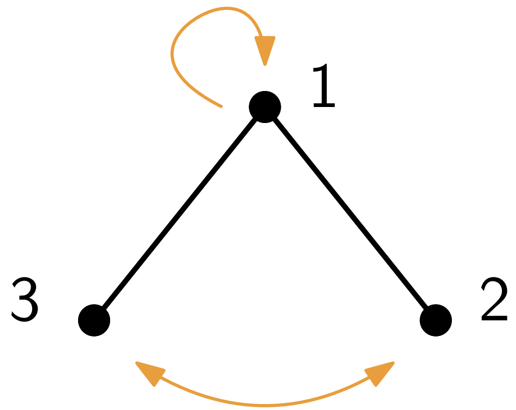

$$(1)(23) \longrightarrow s_1 s_2$$

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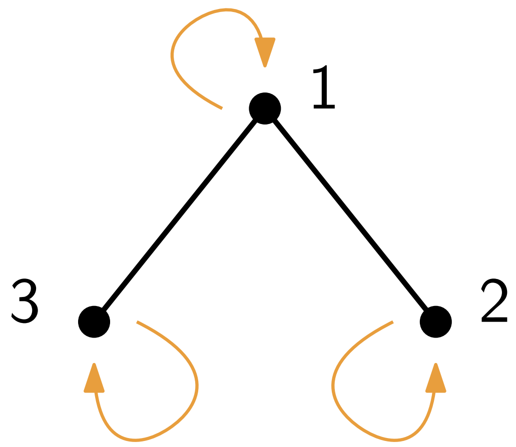

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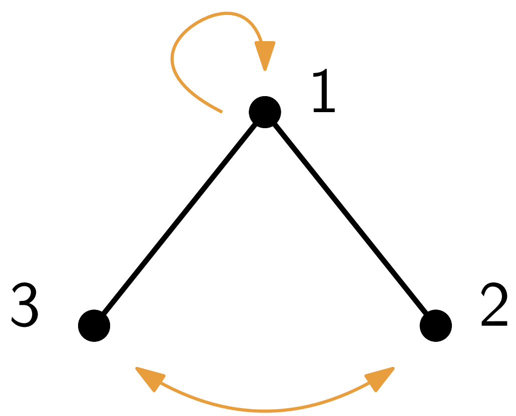

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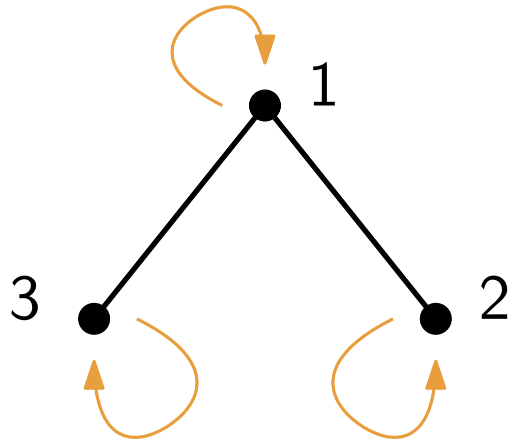

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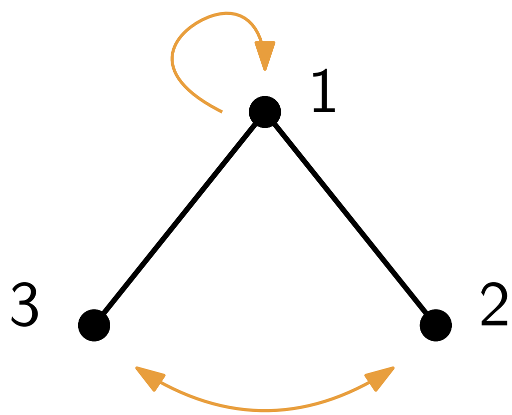

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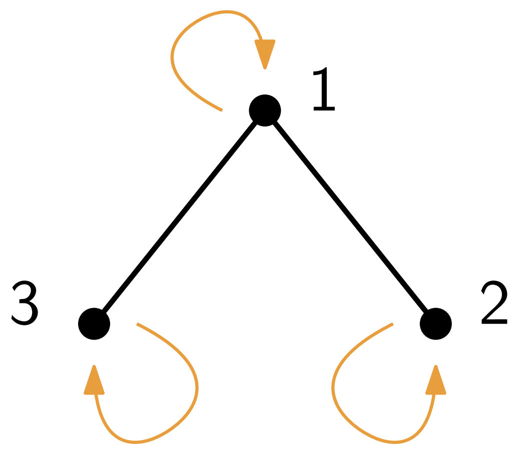

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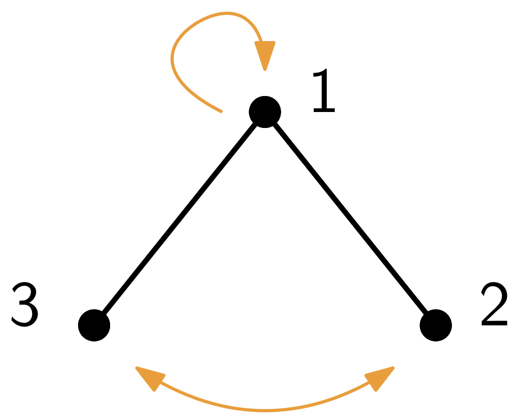

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$Z_{\mathcal{G}}(s_1, s_2, s_3, \dots)$

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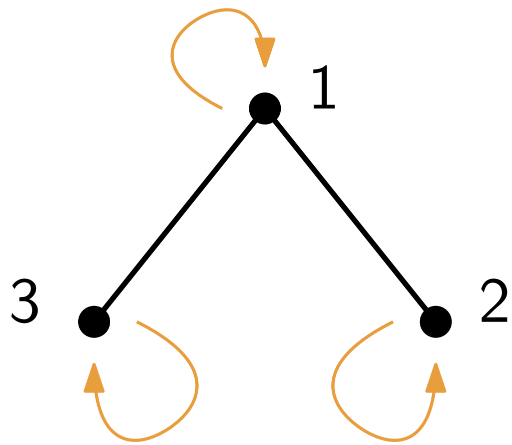

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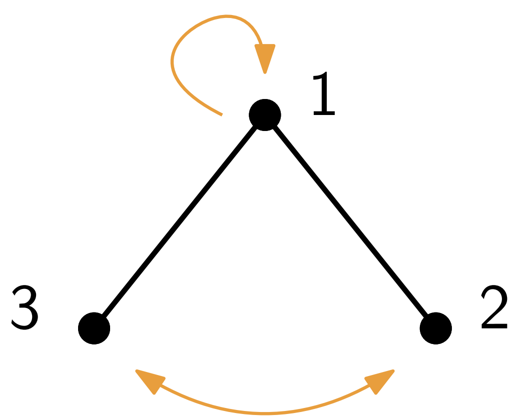

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Cycle index sum
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Theorem [Pólya 1937]
The OGF of the unlabelled class $\tilde{\mathcal{G}}$ is given by

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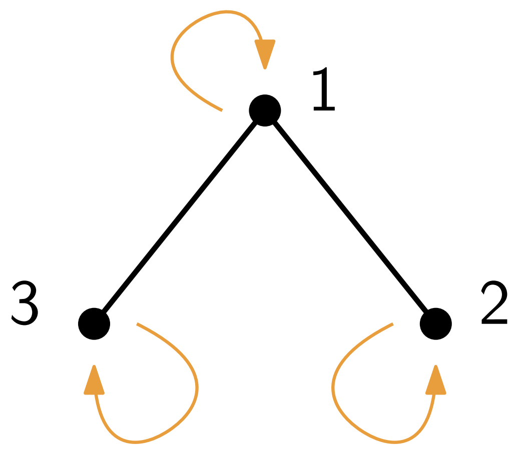

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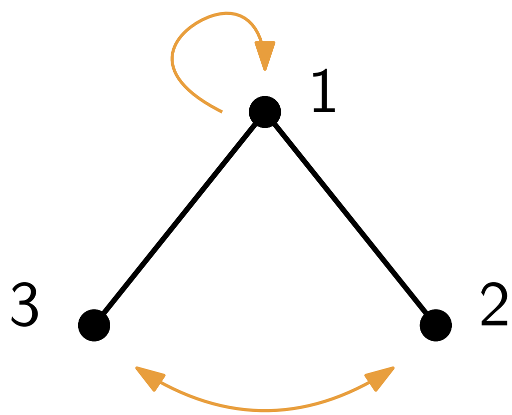

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In our case,

$$G(x) = \frac{3}{3!}(x^3 + x \cdot x^2) = x^3$$

Unlabelled trees

Pólya trees: rooted, unlabelled trees.

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Theorem. [Pólya, 1937]
The OGF $P(x)$ of Pólya trees is given by

$$P(x) = x \exp\left(P(x) + \frac{P(x^2)}{2} + \frac{P(x^3)}{3} + \dots\right).$$

As $n \to \infty$ we have

$$[x^n]P(x) \sim \frac{b\sqrt{\rho}}{2\sqrt{\pi}} \cdot n^{-3/2} \cdot \rho^{-n},$$

with $b \approx 2.681127$ and $\rho \approx 0.338219$.

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What about unrooted unlabelled trees?

Unlabelled trees

Problem!

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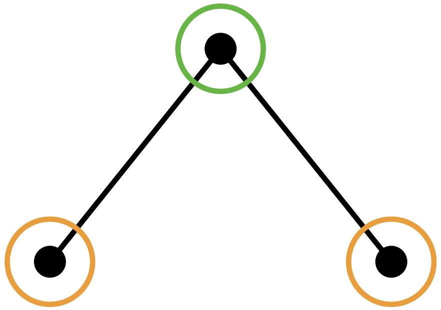
Not every unlabelled graph of size n gives rise to n rooted graphs.



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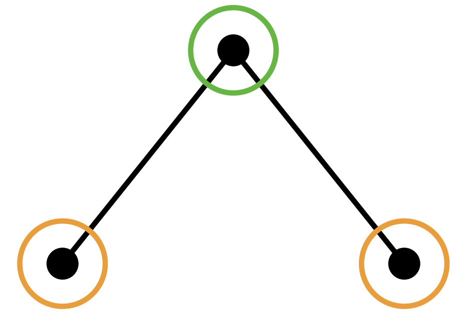

Theorem. [Otter, 1948]
The OGF $U(x)$ of unlabelled trees is given by

$$U(x) = P(x) + \frac{1}{2}(P(x^2) - P(x)^2).$$

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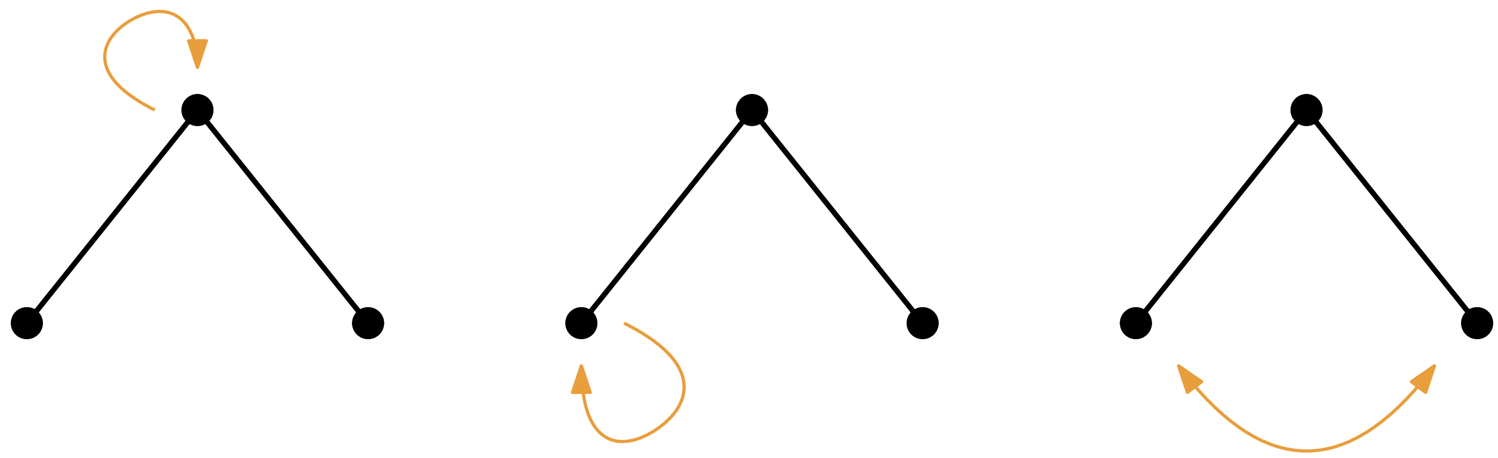
with $b \approx 2.681127$ and $\rho \approx 0.338219$.

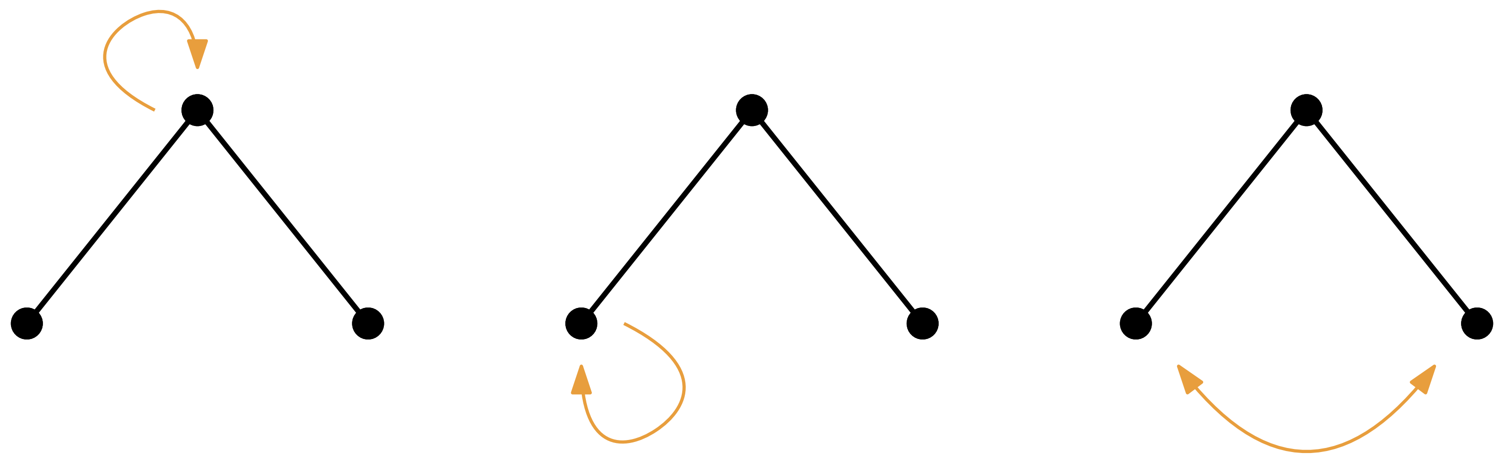
Proof. Using the **dissymmetry theorem**.

Cycle-pointing



Definition. A **cycle-pointed graph** is a pair (G, c) where $G \in \mathcal{G}$ is an unlabelled graph and c is a cycle of some automorphism of G.

Cycle-pointing



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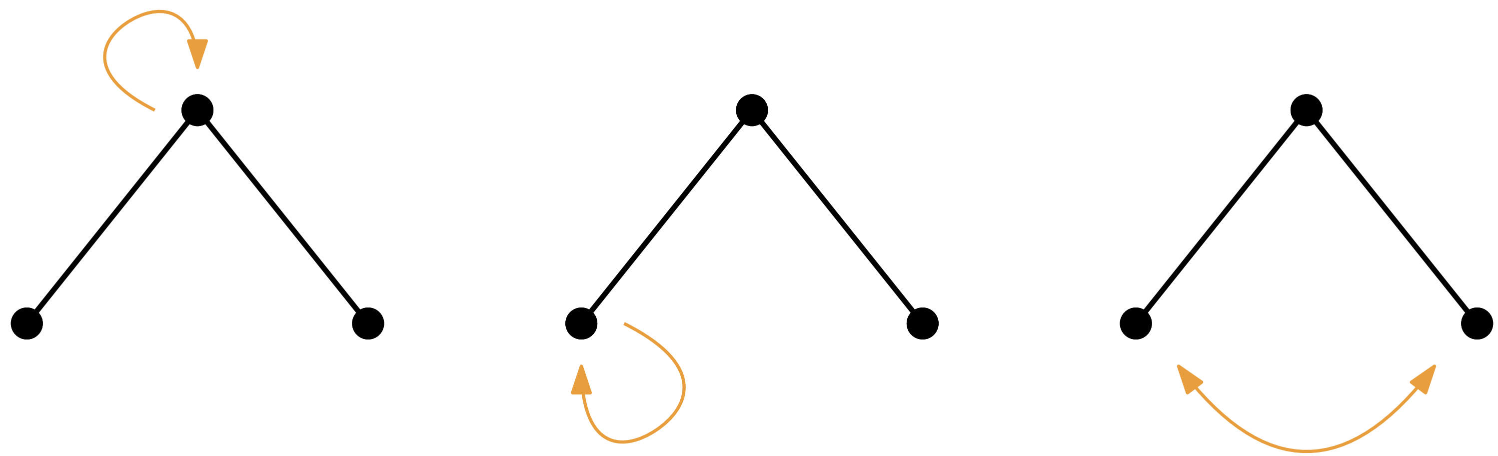

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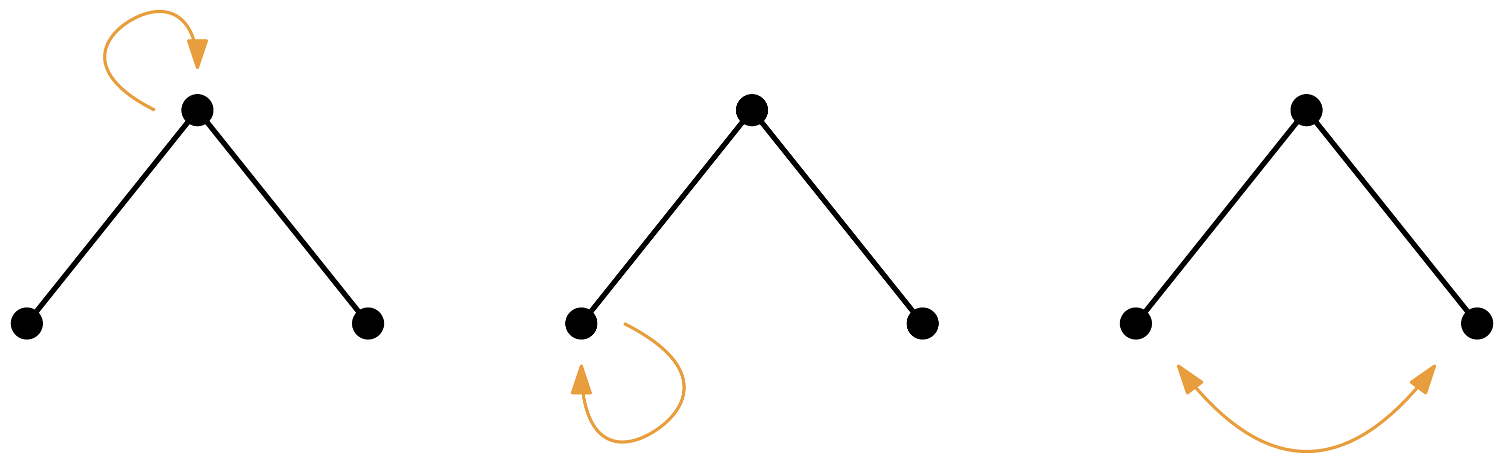

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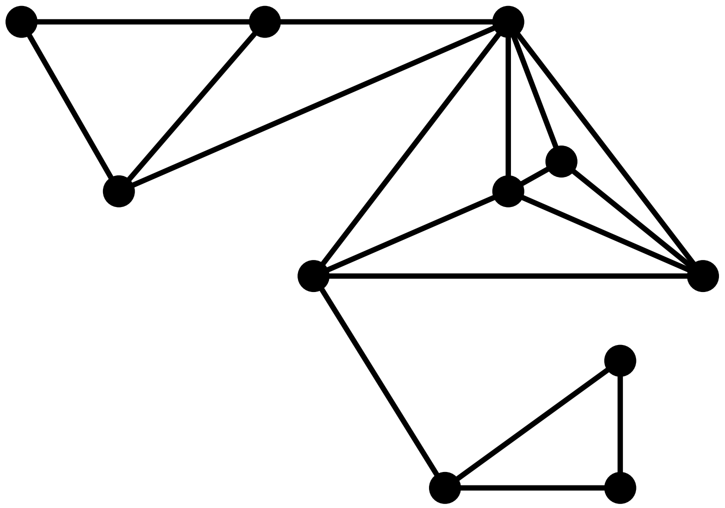
An unbiased rooting (pointing) operator!

They extend Pólya theory to cycle-pointed graphs. In particular, they manage to unroot Pólya trees via cycle-pointing and they recover Otter's formula.

Our class of graphs



Chordal graphs with tree-width at most t

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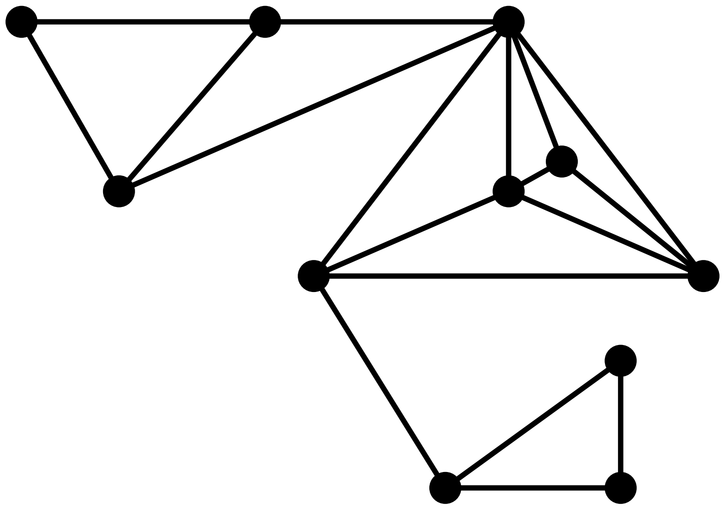

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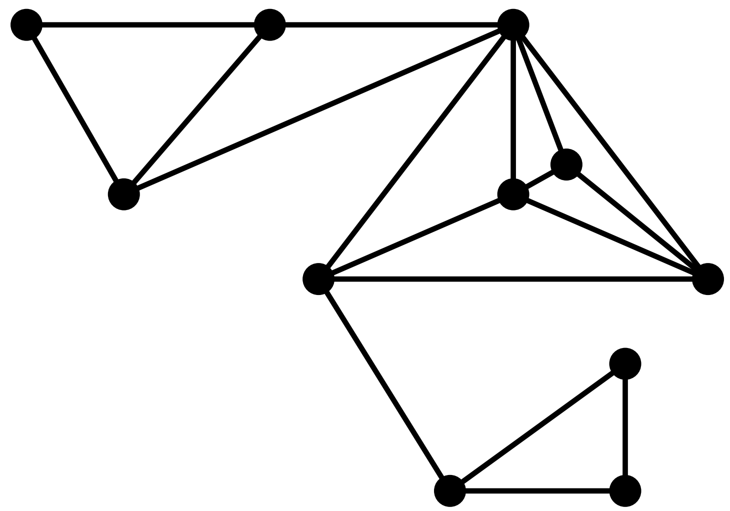

Chordal graphs with tree-width at most t

[Wormald, 1985]: they admit a decomposition into k-connected components.

[C., Drmota, Noy & Requilé, 2023]: assymptotic enumeration of the labelled class.

$$|\mathcal{G}_{t,n}| \sim c_t \cdot n^{-5/2} \cdot \gamma_t^n \cdot n! \qquad \text{as } n \to \infty,$$

for some $c_t > 0$ and $\gamma_t > 1$

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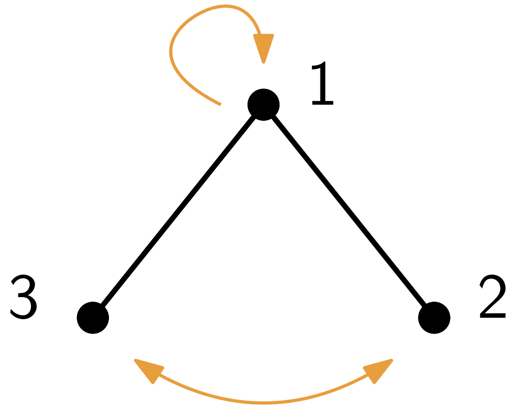

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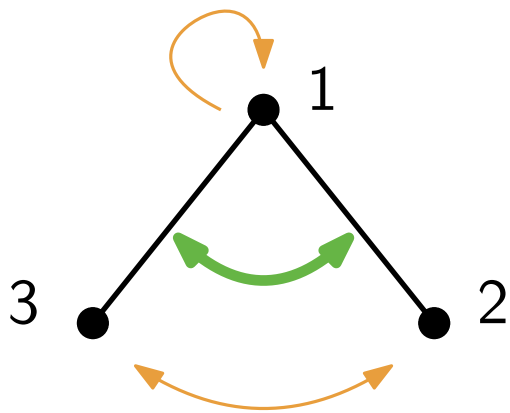

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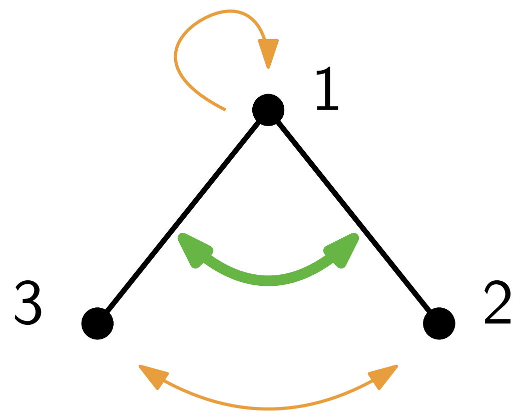

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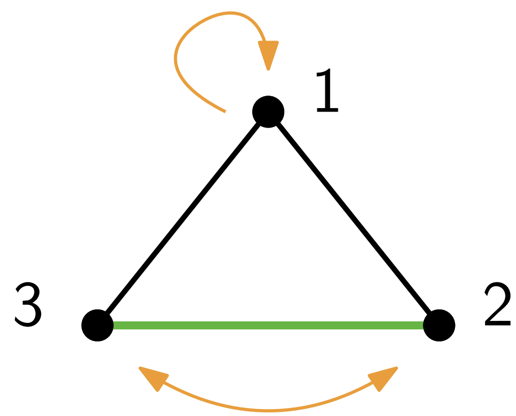

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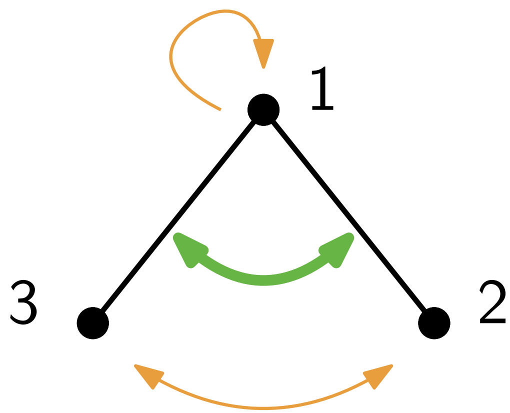

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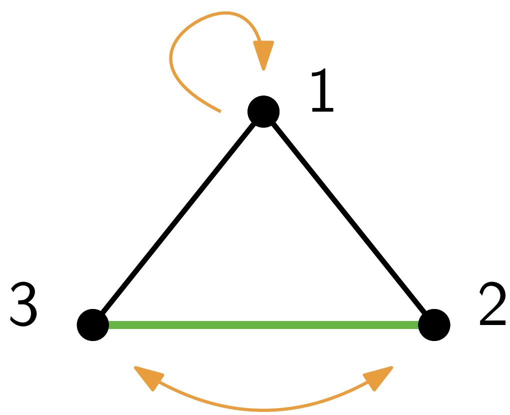

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What we do:

- Refinement of cycle index sums to encode cycles of cliques.
- Extend cycle-pointing to cycles of cliques.

Results

- [Harary & Palmer (1968)], [Fowler, Gessel, Labelle & Leroux (2002)]: asymptotic enumeration of unlabelled (rooted and unrooted) **2-trees**

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This talk: generalisation of previous results.

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This talk: generalisation of previous results.

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Future:

- [C.,Drmota & Requilé (soon?)]: asymptotic enumeration of unlabelled chordal graphs with bounded tree-width.