Chordal graphs with bounded tree-width

Jordi Castellví (UPC)

Work in collaboration with Michael Drmota, Marc Noy and Clément Requilé

EUROCOMB 2023 - Prague

k-trees

Definition. A ***k*-tree** is a graph obtained from a $(k+1)$-clique by successively adding a new vertex connected to all vertices of an existing k-clique.

k-trees

Definition. A ***k*-tree** is a graph obtained from a $(k+1)$-clique by successively adding a new vertex connected to all vertices of an existing k-clique.

- 1-trees are trees

k-trees

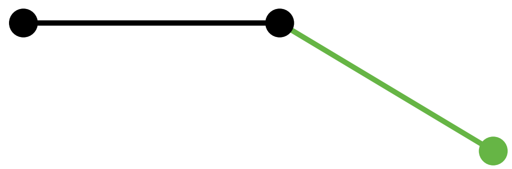
Definition. A **k-tree** is a graph obtained from a $(k+1)$-clique by successively adding a new vertex connected to all vertices of an existing k-clique.

- 1-trees are trees

k-trees

Definition. A ***k*-tree** is a graph obtained from a $(k+1)$-clique by successively adding a new vertex connected to all vertices of an existing k-clique.

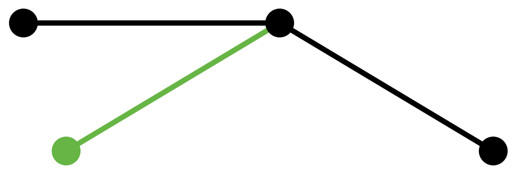
- 1-trees are trees



k-trees

Definition. A ***k*-tree** is a graph obtained from a $(k+1)$-clique by successively adding a new vertex connected to all vertices of an existing k-clique.

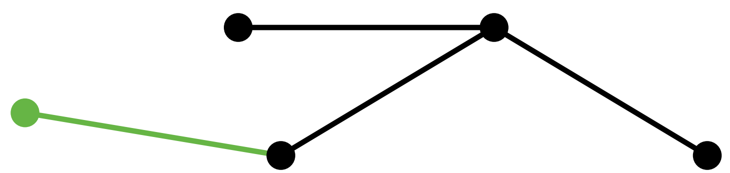
- 1-trees are trees



k-trees

Definition. A ***k*-tree** is a graph obtained from a $(k+1)$-clique by successively adding a new vertex connected to all vertices of an existing k-clique.

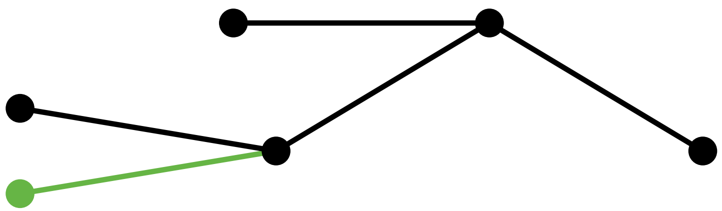
- 1-trees are trees



k-trees

Definition. A ***k*-tree** is a graph obtained from a $(k+1)$-clique by successively adding a new vertex connected to all vertices of an existing k-clique.

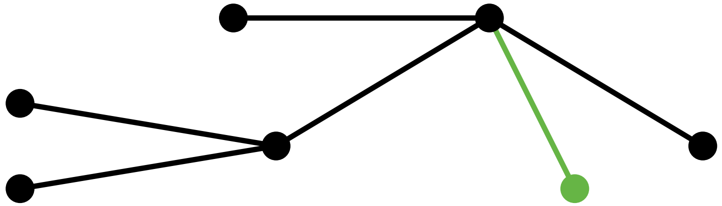
- 1-trees are trees



k-trees

Definition. A ***k*-tree** is a graph obtained from a $(k+1)$-clique by successively adding a new vertex connected to all vertices of an existing k-clique.

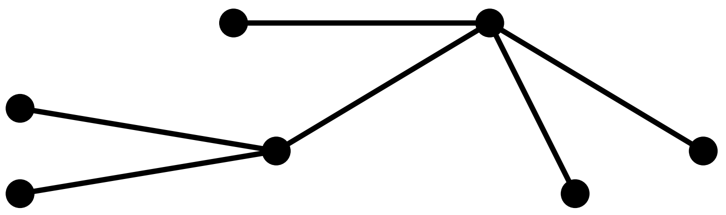
- 1-trees are trees



k-trees

Definition. A ***k*-tree** is a graph obtained from a $(k+1)$-clique by successively adding a new vertex connected to all vertices of an existing k-clique.

- 1-trees are trees

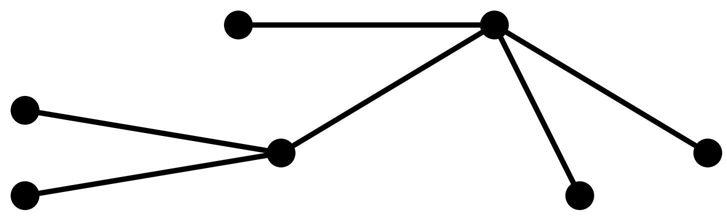


- 2-trees

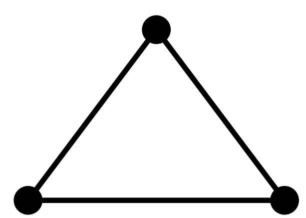
k-trees

Definition. A **k-tree** is a graph obtained from a $(k+1)$-clique by successively adding a new vertex connected to all vertices of an existing k-clique.

- 1-trees are trees



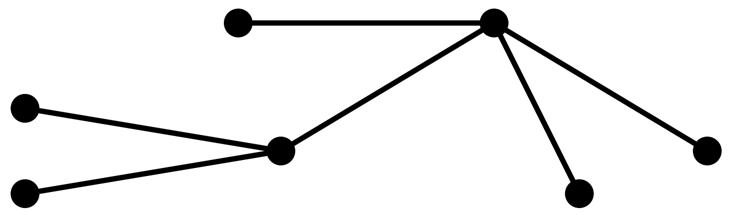
- 2-trees



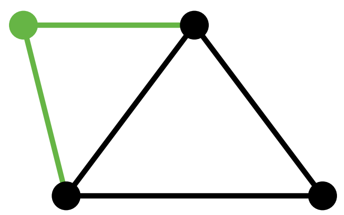
k-trees

Definition. A ***k*-tree** is a graph obtained from a $(k+1)$-clique by successively adding a new vertex connected to all vertices of an existing k-clique.

- 1-trees are trees



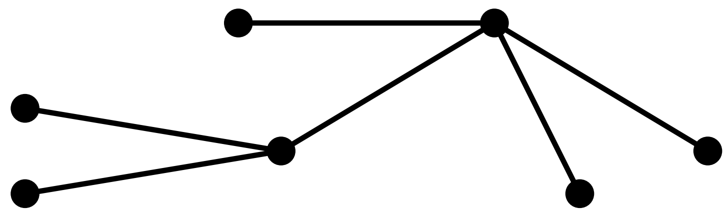
- 2-trees



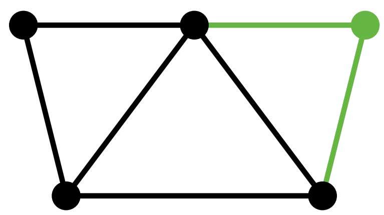
k-trees

Definition. A ***k*-tree** is a graph obtained from a $(k+1)$-clique by successively adding a new vertex connected to all vertices of an existing k-clique.

- 1-trees are trees



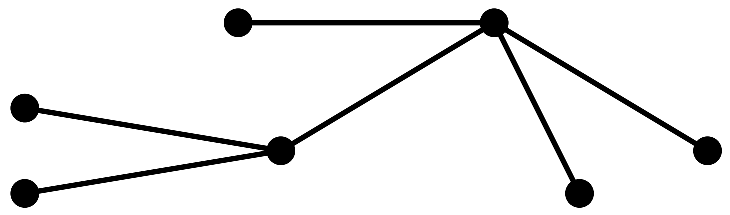
- 2-trees



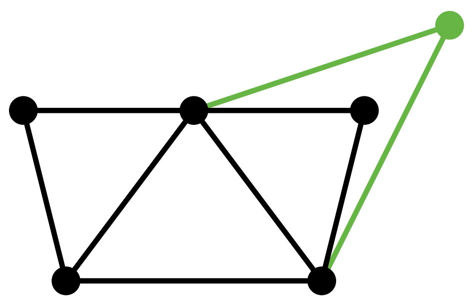
k-trees

Definition. A ***k*-tree** is a graph obtained from a $(k+1)$-clique by successively adding a new vertex connected to all vertices of an existing k-clique.

- 1-trees are trees



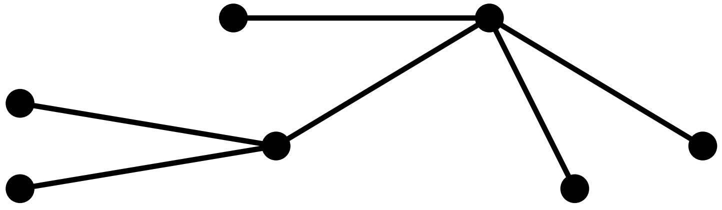
- 2-trees



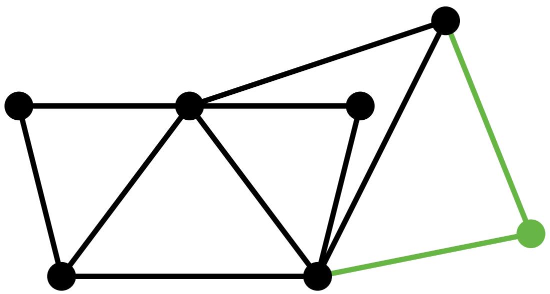
k-trees

Definition. A ***k*-tree** is a graph obtained from a $(k+1)$-clique by successively adding a new vertex connected to all vertices of an existing k-clique.

- 1-trees are trees



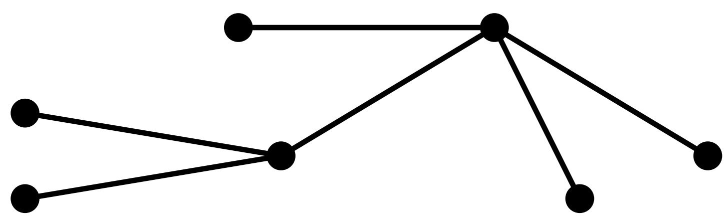
- 2-trees



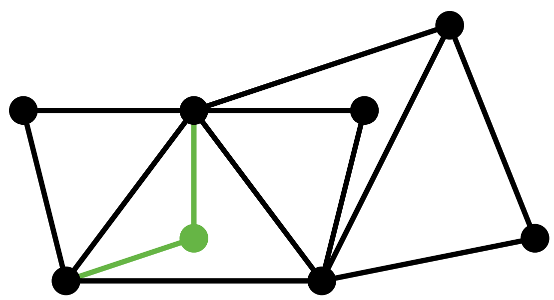
k-trees

Definition. A ***k*-tree** is a graph obtained from a $(k+1)$-clique by successively adding a new vertex connected to all vertices of an existing k-clique.

- 1-trees are trees



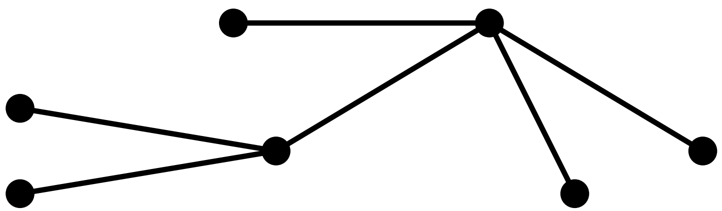
- 2-trees



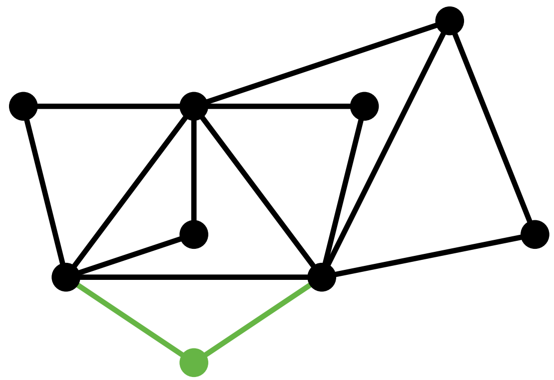
k-trees

Definition. A ***k*-tree** is a graph obtained from a $(k+1)$-clique by successively adding a new vertex connected to all vertices of an existing k-clique.

- 1-trees are trees



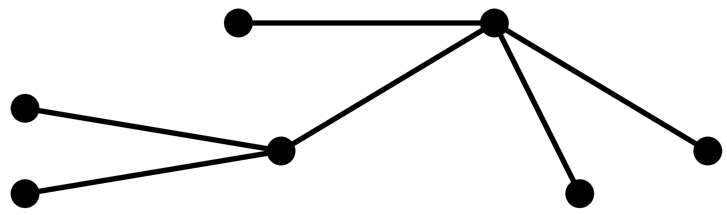
- 2-trees



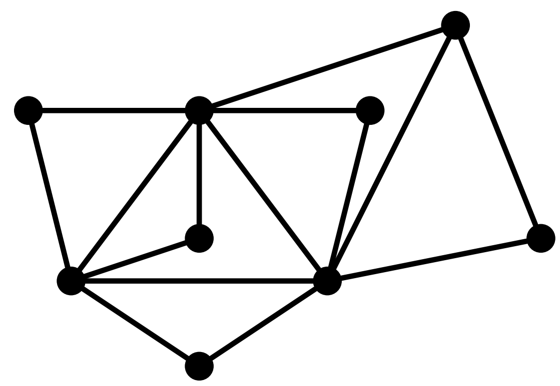
k-trees

Definition. A ***k*-tree** is a graph obtained from a $(k+1)$-clique by successively adding a new vertex connected to all vertices of an existing k-clique.

- 1-trees are trees



- 2-trees



Theorem ([Beineke, Pippert, '69])
The number of labelled k-trees with n vertices is $\binom{n}{k}(kn-k^2+1)^{n-k-2}$.

Tree-width

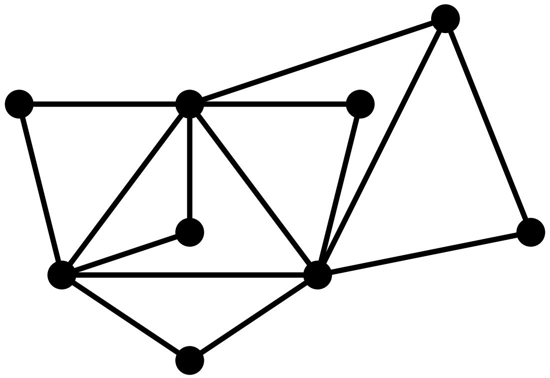
Definition. The **tree-width** of a graph G is the minimum k such that G is the subgraph of a k-tree.

Thus, a graph with **tree-width at most** k is the subgraph of some k-tree.

Tree-width

Definition. The **tree-width** of a graph G is the minimum k such that G is the subgraph of a k-tree.

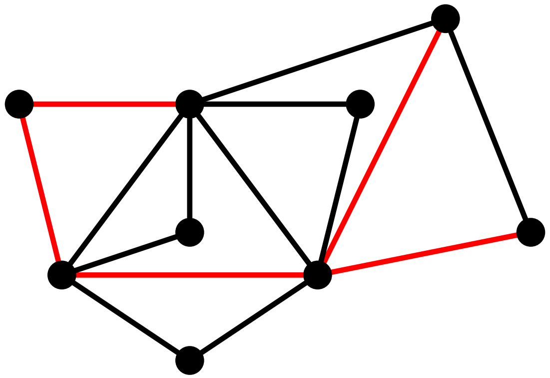
Thus, a graph with **tree-width at most** k is the subgraph of some k-tree.



Tree-width

Definition. The **tree-width** of a graph G is the minimum k such that G is the subgraph of a k-tree.

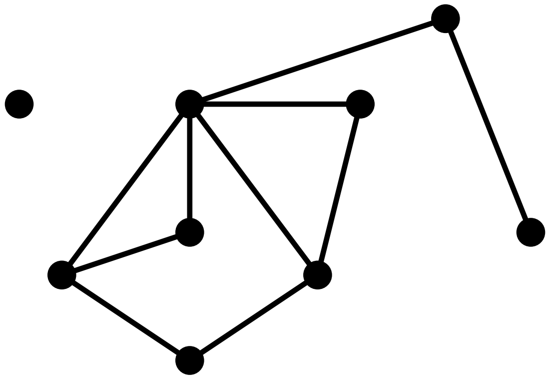
Thus, a graph with **tree-width at most** k is the subgraph of some k-tree.



Tree-width

Definition. The **tree-width** of a graph G is the minimum k such that G is the subgraph of a k-tree.

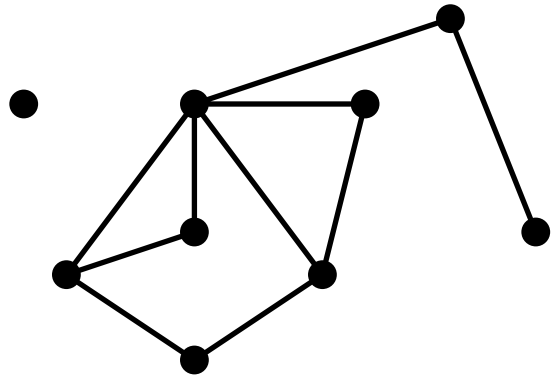
Thus, a graph with **tree-width at most** k is the subgraph of some k-tree.



Tree-width

Definition. The **tree-width** of a graph G is the minimum k such that G is the subgraph of a k-tree.

Thus, a graph with **tree-width at most** k is the subgraph of some k-tree.



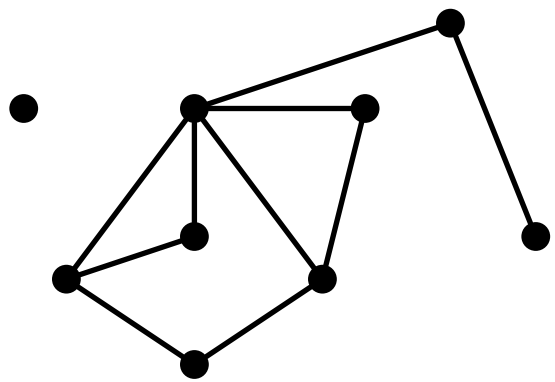
Enumeration of graphs with tree-width at most t.

- $t = 1$ (forests) $\longrightarrow$ **Done!**

Tree-width

Definition. The **tree-width** of a graph G is the minimum k such that G is the subgraph of a k-tree.

Thus, a graph with **tree-width at most** k is the subgraph of some k-tree.



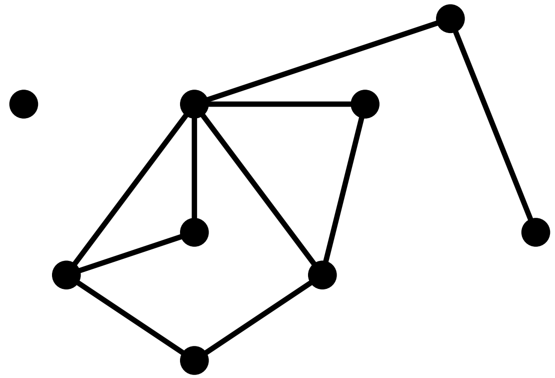
Enumeration of graphs with tree-width at most t.

- $t = 1$ (forests) $\longrightarrow$ **Done!**
- $t = 2 \longrightarrow$ **Done!** ([Bordisky, Giménez, Kang, Noy '07])

Tree-width

Definition. The **tree-width** of a graph G is the minimum k such that G is the subgraph of a k-tree.

Thus, a graph with **tree-width at most** k is the subgraph of some k-tree.



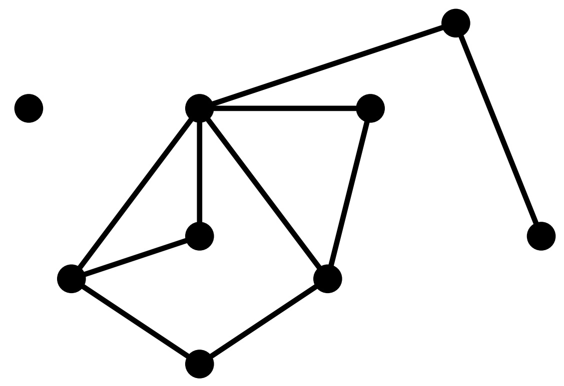
Enumeration of graphs with tree-width at most t.

- $t = 1$ (forests) $\longrightarrow$ **Done!**
- $t = 2 \longrightarrow$ **Done!** ([Bordisky, Giménez, Kang, Noy '07])
- $t \geq 3 \longrightarrow$ **Open problem**

Tree-width

Definition. The **tree-width** of a graph G is the minimum k such that G is the subgraph of a k-tree.

Thus, a graph with **tree-width at most** k is the subgraph of some k-tree.



Enumeration of graphs with tree-width at most t.

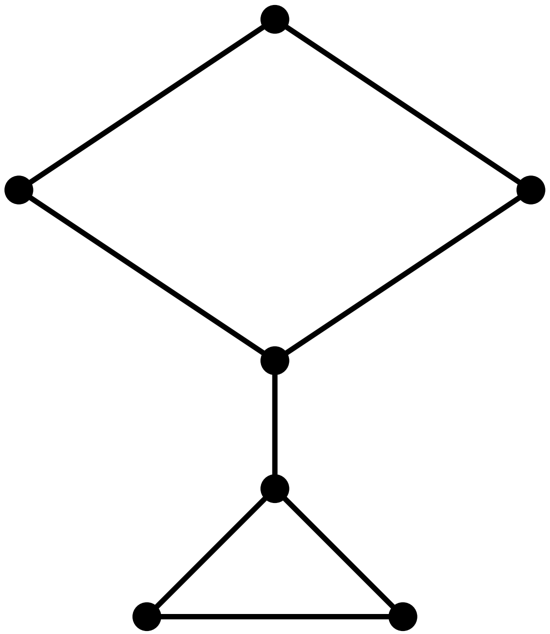
- $t = 1$ (forests) $\longrightarrow$ **Done!**
- $t = 2 \longrightarrow$ **Done!** ([Bordisky, Giménez, Kang, Noy '07])
- $t \geq 3 \longrightarrow$ **Open problem**

Let $g_{n,t}$ be the number of labelled graphs with n vertices and tree-width at most t. Then, $\left(\frac{2^t tn}{\log t}\right)^n \leq g_{n,t} \leq (2^t tn)^n$.

[Baste, Noy, Sau '18]

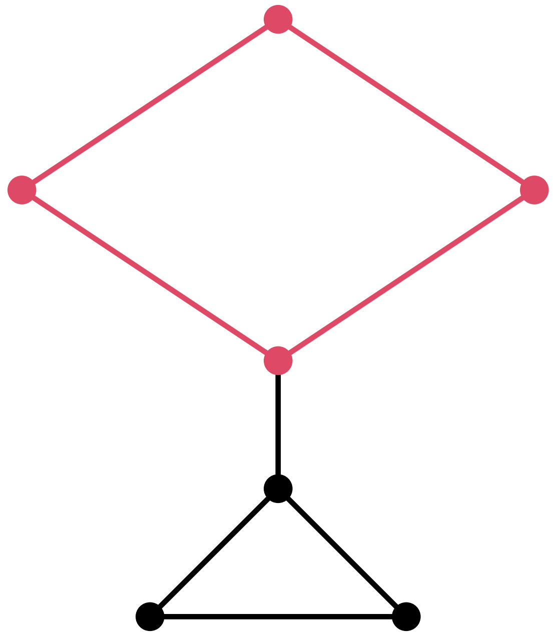
Chordal graphs

Definition. A graph is **chordal** if it has no induced cycle of lengh greater than 3.



Chordal graphs

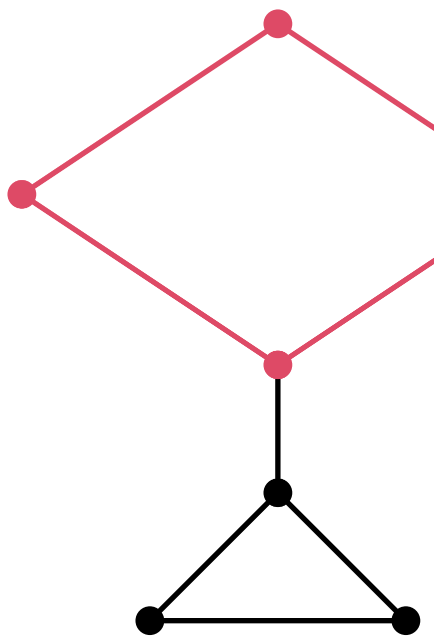
Definition. A graph is **chordal** if it has no induced cycle of lengh greater than 3.

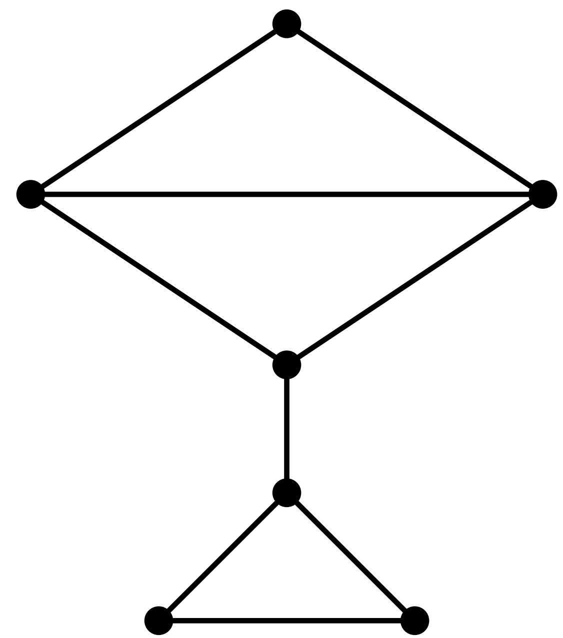


Not chordal

Chordal graphs

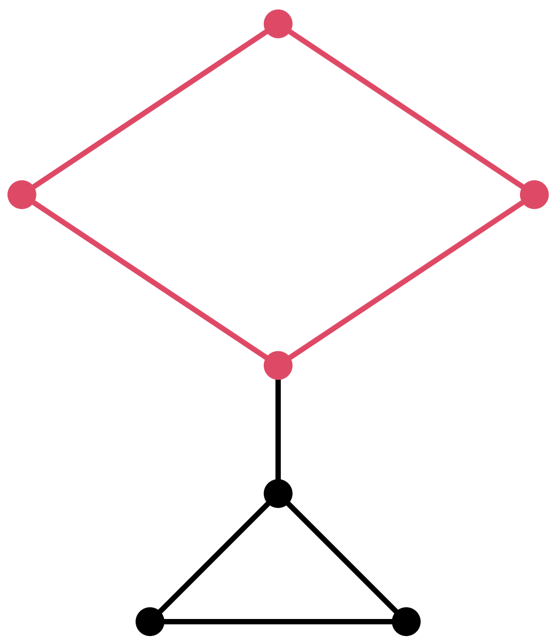
Definition. A graph is **chordal** if it has no induced cycle of lengh greater than 3.



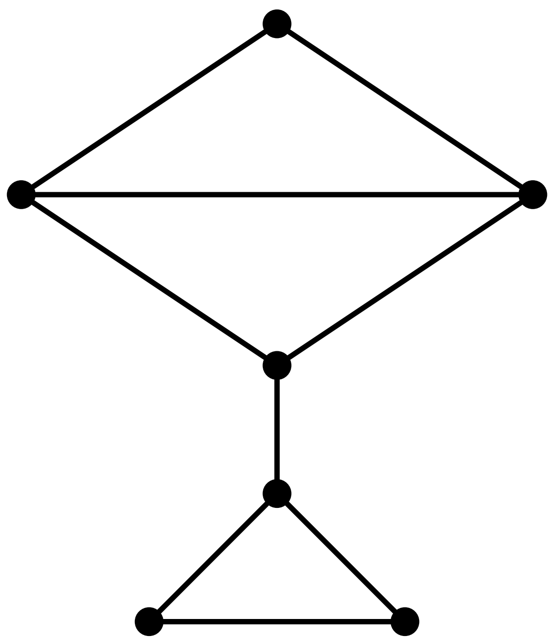


Chordal graphs

Definition. A graph is **chordal** if it has no induced cycle of lengh greater than 3.



Not chordal



Chordal

Theorem. ([Dirac '61])
A graph is chordal iff every minimal separator is a clique.

Main results

Let $\mathcal{G}_{t,k,n}$ be the set of k-connected chordal graphs with n labelled vertices and tree-width at most t. Then, for fixed $t \geq 1$ and $0 \leq k \leq t$:

Theorem 1. ([C., Drmota, Noy, Réquilé '22])
There exist constants $c_{t,k} > 0$ and $\gamma_{t,k} \in (0,1)$ such that

$$|\mathcal{G}_{t,k,n}| \sim c_{t,k} \cdot n^{-5/2} \cdot \gamma_{t,k}^{n} \cdot n!, \qquad \text{as } n \to \infty.$$

Main results

Let $\mathcal{G}_{t,k,n}$ be the set of k-connected chordal graphs with n labelled vertices and tree-width at most t. Then, for fixed $t \geq 1$ and $0 \leq k \leq t$:

Theorem 1. ([C., Drmota, Noy, Réquilé '22])
There exist constants $c_{t,k} > 0$ and $\gamma_{t,k} \in (0,1)$ such that

$$|\mathcal{G}_{t,k,n}| \sim c_{t,k} \cdot n^{-5/2} \cdot \gamma_{t,k}^{n} \cdot n!, \qquad \text{as } n \to \infty.$$

For $i \in [t]$, let X_i be the number of i-cliques in a uniform random graph in $\mathcal{G}_{t,k,n}$.

Theorem 2. ([C., Drmota, Noy, Réquilé '22])
There exist constants $\alpha, \gamma \in (0,1)$ such that

$$\frac{|X_i - \mathbb{E}X_i|}{\sqrt{\mathbb{V}X_i}} \xrightarrow{d} N(0,1), \qquad \text{with} \quad \mathbb{E}X_i \sim \alpha n \quad \text{and} \quad \mathbb{V}X_i \sim \sigma n.$$

Decomposition of graphs into k-connected components

Definition. The **connected components** (or 1-connected components) of a graph are its connected maximal subgraphs.

Decomposition of graphs into k-connected components

Definition. The **connected components** (or 1-connected components) of a graph are its connected maximal subgraphs.

Let $\mathcal{G}$ be a class of labelled graphs and let $\mathcal{C} \subset \mathcal{G}$ be the class of its connected members. Then, their exponential generating functions satisfy

$$G(x) = \exp(C(x)),$$

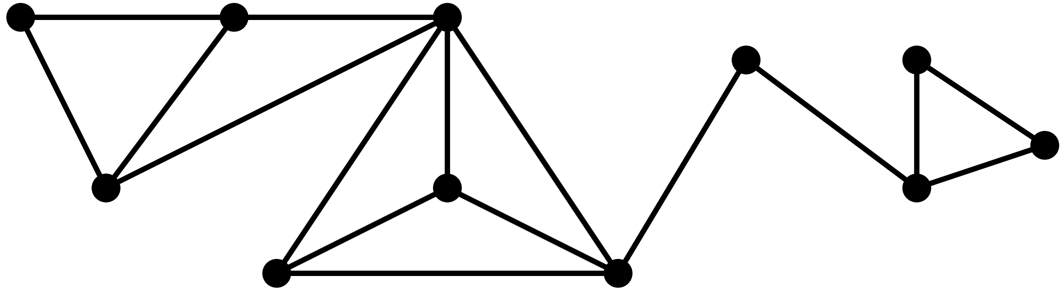
provided that $\mathcal{G}$ is closed under disjoint unions and taking connected components.

Decomposition of graphs into k-connected components

Definition. The **2-connected components** (or blocks) of a connected graph are its maximal 2-connected subgraphs.

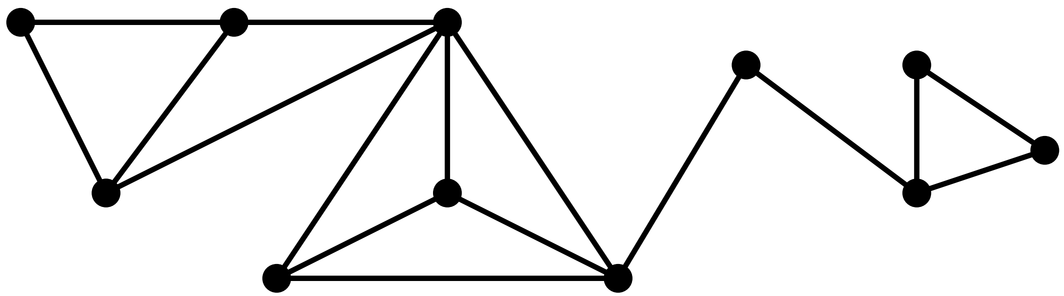
Decomposition of graphs into k-connected components

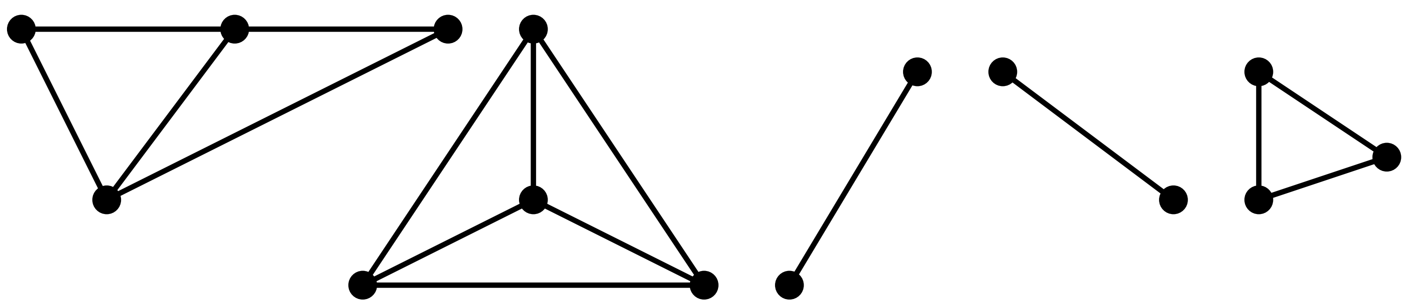
Definition. The **2-connected components** (or blocks) of a connected graph are its maximal 2-connected subgraphs.



Decomposition of graphs into k-connected components

Definition. The **2-connected components** (or blocks) of a connected graph are its maximal 2-connected subgraphs.





Decomposition of graphs into k-connected components

Let $\mathcal{B} \subset \mathcal{C}$ be the class of the 2-connected members of $\mathcal{G}$. Then,

$$C^\bullet(x) = x\exp(B'(C^\bullet(x))), \quad \text{where } C^\bullet(x) = xC'(x),$$

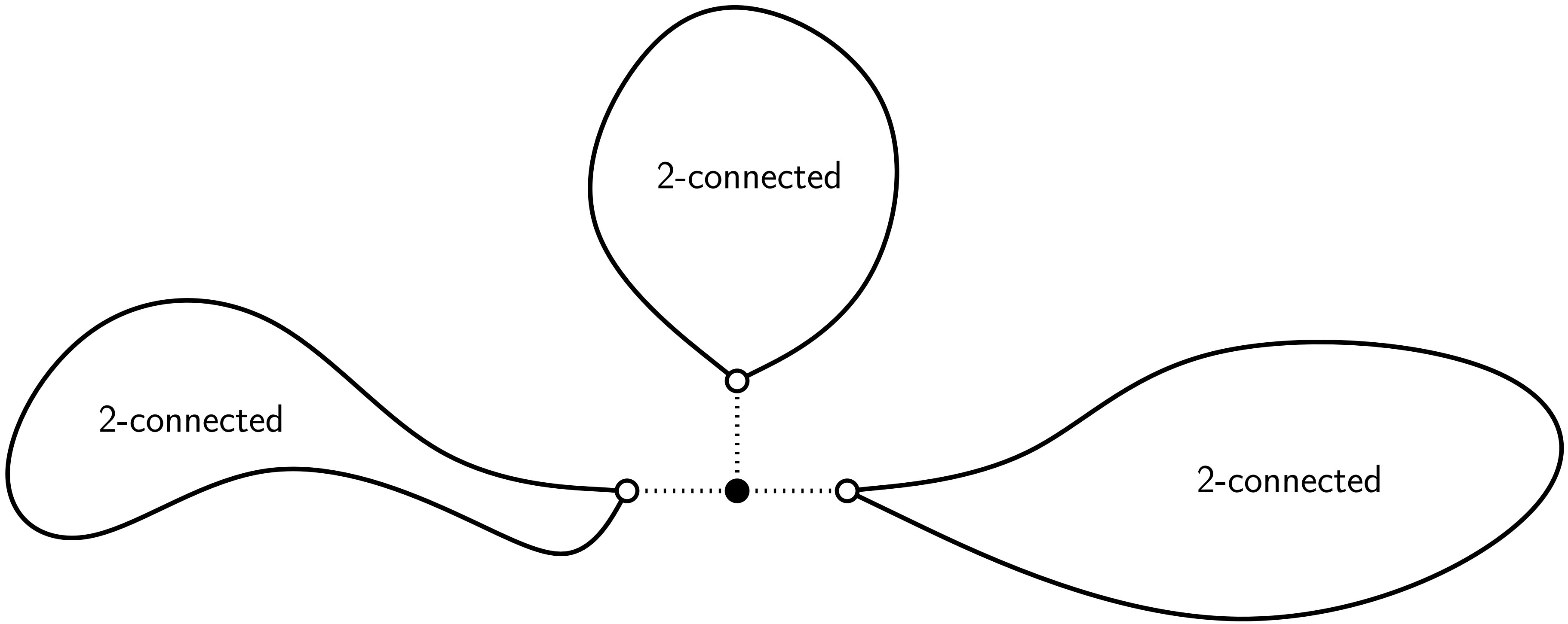
provided that $\mathcal{G}$ is **block-stable**, i.e., that a graph belongs to $\mathcal{C}$ iff its blocks belong to $\mathcal{B}$.

Decomposition of graphs into k-connected components

Let $\mathcal{B} \subset \mathcal{C}$ be the class of the 2-connected members of $\mathcal{G}$. Then,

$$C^\bullet(x) = x \exp(B'(C^\bullet(x))), \quad \text{where } C^\bullet(x) = xC'(x),$$

provided that $\mathcal{G}$ is **block-stable**, i.e., that a graph belongs to $\mathcal{C}$ iff its blocks belong to $\mathcal{B}$.

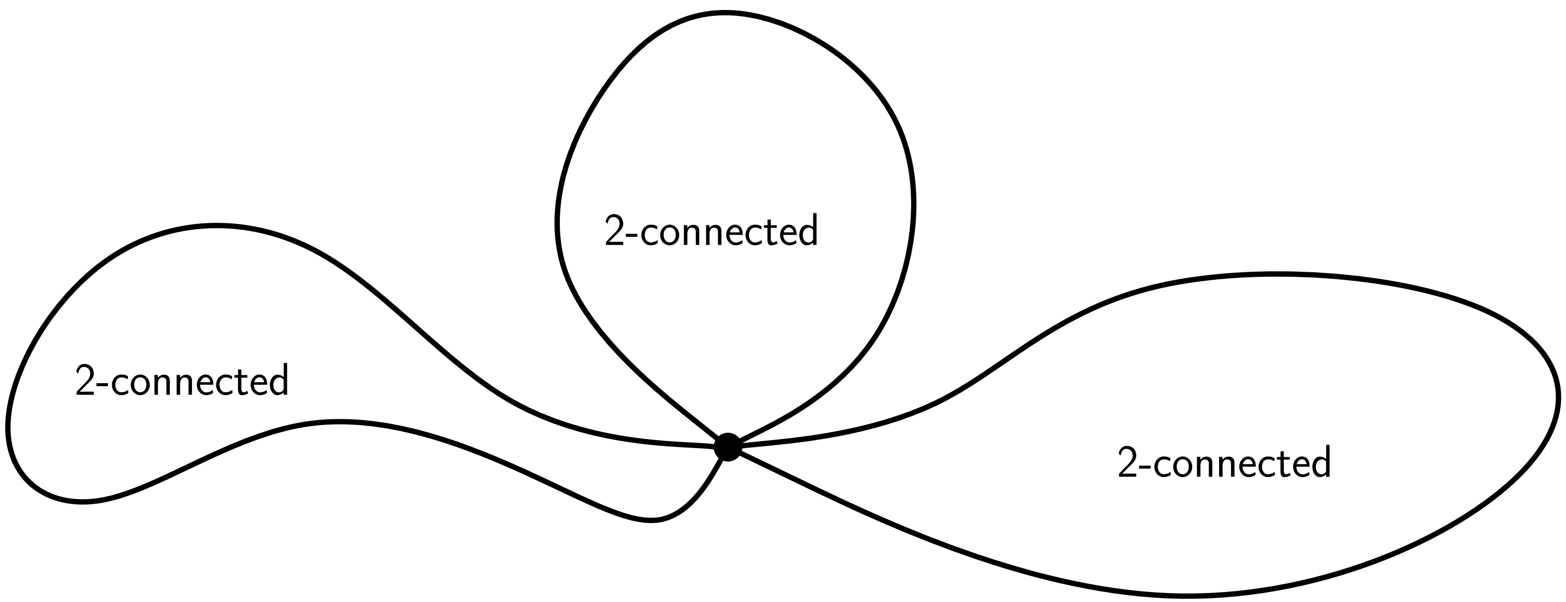


Decomposition of graphs into k-connected components

Let $\mathcal{B} \subset \mathcal{C}$ be the class of the 2-connected members of $\mathcal{G}$. Then,

$$C^\bullet(x) = x\exp(B'(C^\bullet(x))), \quad \text{where } C^\bullet(x) = xC'(x),$$

provided that $\mathcal{G}$ is **block-stable**, i.e., that a graph belongs to $\mathcal{C}$ iff its blocks belong to $\mathcal{B}$.

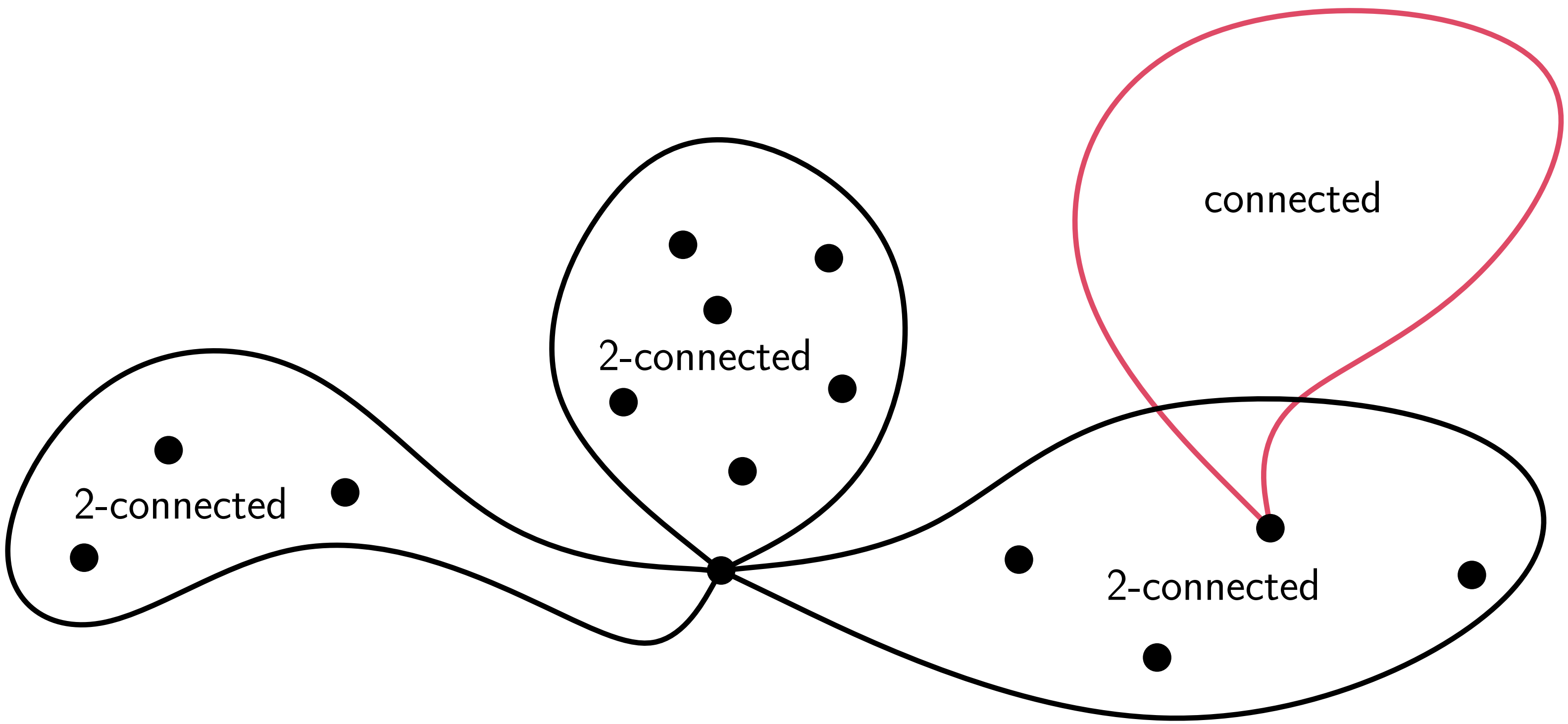


Decomposition of graphs into k-connected components

Let $\mathcal{B} \subset \mathcal{C}$ be the class of the 2-connected members of $\mathcal{G}$. Then,

$$C^\bullet(x) = x\exp(B'(C^\bullet(x))), \quad \text{where } C^\bullet(x) = xC'(x),$$

provided that $\mathcal{G}$ is **block-stable**, i.e., that a graph belongs to $\mathcal{C}$ iff its blocks belong to $\mathcal{B}$.



Decomposition of graphs into k-connected components

The 3-connected components of a 2-connected graph can also be defined, but the details are a little more involved.
There is also a relation between their generating functions.

Decomposition of graphs into k-connected components

The 3-connected components of a 2-connected graph can also be defined, but the details are a little more involved.
There is also a relation between their generating functions.

Unfortunately, the 4-connected components of 3-connected graphs cannot be defined in general (in such a way that the decomposition is unambiguous and well-defined).

Decomposition of graphs into k-connected components

The 3-connected components of a 2-connected graph can also be defined, but the details are a little more involved.
There is also a relation between their generating functions.

Unfortunately, the 4-connected components of 3-connected graphs cannot be defined in general (in such a way that the decomposition is unambiguous and well-defined).

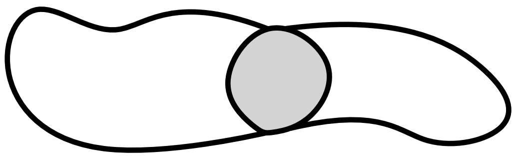
However, any k-connected **chordal** graph admits a decomposition into $(k+1)$-connected components!

Decomposition of chordal graphs into k-connected components

"Definition". **Slicing** through a k-separator:

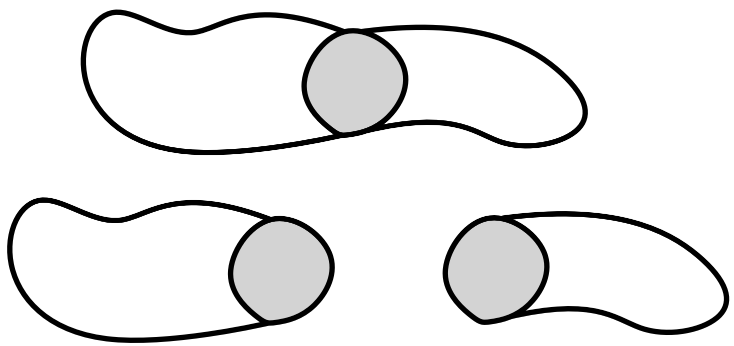
Decomposition of chordal graphs into k-connected components

"Definition". **Slicing** through a k-separator:



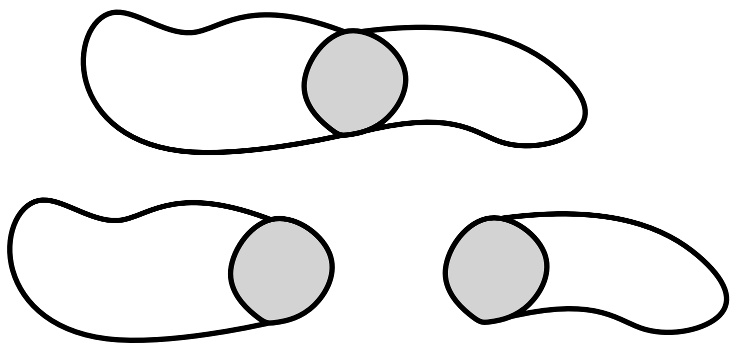
Decomposition of chordal graphs into k-connected components

"Definition". **Slicing** through a k-separator:



Decomposition of chordal graphs into k-connected components

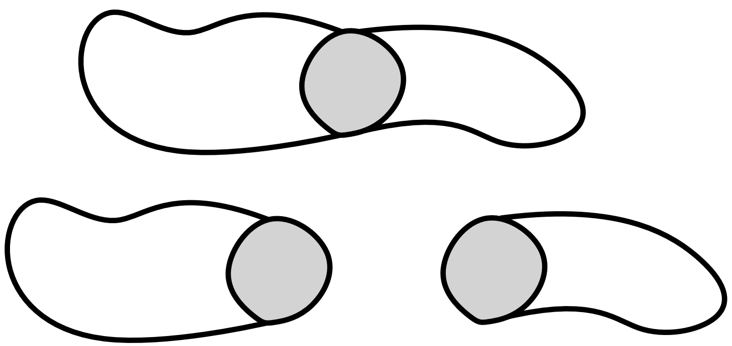
"Definition". **Slicing** through a k-separator:



"Definition". The $(k+1)$-**connected components** of a k-connected chordal graph are obtained by slicing it through all its k-separators (which are k-cliques).

Decomposition of chordal graphs into k-connected components

"Definition". **Slicing** through a k-separator:

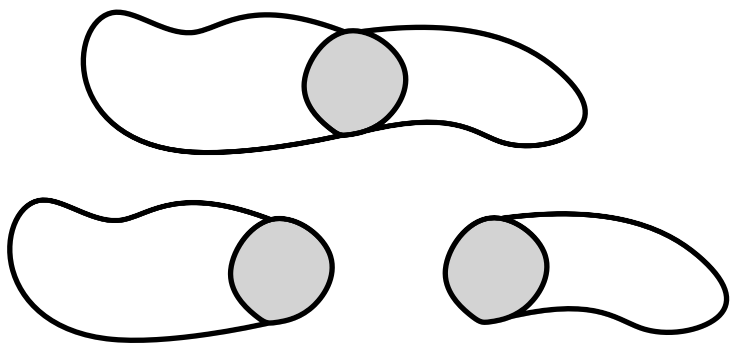


"Definition". The $(k+1)$-**connected components** of a k-connected chordal graph are obtained by slicing it through all its k-separators (which are k-cliques).

Proposition. This is well defined (the order does not matter, no k-separators appear or disappear in the process).

Decomposition of chordal graphs into k-connected components

“Definition”. **Slicing** through a k-separator:

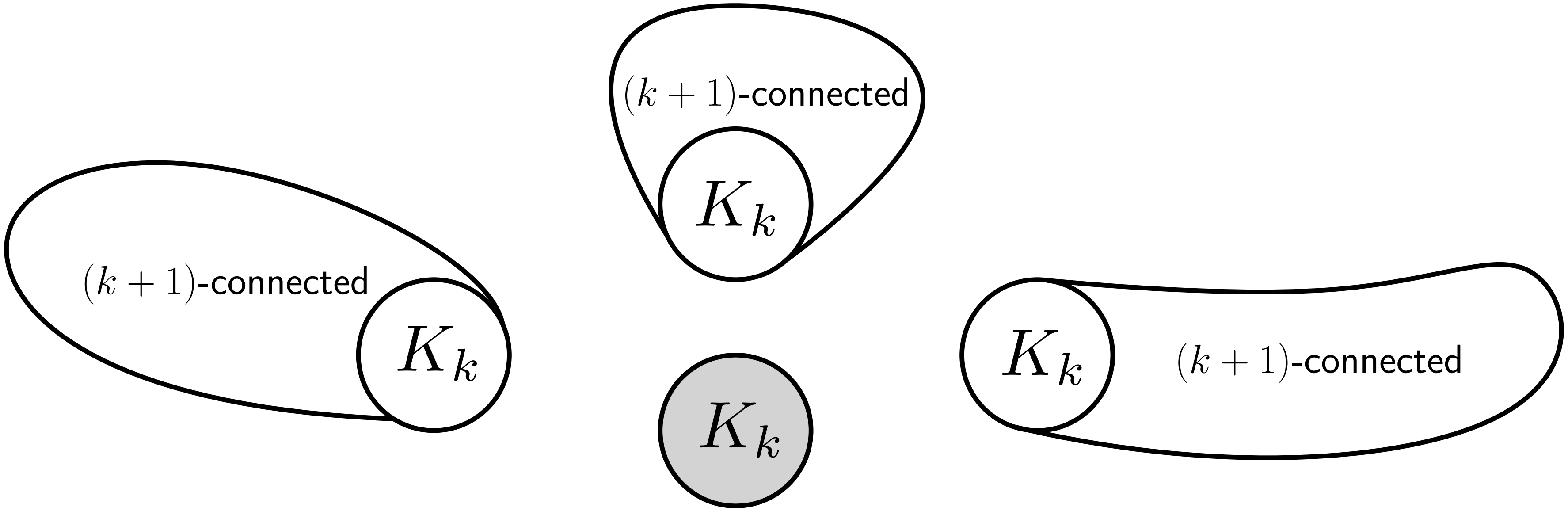


“Definition”. The $(k+1)$-**connected components** of a k-connected chordal graph are obtained by slicing it through all its k-separators (which are k-cliques).

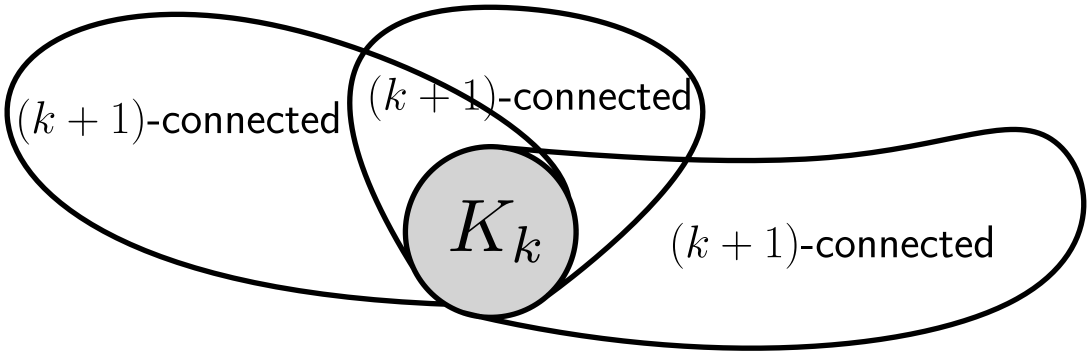
Proposition. This is well defined (the order does not matter, no k-separators appear or disappear in the process).

$\rightarrow$ Note that the $(k+1)$-connected components are the **maximal** $(k+1)$**-connected subgraphs**.

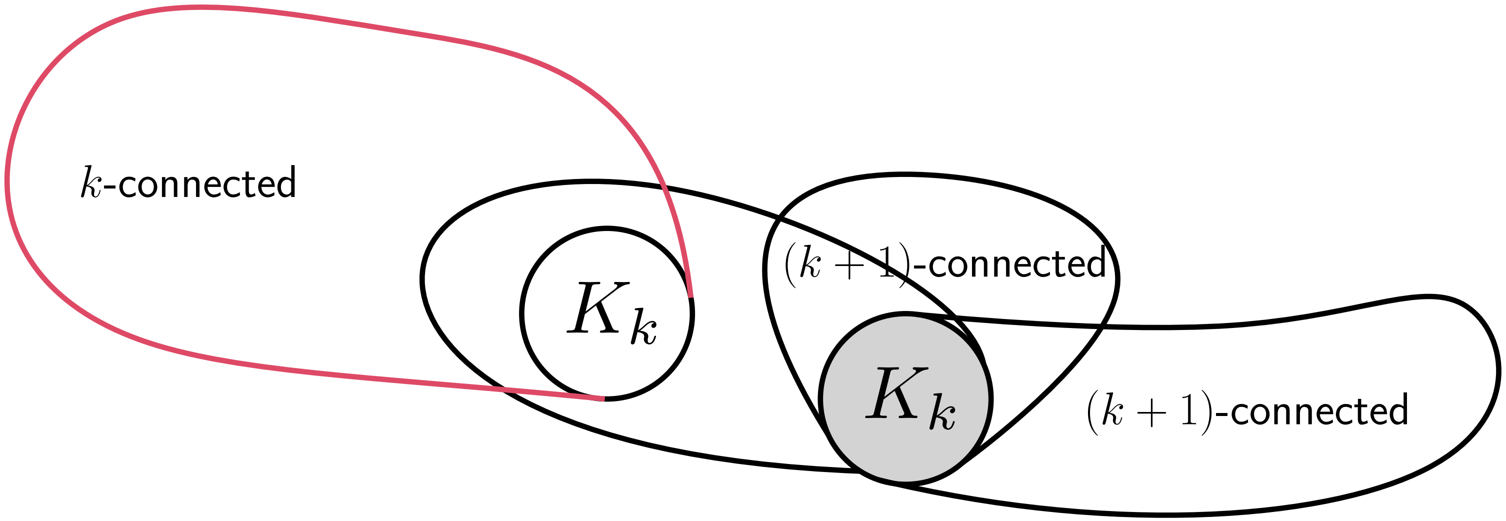
Decomposition of chordal graphs into k-connected components



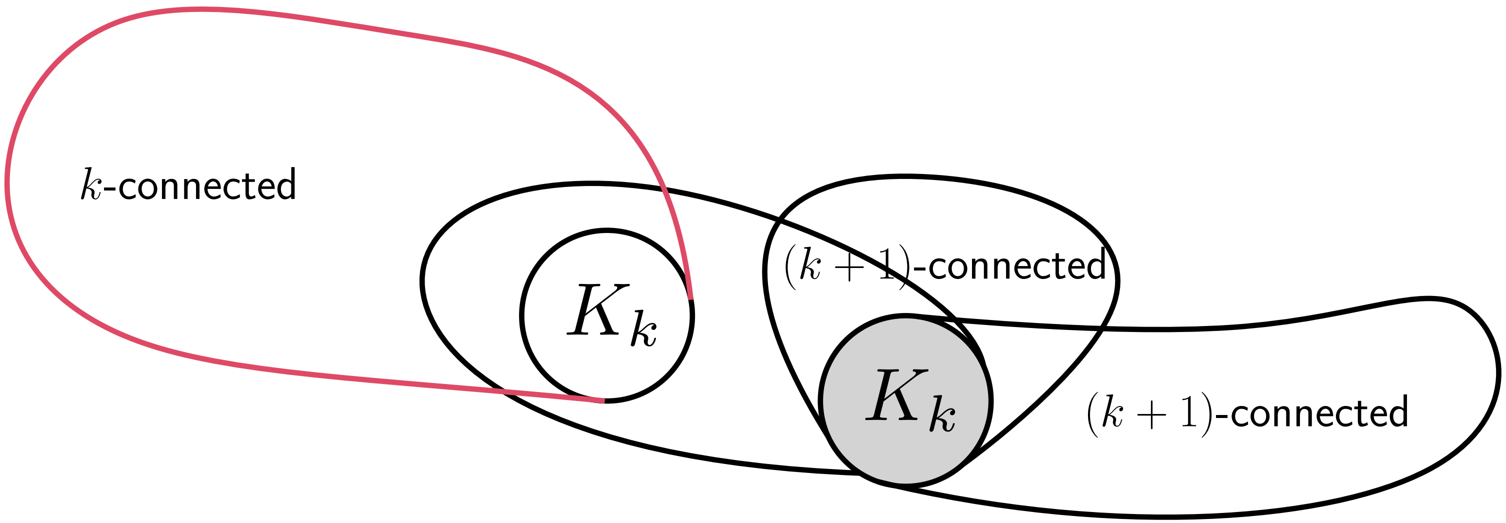
Decomposition of chordal graphs into k-connected components



Decomposition of chordal graphs into k-connected components



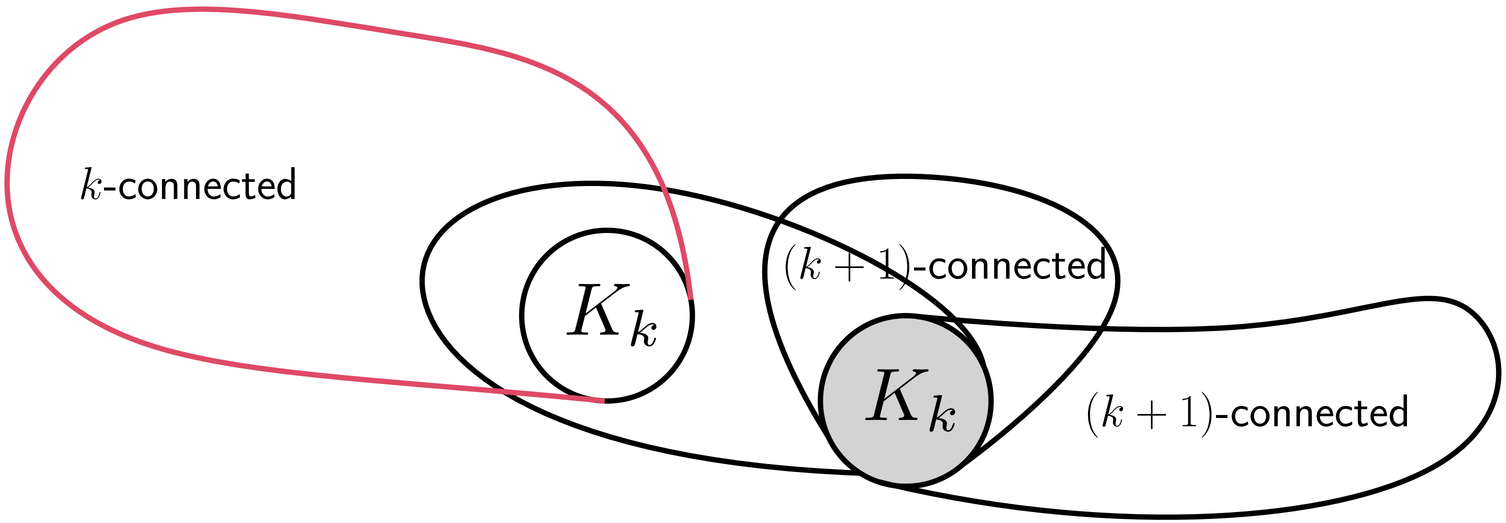
Decomposition of chordal graphs into k-connected components



Let $\mathcal{G}_k^{(i)}$ be the class of k-connected chordal graphs rooted at an unlabelled, ordered i-clique.
Consider its multivariate exponential generating function $G_k^{(j)}(x, x_k)$, where the variable x_k marks the number of k-cliques. Then, we have that

$$G_k^{(k)}(x, x_k) = \exp\left(G_{k+1}^{(k)}(x, x_k G_k^{(k)}(x, x_k))\right).$$

Decomposition of chordal graphs into k-connected components



Let $\mathcal{G}_k^{(i)}$ be the class of k-connected chordal graphs rooted at an unlabelled, ordered i-clique.
Consider its multivariate exponential generating function $G_k^{(j)}(x, x_k)$, where the variable x_k marks the number of k-cliques. Then, we have that

$$G_k^{(k)}(x, x_k) = \exp\left(G_{k+1}^{(k)}(x, x_k G_k^{(k)}(x, x_k))\right).$$

This generalizes the classical decomposition of connected graphs into 2-connected components.

Chordal graphs with bounded tree-width

We start with the $(t+1)$-connected members: only K_{t+1}.

$$G_{t+1}(\mathbf{x}) = \frac{1}{(t+1)!} \prod_{j \in [t]} x_j^{\binom{t+1}{j}}.$$

Chordal graphs with bounded tree-width

We start with the $(t+1)$-connected members: only K_{t+1}.

$$G_{t+1}(\mathbf{x}) = \frac{1}{(t+1)!} \prod_{j \in [t]} x_j^{\binom{t+1}{j}}.$$

And we go down the stairs.

$$\begin{array}{ccccccccccc}
G_{t+1} & \rightarrow & G_{t+1}^{(t)} & & & & & & & & \\
 & & \downarrow & & & & & & & & \\
 & & G_t^{(t)} & \rightarrow & G_t & \rightarrow & G_t^{(t-1)} & & & & \\
 & & & & & & \downarrow & & & & \\
 & & & & & & \vdots & & & & \\
 & & & & & & \downarrow & & & & \\
 & & & & & & G_2^{(2)} & \rightarrow & G_2 & \rightarrow & G_2^{(1)} \\
 & & & & & & & & & & \downarrow \\
 & & & & & & & & & & G_1^{(1)} \rightarrow G_1
\end{array}$$

Chordal graphs with bounded tree-width

We start with the $(t+1)$-connected members: only K_{t+1}.

$$G_{t+1}(\mathbf{x}) = \frac{1}{(t+1)!} \prod_{j \in [t]} x_j^{\binom{t+1}{j}}.$$

And we go down the stairs.

$$\begin{array}{ccccccccc}
G_{t+1} & \rightarrow & G_{t+1}^{(t)} & & & & & & \\
 & & \downarrow & & & & & & \\
 & & G_t^{(t)} & \rightarrow & G_t & \rightarrow & G_t^{(t-1)} & \longrightarrow & t\text{-trees} \\
 & & & & & & \downarrow & & \\
 & & & & & & \vdots & & \\
 & & & & & & \downarrow & & \\
 & & & & & & G_2^{(2)} & \rightarrow \; G_2 \; \rightarrow & G_2^{(1)} \\
 & & & & & & & & \downarrow \\
 & & & & & & & & G_1^{(1)} \; \rightarrow \; G_1
\end{array}$$

Chordal graphs with bounded tree-width

We start with the $(t+1)$-connected members: only K_{t+1}.

$$G_{t+1}(\mathbf{x}) = \frac{1}{(t+1)!} \prod_{j \in [t]} x_j^{\binom{t+1}{j}}.$$

And we go down the stairs.

$$\begin{array}{ccccccccccc}
G_{t+1} & \rightarrow & G_{t+1}^{(t)} & & & & & & & & \\
 & & \downarrow & & & & & & & & \\
 & & G_t^{(t)} & \rightarrow & G_t & \rightarrow & G_t^{(t-1)} & & \longrightarrow & t\text{-}\mathbf{trees} & \\
 & & & & & & \downarrow & & & & \\
 & & & & & & \vdots & & & & \\
 & & & & & & \downarrow & & & & \\
 & & & & & & G_2^{(2)} & \rightarrow & G_2 & \rightarrow & G_2^{(1)} \\
 & & & & & & & & & & \downarrow \\
 & & & & & & & & & & G_1^{(1)} \rightarrow G_1
\end{array}$$

Finally, $G(x) = G_0(x) = \exp(G_1(x, 1, \ldots, 1))$.

Chordal graphs with bounded tree-width

We start with the $(t+1)$-connected members: only K_{t+1}.

$$G_{t+1}(\mathbf{x}) = \frac{1}{(t+1)!} \prod_{j \in [t]} x_j^{\binom{t+1}{j}}.$$

And we go down the stairs.

$$\begin{array}{llllll}
G_{t+1} \rightarrow G_{t+1}^{(t)} & & & & & \\
\qquad\quad \downarrow & & & & & \\
\qquad\quad G_t^{(t)} \rightarrow G_t \rightarrow G_t^{(t-1)} & & & & & \longrightarrow t\text{-trees} \\
\qquad\qquad\qquad\qquad\quad \downarrow & & & & & \\
\qquad\qquad\qquad\qquad\quad \vdots & & & & & \\
\qquad\qquad\qquad\qquad\quad \downarrow & & & & & \\
\qquad\qquad\qquad\qquad\quad G_2^{(2)} \rightarrow G_2 \rightarrow G_2^{(1)} & & & & & \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \downarrow & & & & & \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad G_1^{(1)} \rightarrow G_1 & & & & &
\end{array}$$

[Wormald '85]

Finally, $G(x) = G_0(x) = \exp(G_1(x, 1, \ldots, 1))$.

Analytic combinatorics

First principle. The **location** of a function's singularities dictates the **exponential growth** of its coefficients.

$$[x^n]F(x) \propto \rho^{-n}, \qquad \text{if } \rho > 0.$$

Analytic combinatorics

First principle. The **location** of a function's singularities dictates the **exponential growth** of its coefficients.

$$[x^n]F(x) \propto \rho^{-n}, \qquad \text{if } \rho > 0.$$

Second principle. The **nature** of a function's singularities determines the **subexponential factor**.

Analytic combinatorics

First principle. The **location** of a function's singularities dictates the **exponential growth** of its coefficients.

$$[x^n]F(x) \propto \rho^{-n}, \qquad \text{if } \rho > 0.$$

Second principle. The **nature** of a function's singularities determines the **subexponential factor**.

Transfer theorem. ([Floajolet, Odlyzko '82])
Suppose that $F(x)$ is analytic in a Δ-domain where it admits a singular expansion

$$F(x) = f_1(x) + f_2(x)\left(1 - \frac{x}{x_0}\right)^{-\alpha}.$$

for analytic functions f_1, f_2 with $f_2(x_0) \neq 0$. Then, as $n \to \infty$,

$$[x^n]F(x) \sim f_2(x_0)\frac{n^{\alpha-1}}{\Gamma(\alpha)}x_0^{-n}.$$

Analytic combinatorics

First principle. The **location** of a function's singularities dictates the **exponential growth** of its coefficients.

$$[x^n]F(x) \propto \rho^{-n}, \qquad \text{if } \rho > 0.$$

Second principle. The **nature** of a function's singularities determines the **subexponential factor**.

Transfer theorem. ([Floajolet, Odlyzko '82])
Suppose that $F(x)$ is analytic in a Δ-domain where it admits a singular expansion

$$F(x) = f_1(x) + f_2(x)\left(1 - \frac{x}{x_0}\right)^{-\alpha}.$$

for analytic functions f_1, f_2 with $f_2(x_0) \neq 0$. Then, as $n \to \infty$,

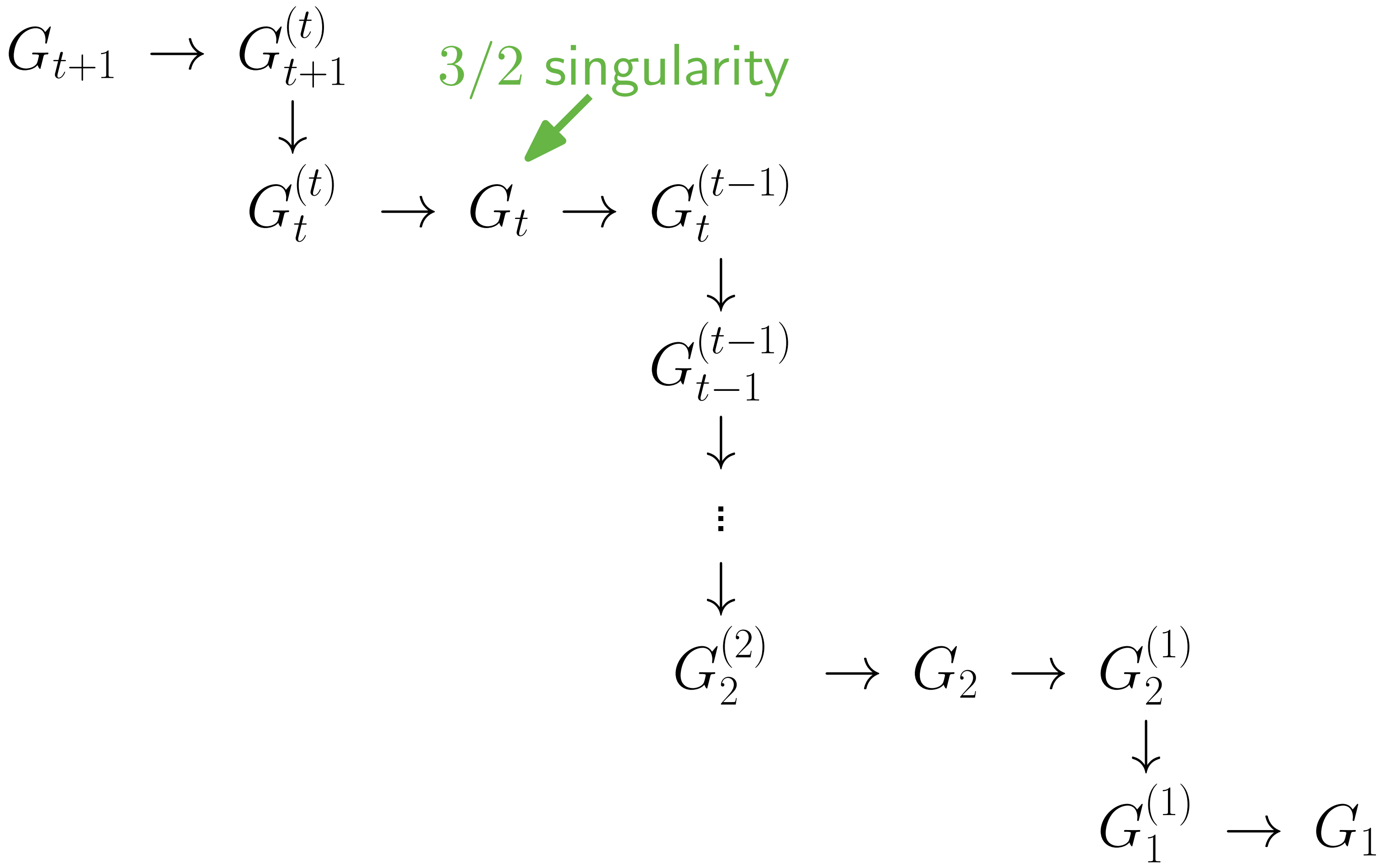
$$[x^n]F(x) \sim f_2(x_0)\frac{n^{\alpha-1}}{\Gamma(\alpha)}x_0^{-n}.$$

In our case, $\alpha = -3/2$.

Analysis of the system

$$
\begin{array}{ccccccccc}
G_{t+1} & \rightarrow & G_{t+1}^{(t)} & & & & & & \\
 & & \downarrow & & & & & & \\
 & & G_t^{(t)} & \rightarrow & G_t & \rightarrow & G_t^{(t-1)} & & \\
 & & & & & & \downarrow & & \\
 & & & & & & G_{t-1}^{(t-1)} & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \vdots & & \\
 & & & & & & \downarrow & & \\
 & & & & & & G_2^{(2)} \rightarrow G_2 \rightarrow G_2^{(1)} & & \\
 & & & & & & \qquad\qquad\qquad\quad \downarrow & & \\
 & & & & & & \qquad\qquad\qquad\quad G_1^{(1)} \rightarrow G_1 & &
\end{array}
$$

Analysis of the system



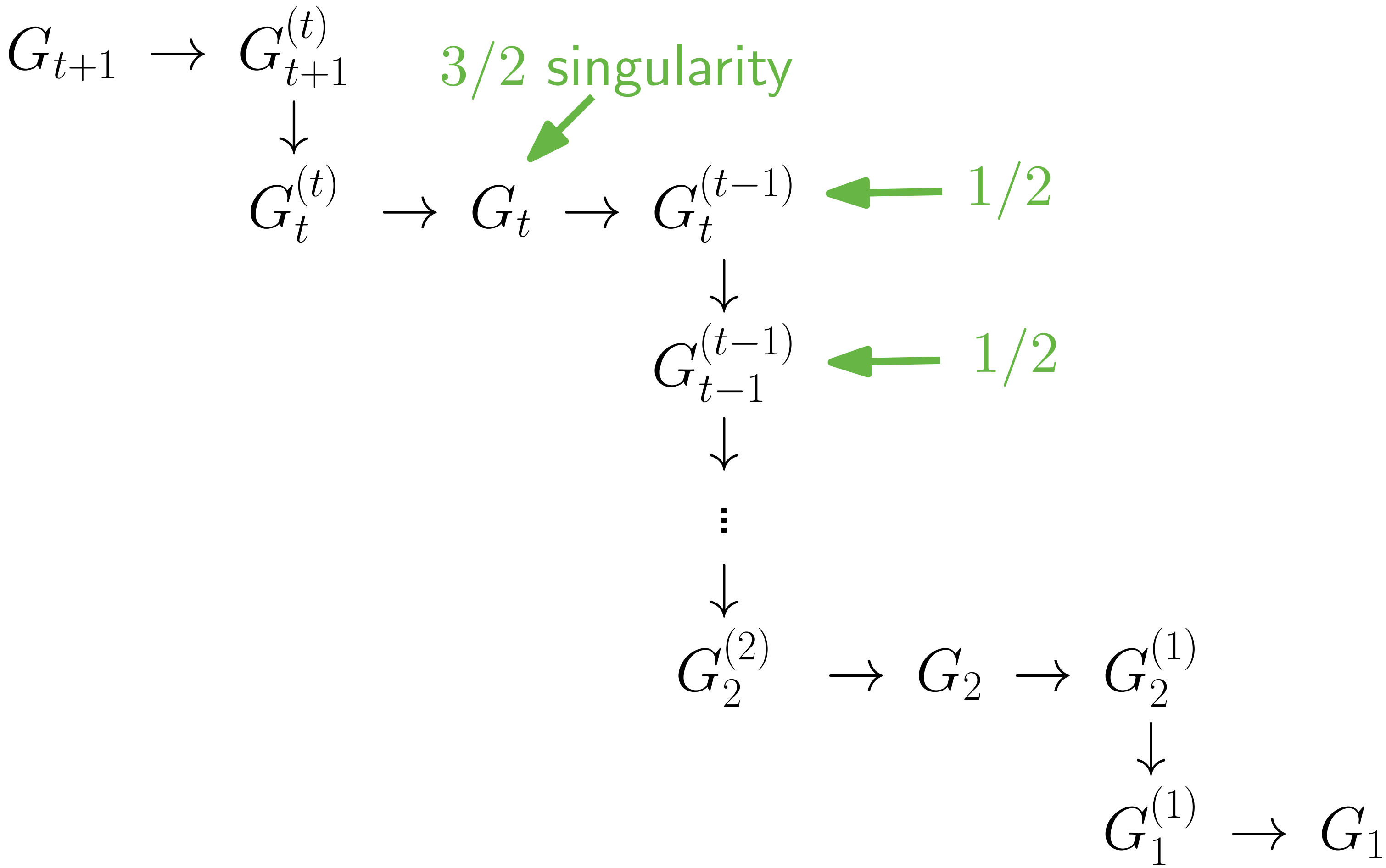
Analysis of the system

3/2 singularity (arrow pointing to G_t)

1/2 (arrow pointing to $G_t^{(t-1)}$)

$$
\begin{array}{ccccccccc}
G_{t+1} & \rightarrow & G_{t+1}^{(t)} & & & & & & \\
 & & \downarrow & & & & & & \\
 & & G_t^{(t)} & \rightarrow & G_t & \rightarrow & G_t^{(t-1)} & & \\
 & & & & & & \downarrow & & \\
 & & & & & & G_{t-1}^{(t-1)} & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \vdots & & \\
 & & & & & & \downarrow & & \\
 & & & & & & G_2^{(2)} & \rightarrow & G_2 & \rightarrow & G_2^{(1)} \\
 & & & & & & & & & & \downarrow \\
 & & & & & & & & & & G_1^{(1)} & \rightarrow & G_1
\end{array}
$$

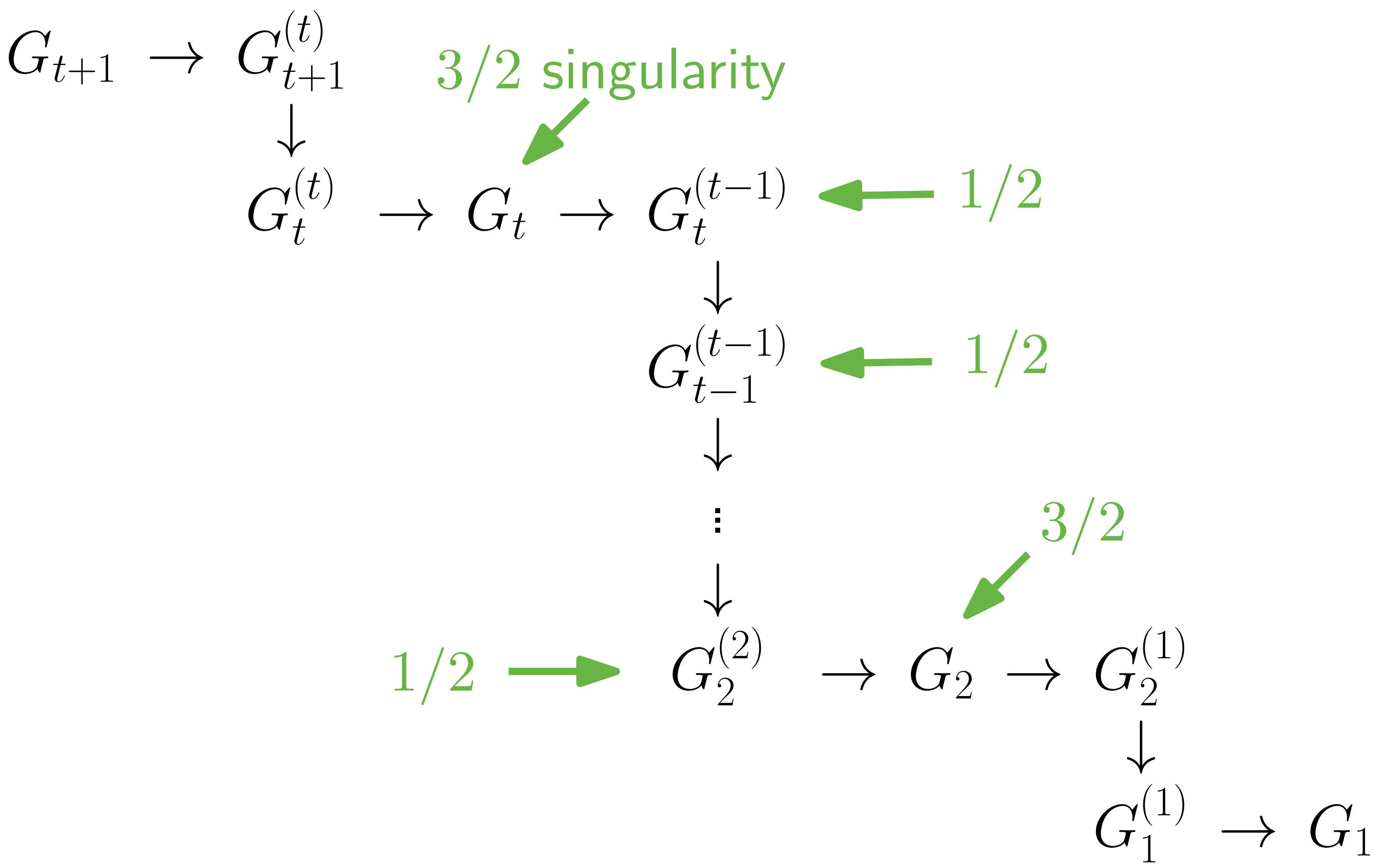
Analysis of the system



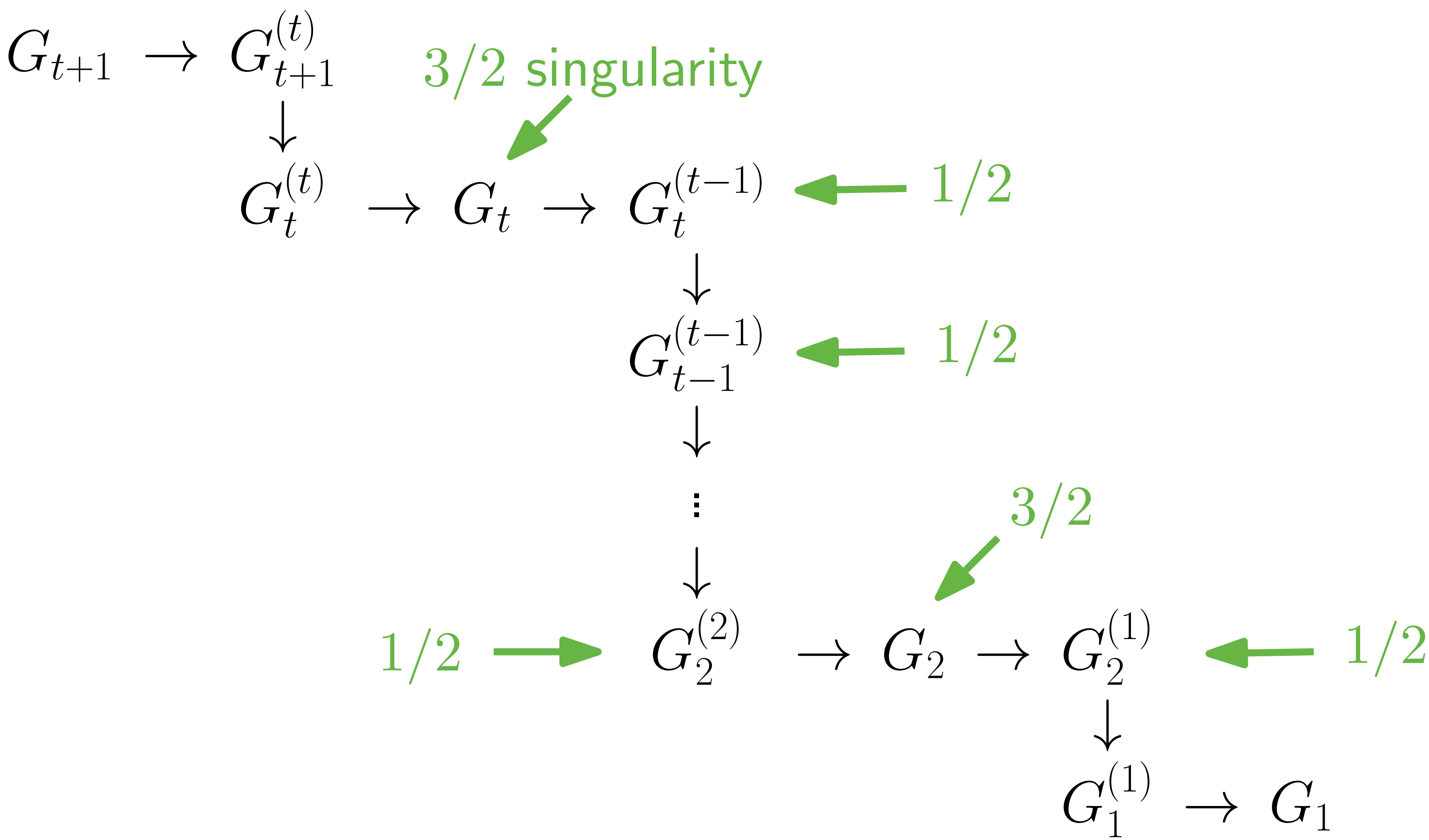
Analysis of the system

$$
\begin{array}{ccccccccc}
G_{t+1} & \to & G_{t+1}^{(t)} & & & & & & \\
 & & \downarrow & & \text{3/2 singularity} \swarrow & & & & \\
 & & G_t^{(t)} & \to & G_t & \to & G_t^{(t-1)} & \leftarrow 1/2 & \\
 & & & & & & \downarrow & & \\
 & & & & & & G_{t-1}^{(t-1)} & \leftarrow 1/2 & \\
 & & & & & & \downarrow & & \\
 & & & & & & \vdots & & \\
 & & & & & & \downarrow & & \\
 & & & & & 1/2 \to & G_2^{(2)} & \to G_2 \to & G_2^{(1)} \\
 & & & & & & & & \downarrow \\
 & & & & & & & & G_1^{(1)} \to G_1
\end{array}
$$

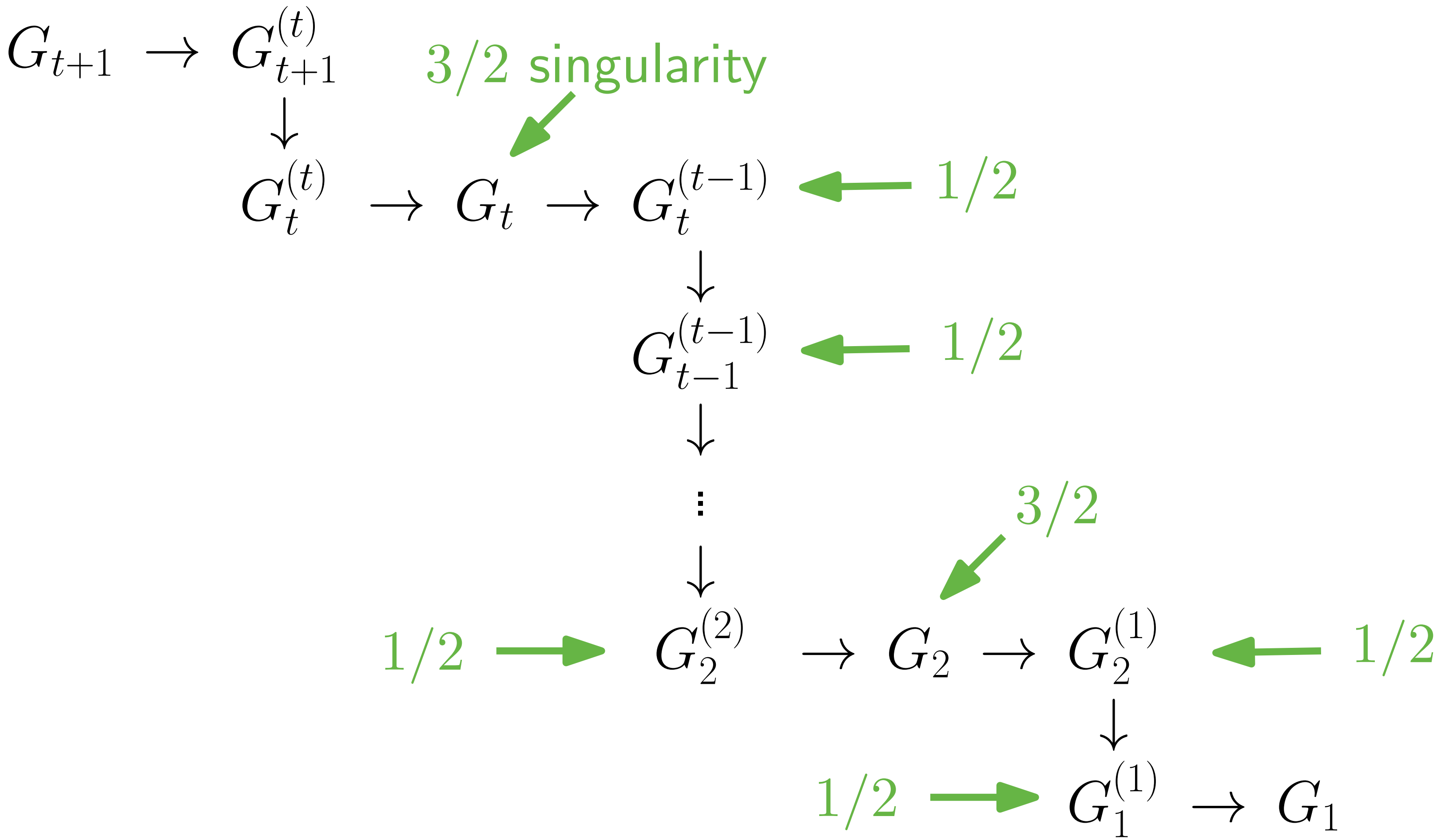
Analysis of the system



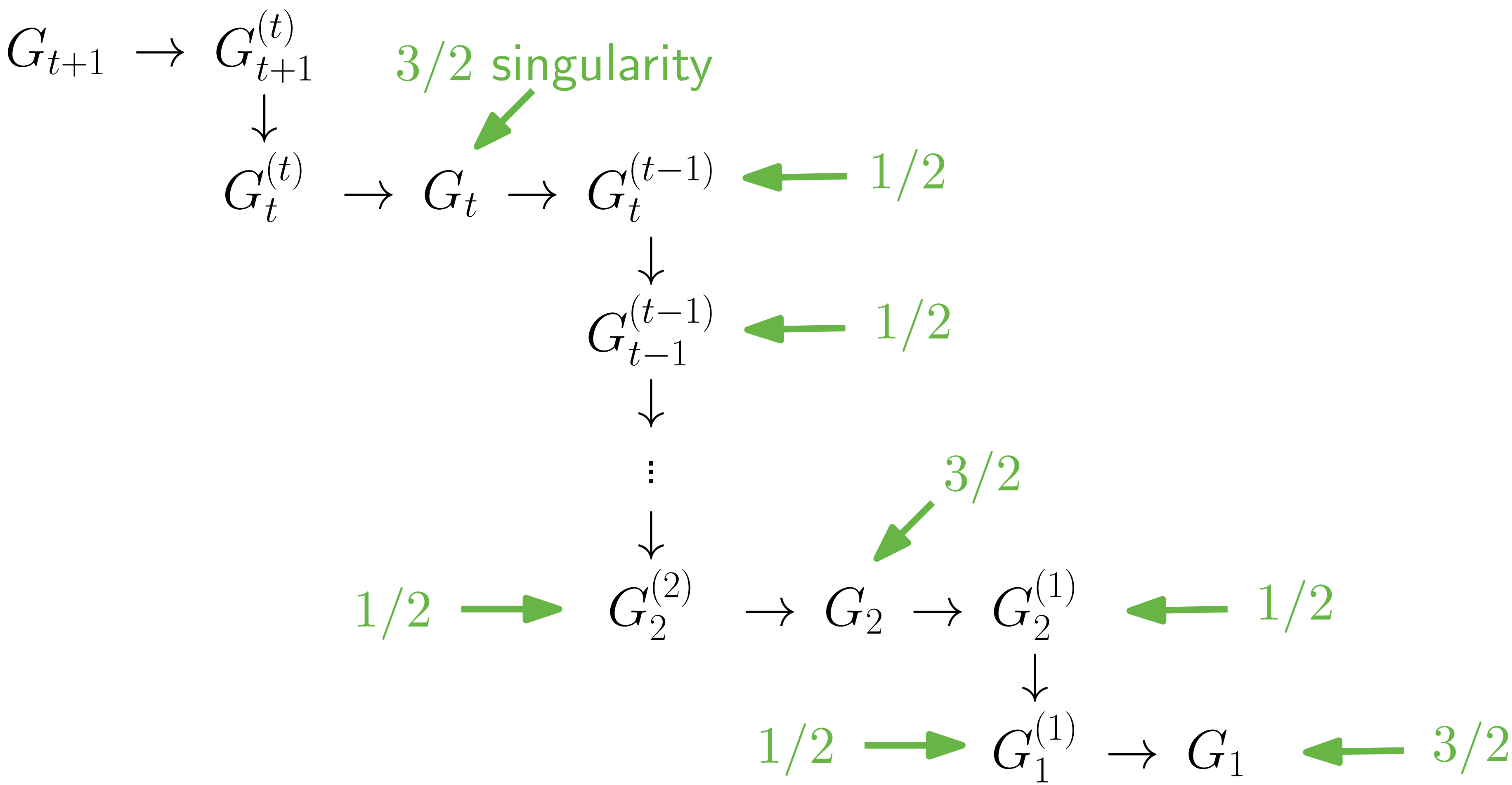
Analysis of the system



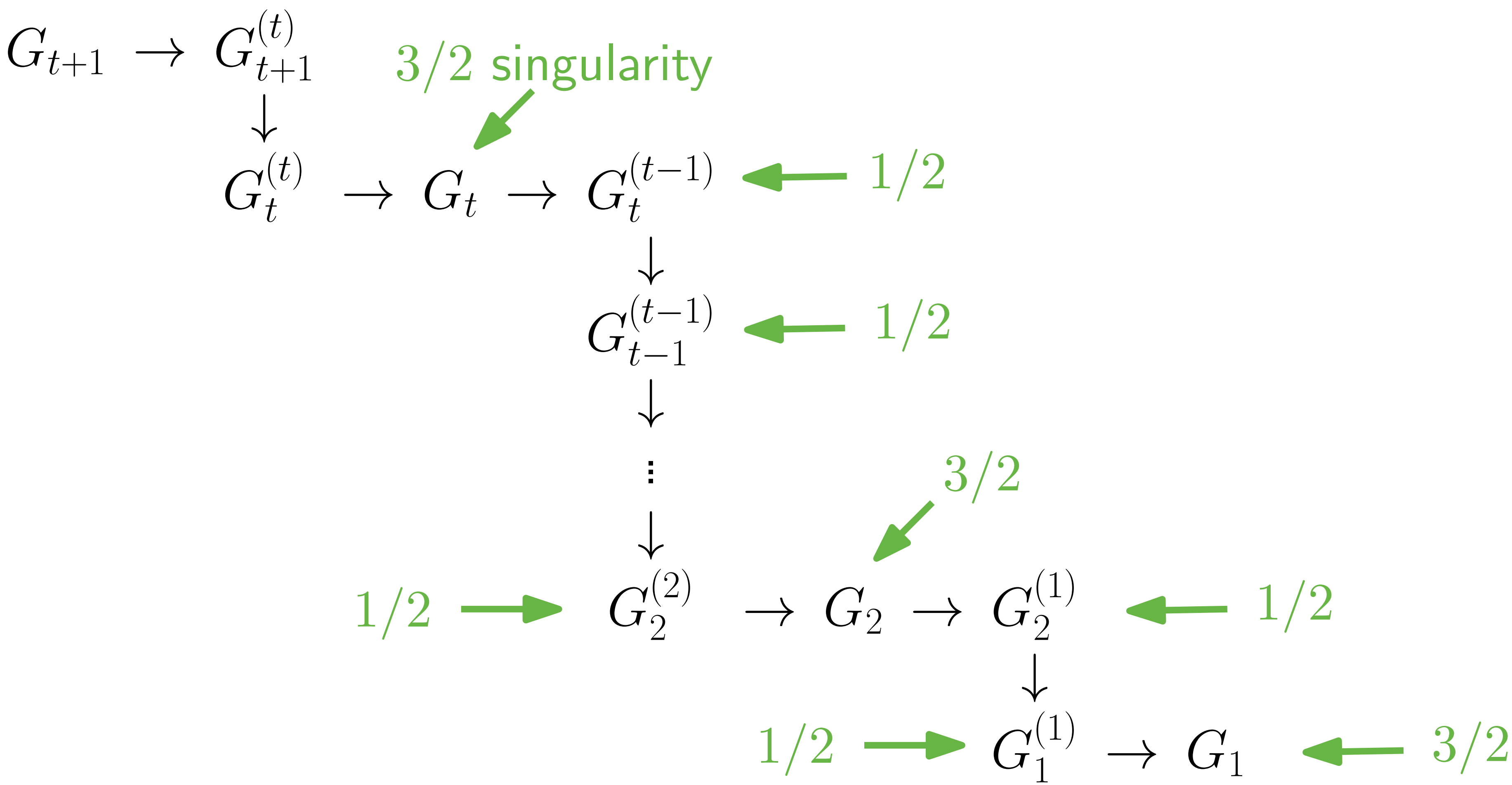
Analysis of the system



Analysis of the system

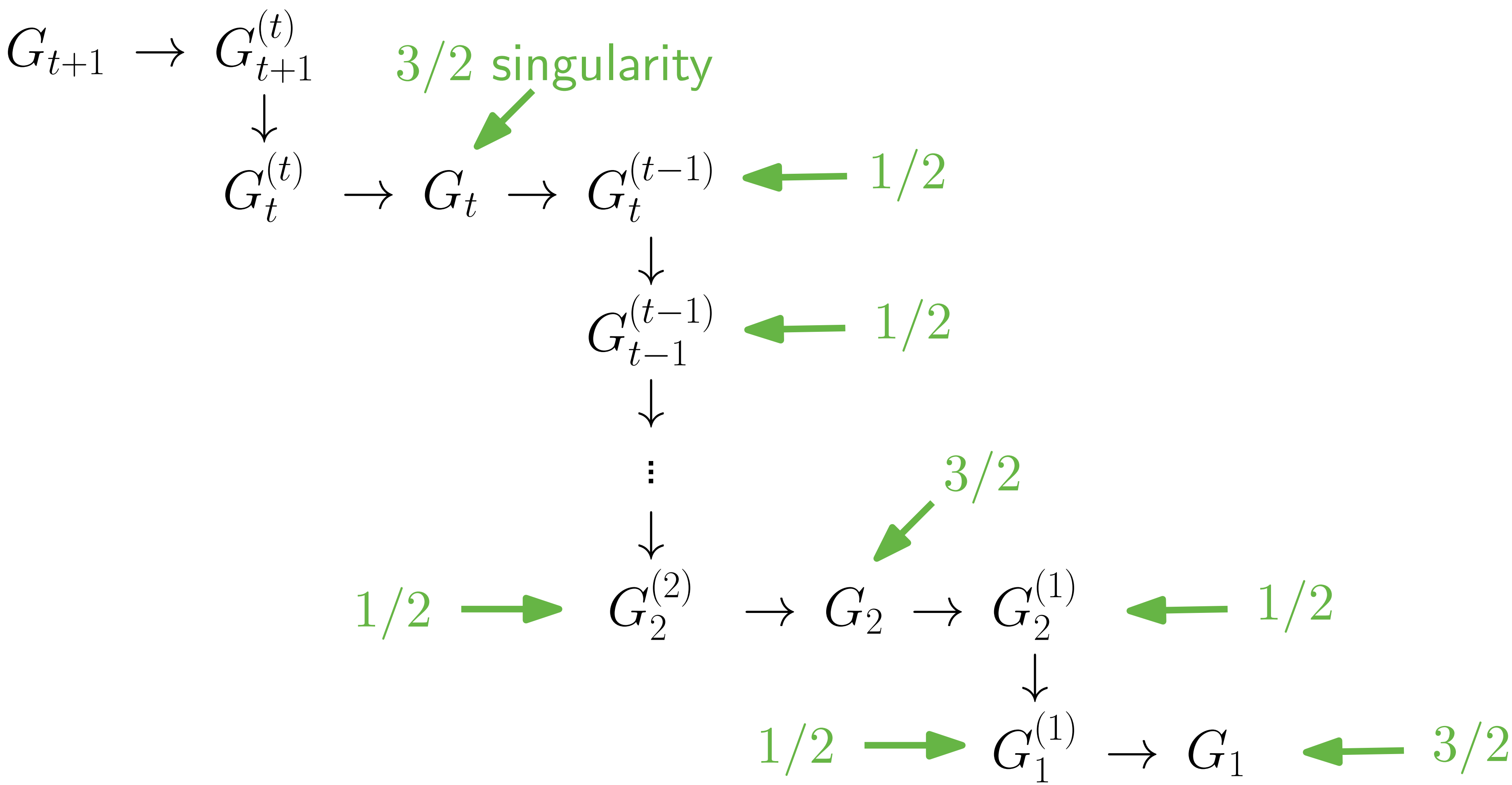


Analysis of the system



Lemma 1. We have singular expansions of the same type in any variable.

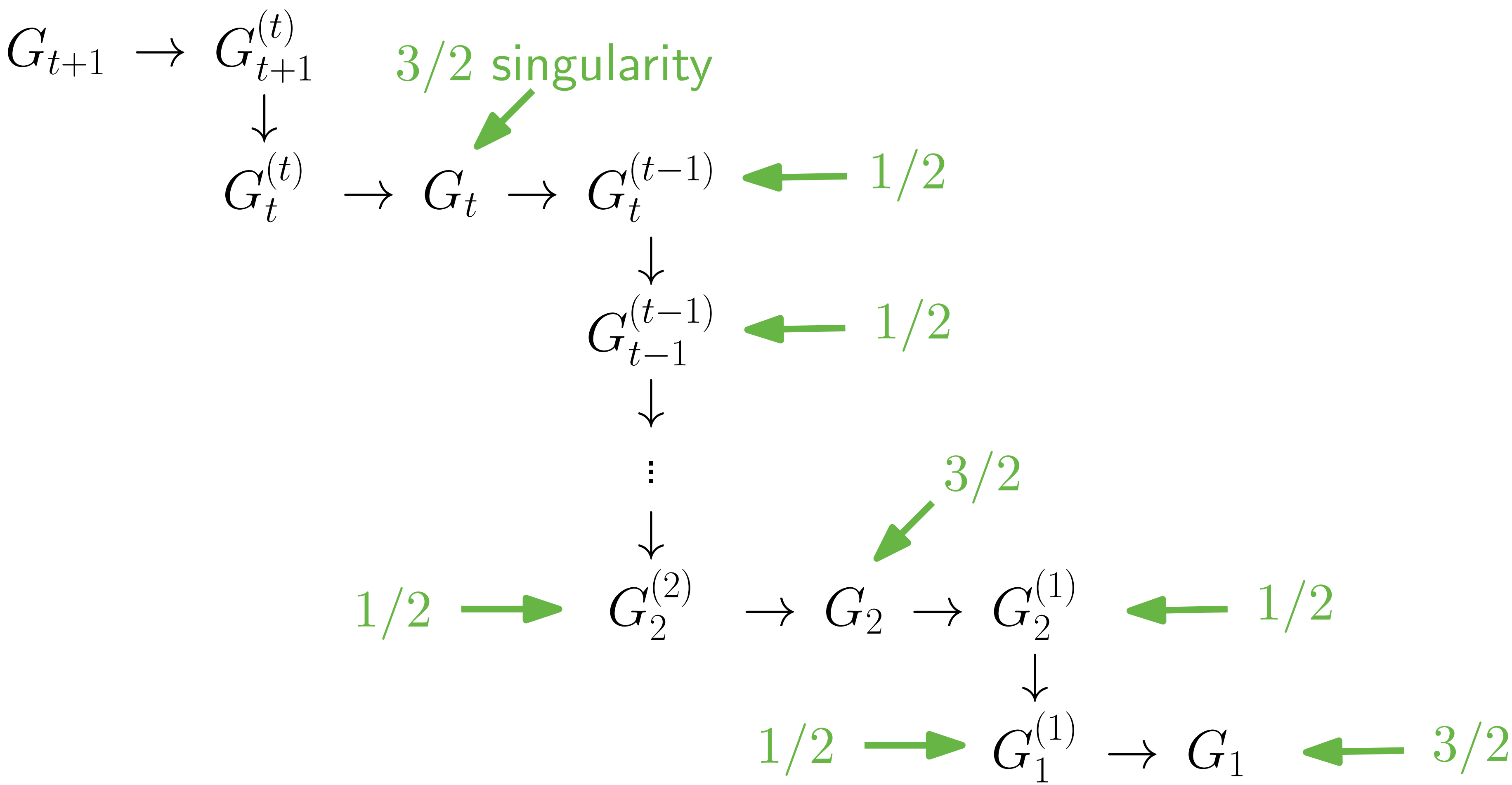
Analysis of the system



Lemma 1. We have singular expansions of the same type in any variable.

Lemma 2. Derivating a $3/2$-singularity yields a $1/2$-singularity and integrating a $1/2$-singularity yields a $3/2$-singularity.

Analysis of the system



Lemma 1. We have singular expansions of the same type in any variable.

Lemma 2. Derivating a $3/2$-singularity yields a $1/2$-singularity and integrating a $1/2$-singularity yields a $3/2$-singularity.

Lemma 3. The solution to the implicit equation has a $1/2$-singularity.

Analysis of the system

Theorem 2 follows from these singular expansions by an application of the so-called Quasi Power Theorem.

Analysis of the system

Theorem 2 follows from these singular expansions by an application of the so-called Quasi Power Theorem.

Some values of the singularities $\rho_{t,k}$

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$t=1$	0.36788						
$t=2$	0.14665	0.18394					
$t=3$	0.07703	0.08421	0.12263				
$t=4$	0.04444	0.04662	0.05664	0.09197			
$t=5$	0.02657	0.02732	0.03092	0.04152	0.07358		
$t=6$	0.01608	0.01635	0.01773	0.02184	0.03214	0.06131	
$t=7$	0.00974	0.00984	0.01038	0.01204	0.01614	0.02583	0.05255

Analysis of the system

Theorem 2 follows from these singular expansions by an application of the so-called Quasi Power Theorem.

Some values of the singularities $\rho_{t,k}$

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$t=1$	0.36788						
$t=2$	0.14665	0.18394					
$t=3$	0.07703	0.08421	0.12263				
$t=4$	0.04444	0.04662	0.05664	0.09197			
$t=5$	0.02657	0.02732	0.03092	0.04152	0.07358		
$t=6$	0.01608	0.01635	0.01773	0.02184	0.03214	0.06131	
$t=7$	0.00974	0.00984	0.01038	0.01204	0.01614	0.02583	0.05255

Theorem. ([Bender, Richmond, Wormald '85])
Almost all chordal graphs are split.

Therefore, the number of chordal graphs grows like $2^{n^2/4}$ and $\rho_t \to 0$ as $t \to \infty$.

Open questions

1. At what rate does ρ_t go to 0?

$$c_1 \frac{1}{t2^t} < \rho_t < c_2 \frac{1}{t}.$$

tw at most t

t-trees

Open questions

1. At what rate does ρ_t go to 0?

$$c_1 \frac{1}{t2^t} < \rho_t < c_2 \frac{1}{t}.$$

tw at most t

t-trees

2. What happens if we let $t = t(n)$ grow with n?

If $t = (1+\varepsilon)\log n$, then the class is large.

Consider split graphs with a clique of size t. There are $2^{(1+\varepsilon)\log n(n-(1+\varepsilon)\log n)}$ such graphs. This number grows faster that $c^n n!$ for any c.

At which point between $t = O(1)$ and $t = \log n$ the class ceases to be small?