Chordal graphs with bounded tree-width

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Work in collaboration with Michael Drmota, Marc Noy and Clément Requilé

EUROCOMB 2023 - Prague

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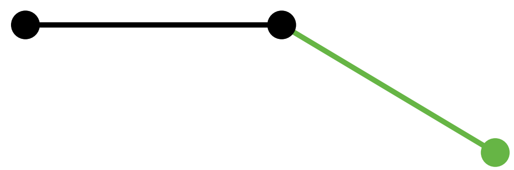

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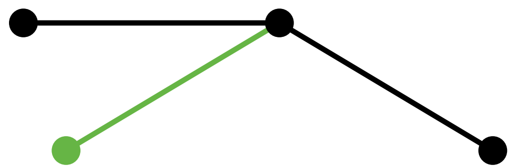

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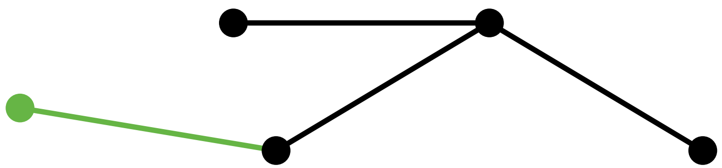

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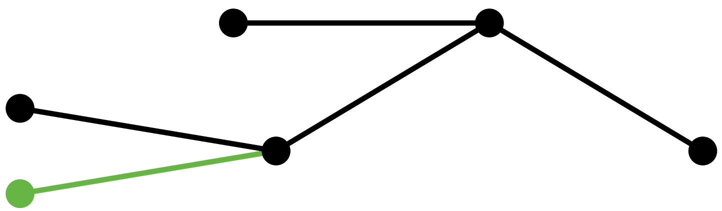

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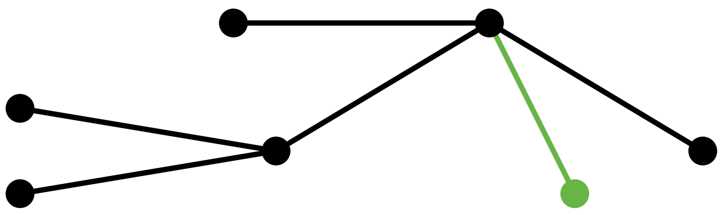

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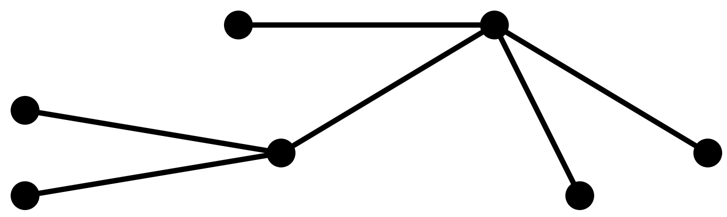

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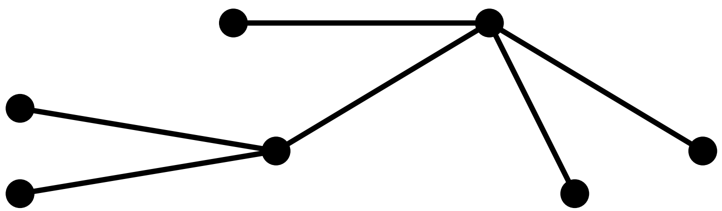

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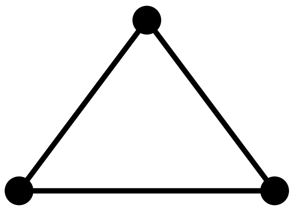

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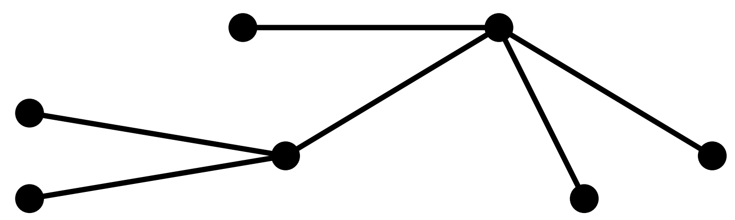

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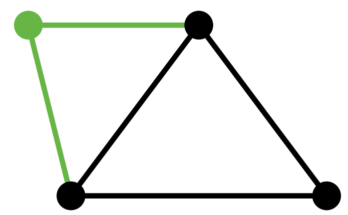

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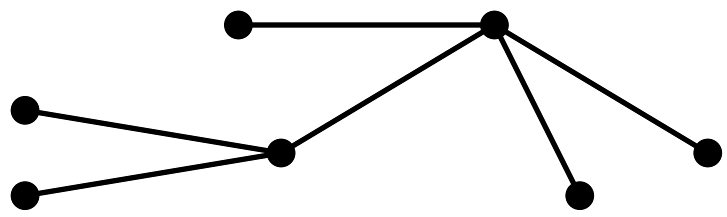

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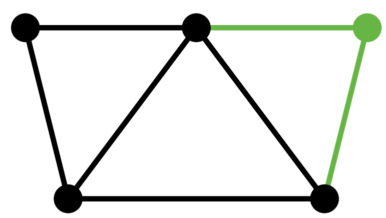

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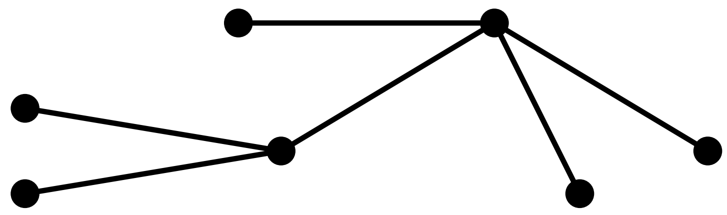

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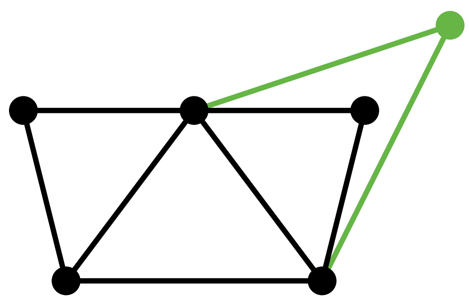

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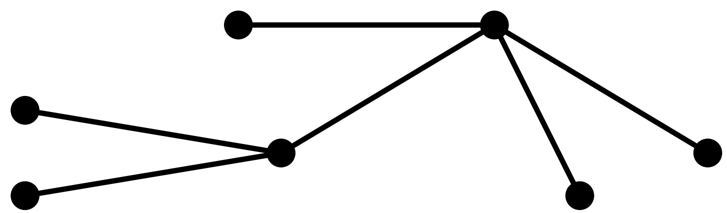

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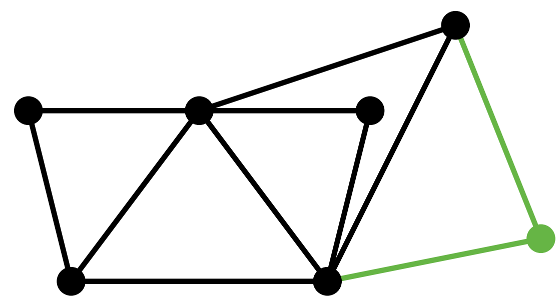

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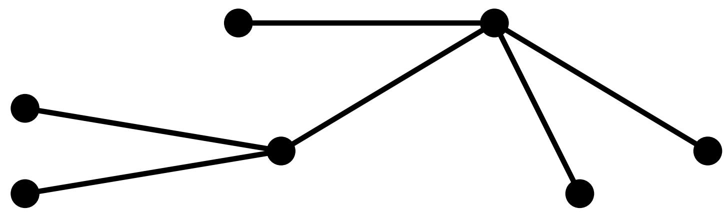

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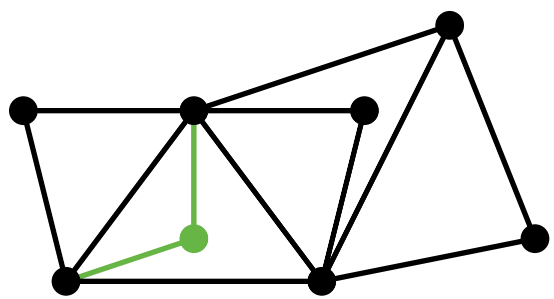

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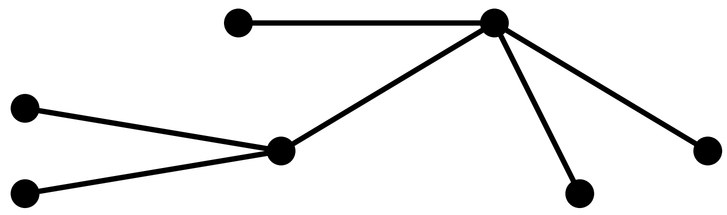

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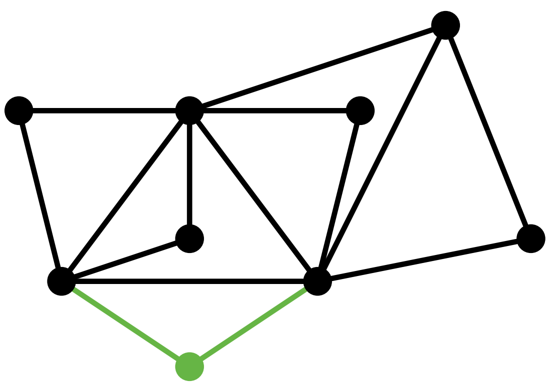

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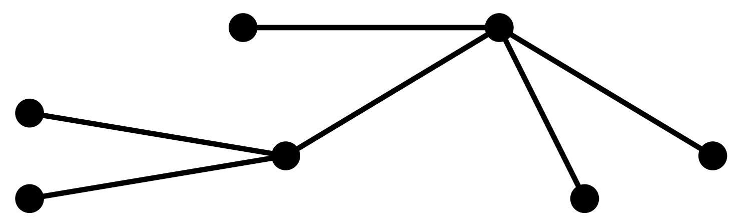

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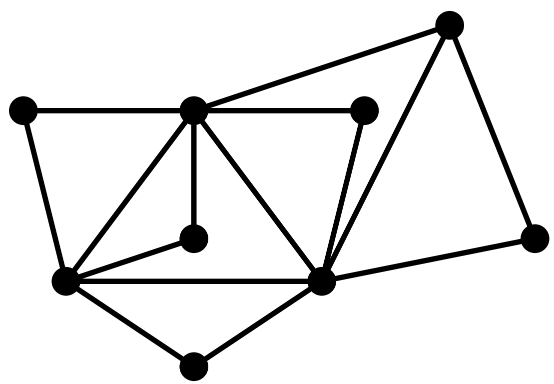

Theorem ([Beineke, Pippert, '69])
The number of labelled k-trees with n vertices is $\binom{n}{k}(kn-k^2+1)^{n-k-2}$.

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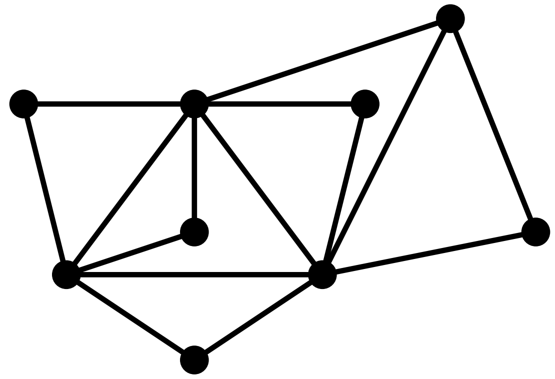

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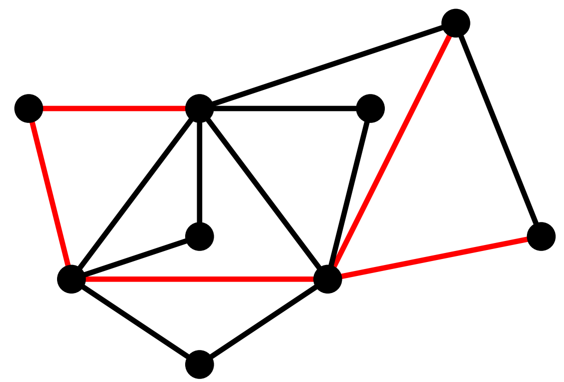

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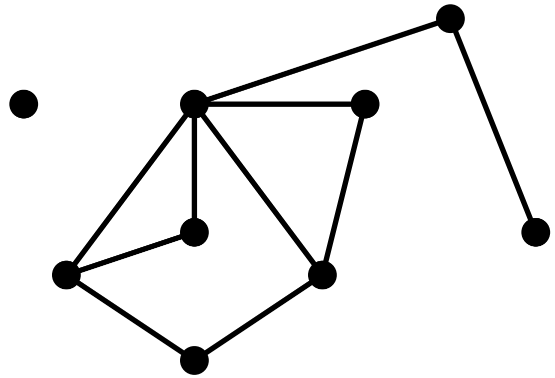

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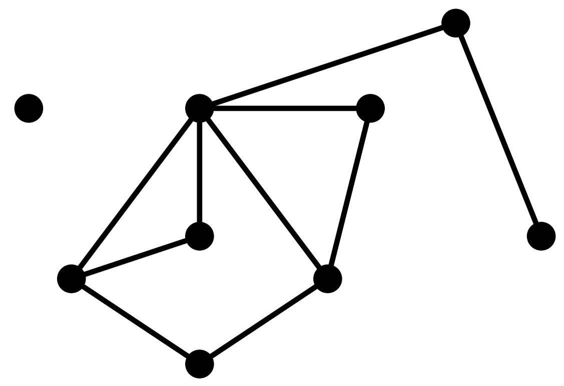

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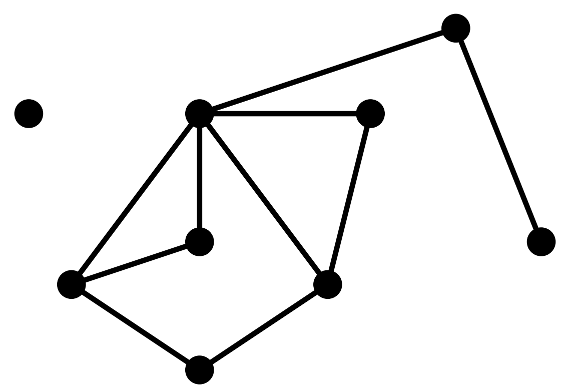

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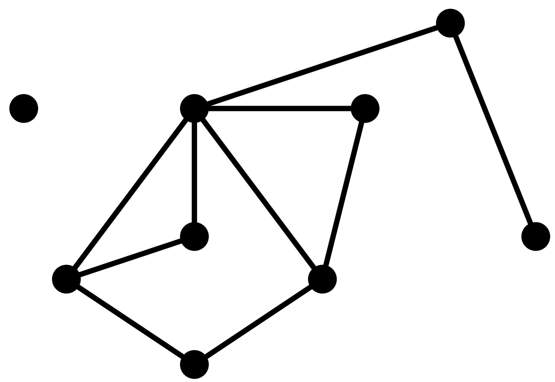

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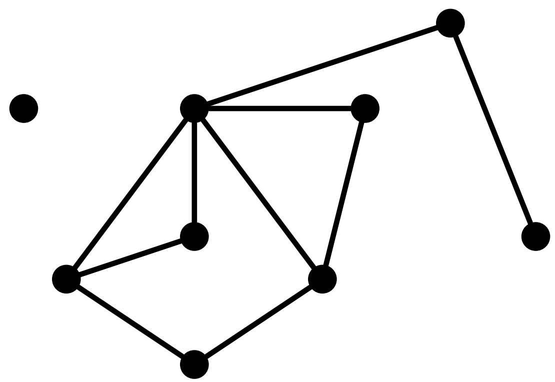

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Let $g_{n,t}$ be the number of labelled graphs with n vertices and tree-width at most t. Then, $\left(\frac{2^t tn}{\log t}\right)^n \leq g_{n,t} \leq (2^t tn)^n$.

[Baste, Noy, Sau '18]

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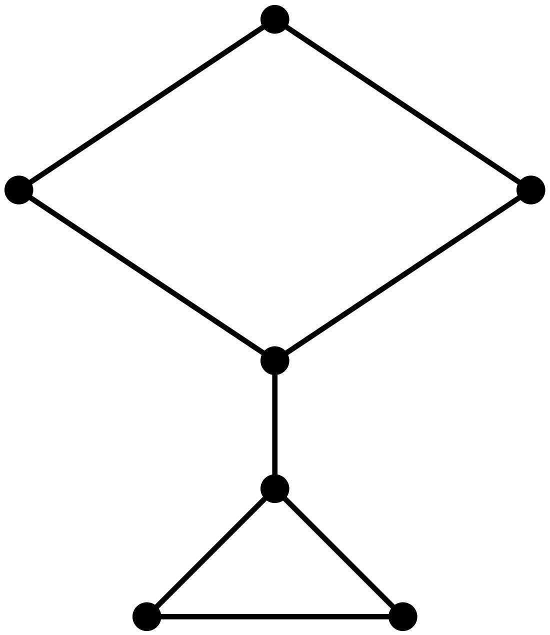

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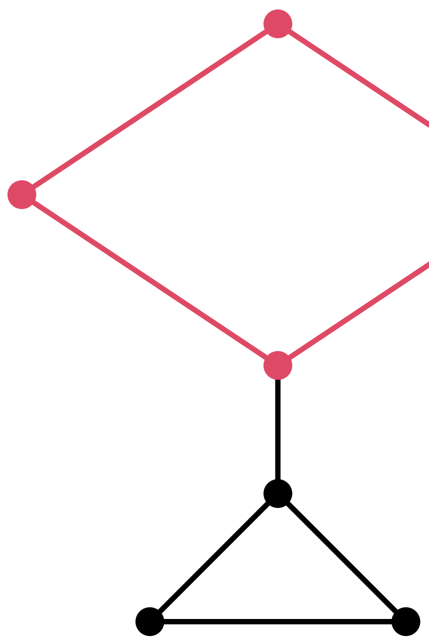

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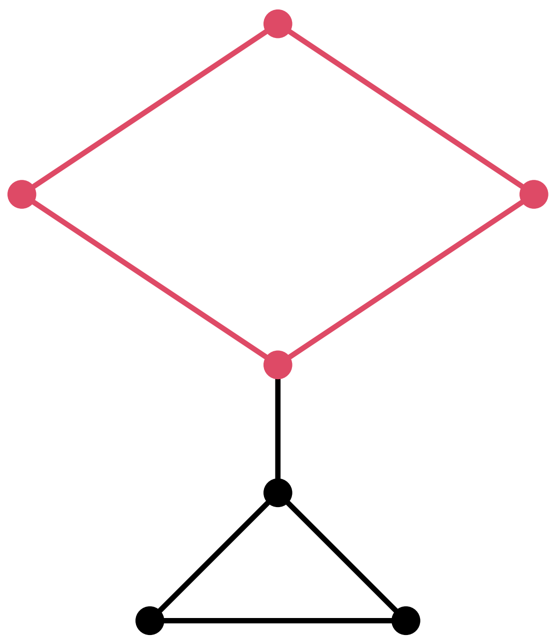

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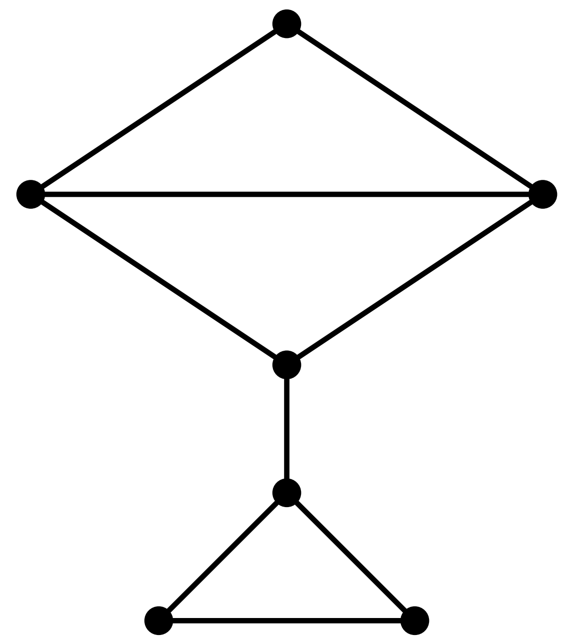

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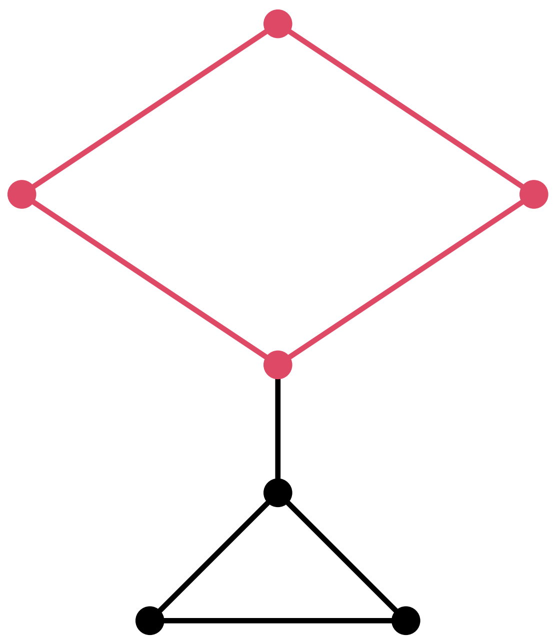

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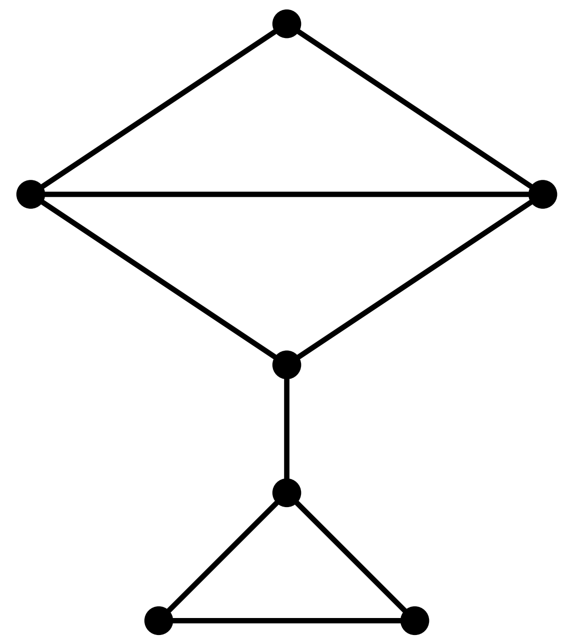

Chordal

Theorem. ([Dirac '61])
A graph is chordal iff every minimal separator is a clique.

Main results

Let $\mathcal{G}_{t,k,n}$ be the set of k-connected chordal graphs with n labelled vertices and tree-width at most t. Then, for fixed $t \geq 1$ and $0 \leq k \leq t$:

Theorem 1. ([C., Drmota, Noy, Réquilé '22])
There exist constants $c_{t,k} > 0$ and $\gamma_{t,k} \in (0,1)$ such that

$$|\mathcal{G}_{t,k,n}| \sim c_{t,k} \cdot n^{-5/2} \cdot \gamma_{t,k}^{n} \cdot n!, \qquad \text{as } n \to \infty.$$

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For $i \in [t]$, let X_i be the number of i-cliques in a uniform random graph in $\mathcal{G}_{t,k,n}$.

Theorem 2. ([C., Drmota, Noy, Réquilé '22])
There exist constants $\alpha, \gamma \in (0,1)$ such that

$$\frac{|X_i - \mathbb{E}X_i|}{\sqrt{\mathbb{V}X_i}} \xrightarrow{d} N(0,1), \qquad \text{with} \quad \mathbb{E}X_i \sim \alpha n \quad \text{and} \quad \mathbb{V}X_i \sim \sigma n.$$

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Definition. The **connected components** (or 1-connected components) of a graph are its connected maximal subgraphs.

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Let $\mathcal{G}$ be a class of labelled graphs and let $\mathcal{C} \subset \mathcal{G}$ be the class of its connected members. Then, their exponential generating functions satisfy

$$G(x) = \exp(C(x)),$$

provided that $\mathcal{G}$ is closed under disjoint unions and taking connected components.

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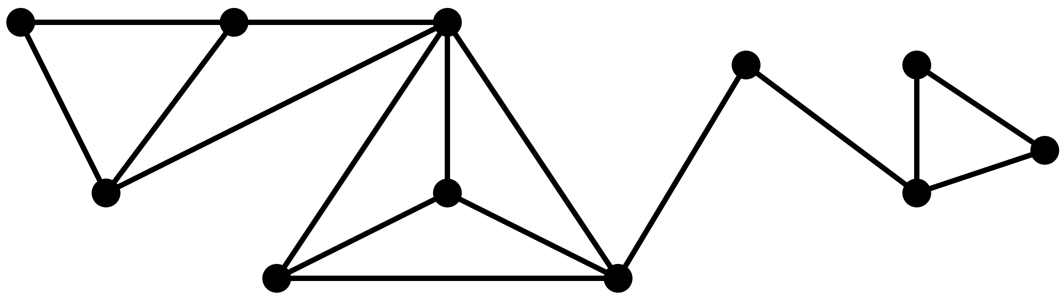

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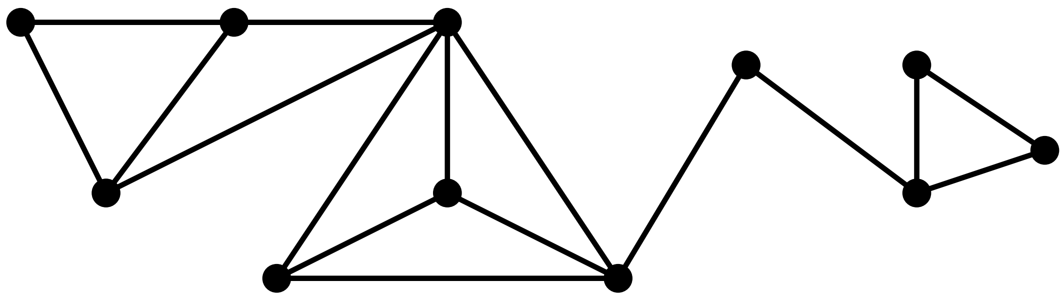

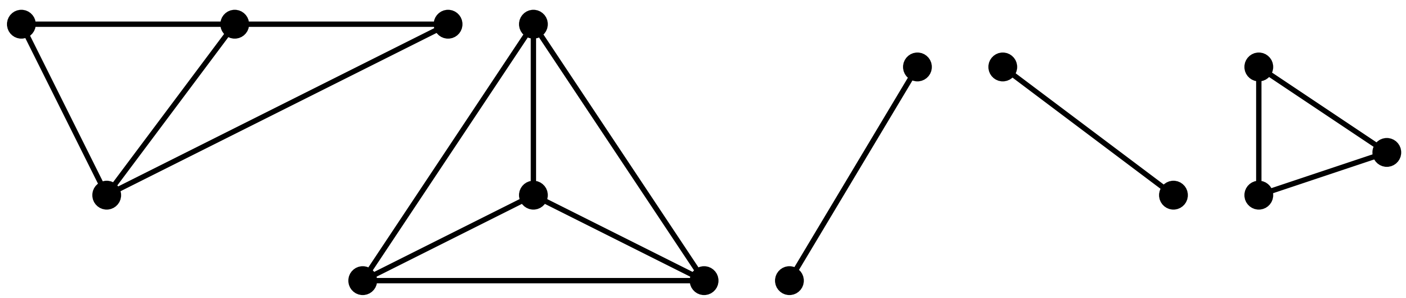

Decomposition of graphs into k-connected components

Let $\mathcal{B} \subset \mathcal{C}$ be the class of the 2-connected members of $\mathcal{G}$. Then,

$$C^{\bullet}(x) = x \exp(B'(C^{\bullet}(x))), \quad \text{where } C^{\bullet}(x) = xC'(x),$$

provided that $\mathcal{G}$ is **block-stable**, i.e., that a graph belongs to $\mathcal{C}$ iff its blocks belong to $\mathcal{B}$.

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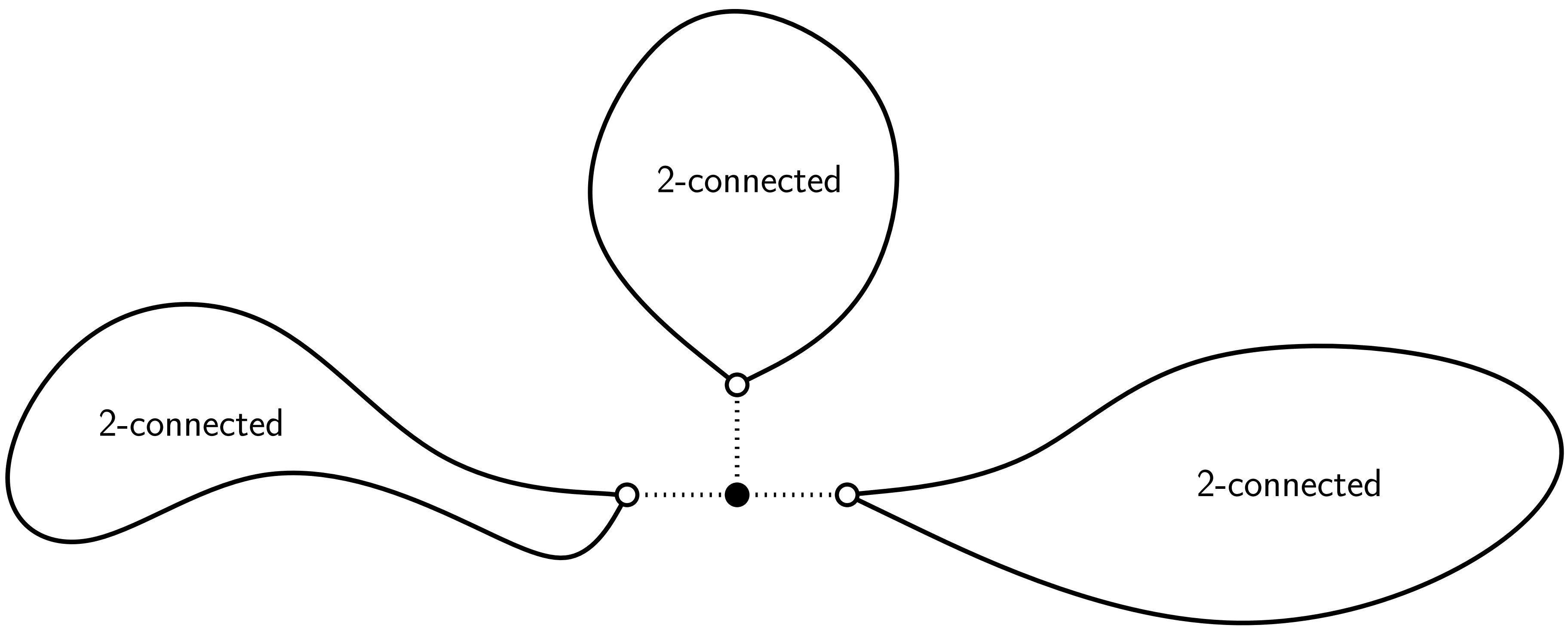

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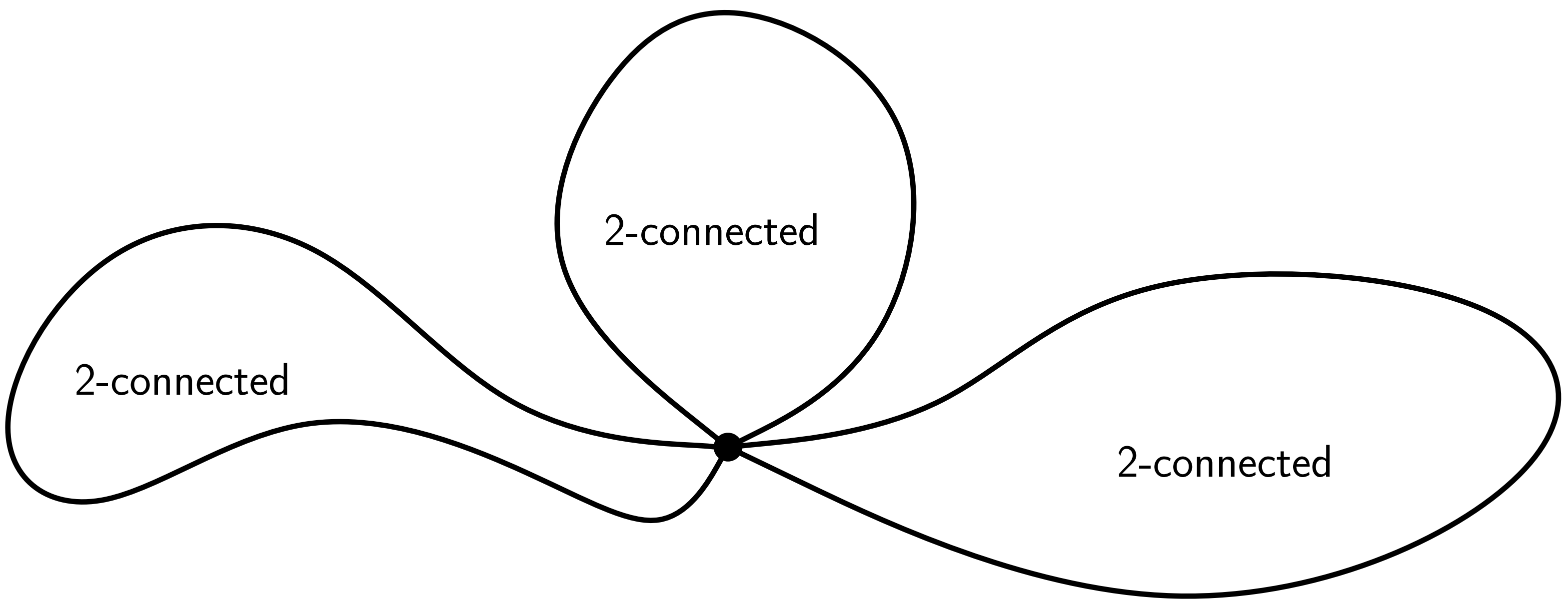

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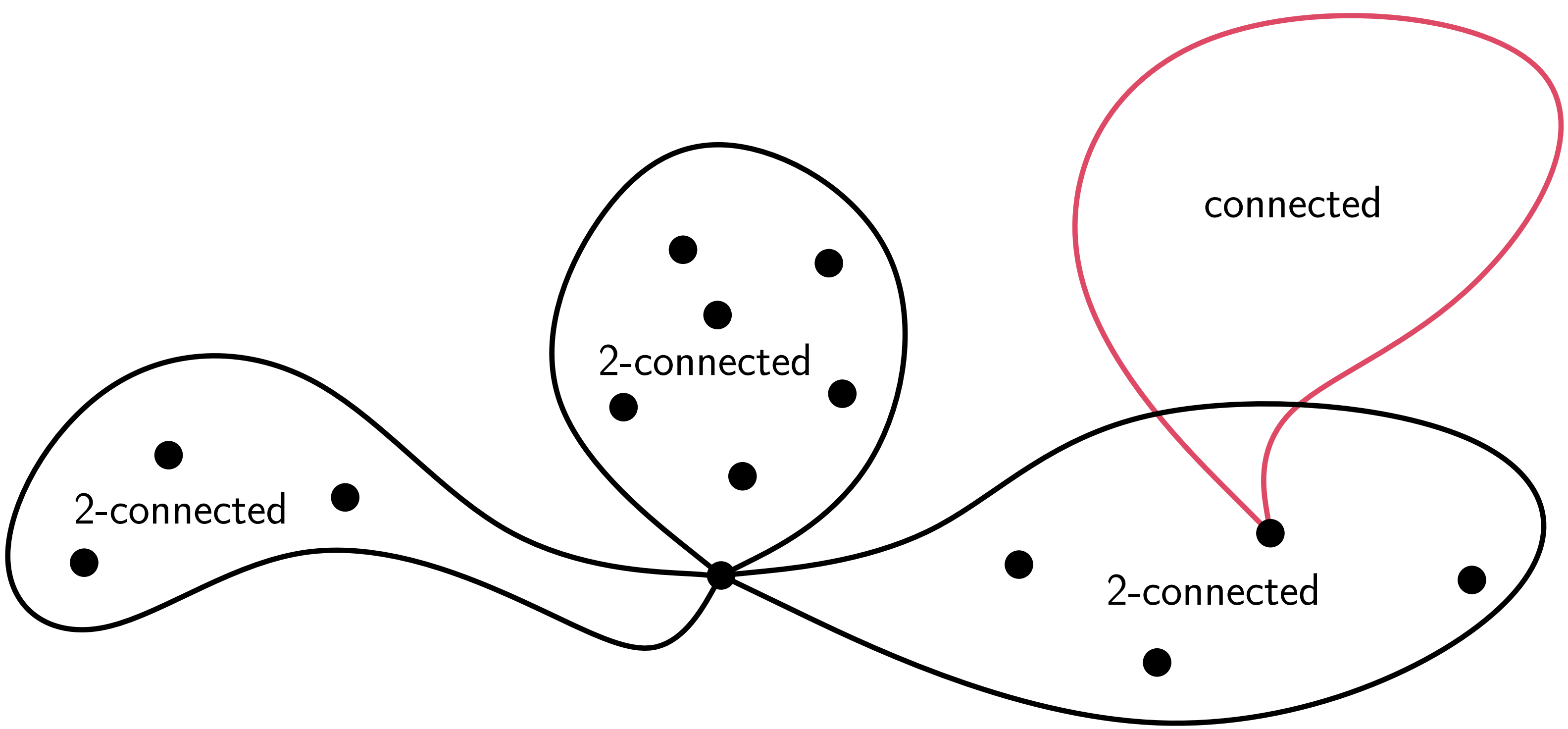

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However, any k-connected **chordal** graph admits a decomposition into $(k+1)$-connected components!

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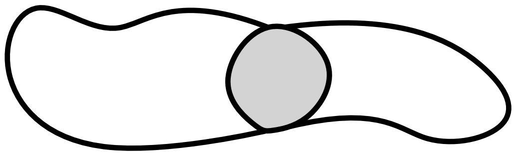

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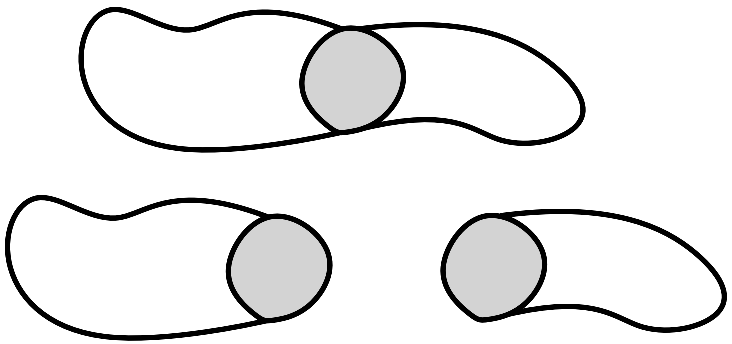

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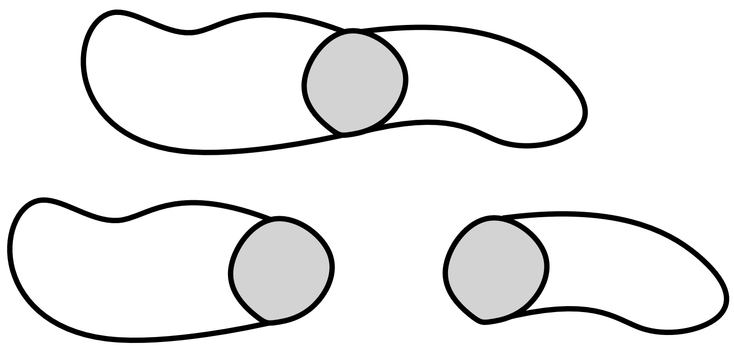
"Definition". **Slicing** through a k-separator:



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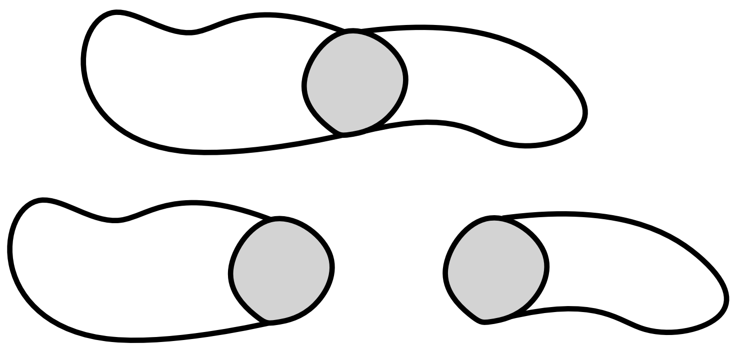

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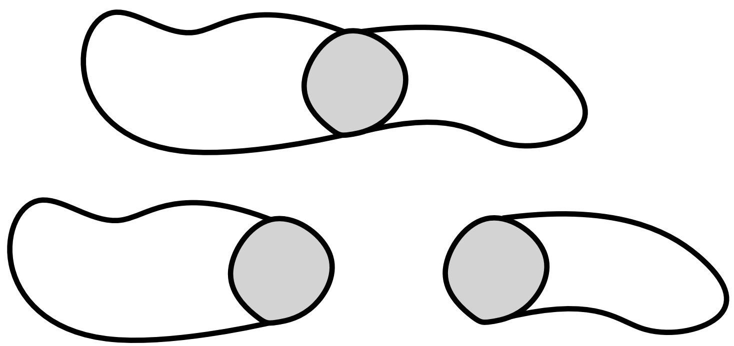

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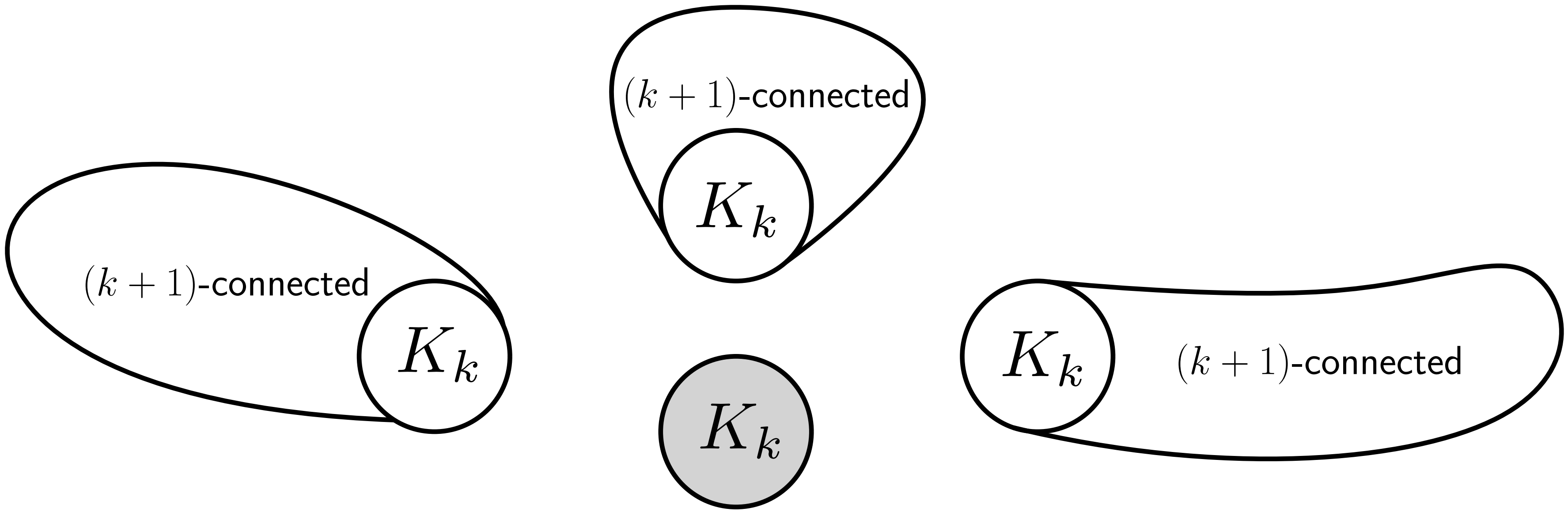
$\to$ Note that the $(k+1)$-connected components are the **maximal** $(k+1)$-**connected subgraphs**.

Decomposition of chordal graphs into k-connected components



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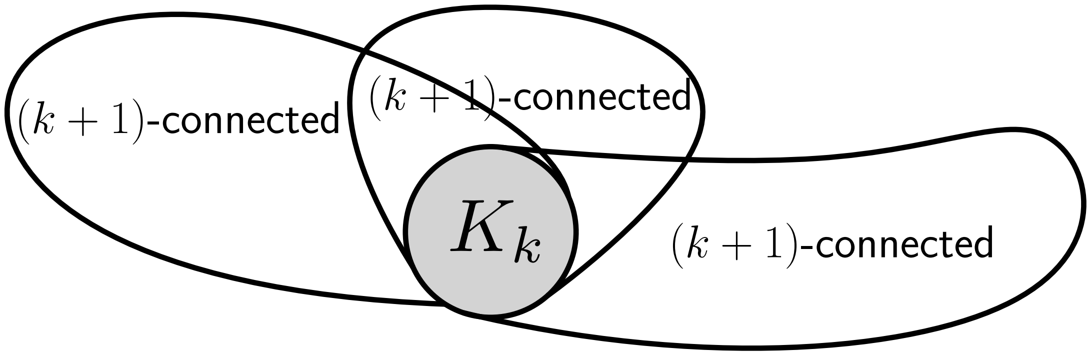

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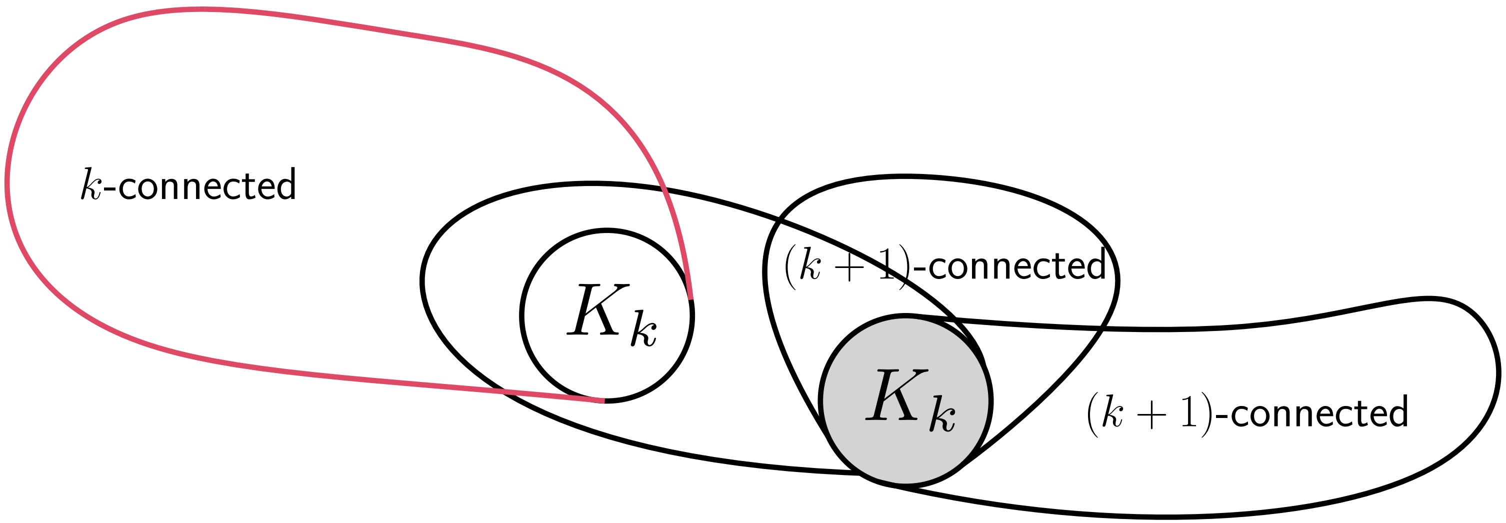

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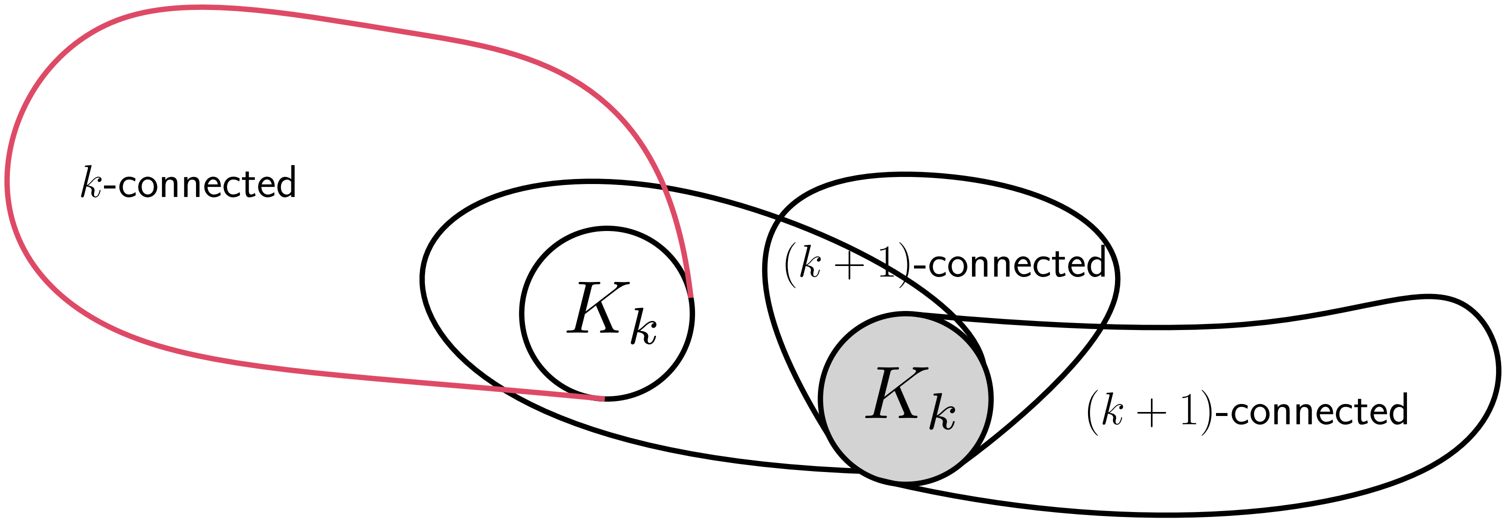

Let $\mathcal{G}_k^{(i)}$ be the class of k-connected chordal graphs rooted at an unlabelled, ordered i-clique.

Consider its multivariate exponential generating function $G_k^{(j)}(x, x_k)$, where the variable x_k marks the number of k-cliques. Then, we have that

$$G_k^{(k)}(x, x_k) = \exp\left(G_{k+1}^{(k)}(x, x_k G_k^{(k)}(x, x_k))\right).$$

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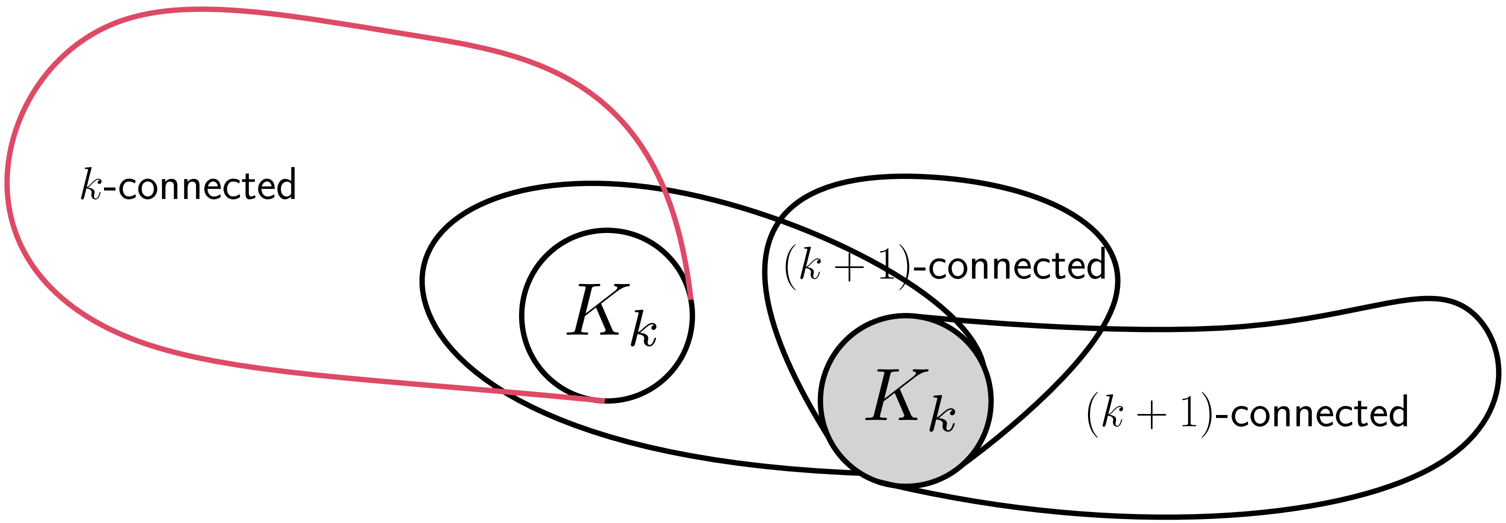

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This generalizes the classical decomposition of connected graphs into 2-connected components.

Chordal graphs with bounded tree-width

We start with the $(t+1)$-connected members: only K_{t+1}.

$$G_{t+1}(\mathbf{x}) = \frac{1}{(t+1)!} \prod_{j\in[t]} x_j^{\binom{t+1}{j}}.$$

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 & & \downarrow & & & & & & & & \\
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 & & & & & & \downarrow & & & & \\
 & & & & & & \vdots & & & & \\
 & & & & & & \downarrow & & & & \\
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 & & & & & & \downarrow & & \\
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$\longrightarrow$ **t-trees**

[Wormald '85]

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Analytic combinatorics

First principle. The **location** of a function's singularities dictates the **exponential growth** of its coefficients.

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Transfer theorem. ([Floajolet, Odlyzko '82])
Suppose that $F(x)$ is analytic in a Δ-domain where it admits a singular expansion

$$F(x) = f_1(x) + f_2(x)\left(1 - \frac{x}{x_0}\right)^{-\alpha}.$$

for analytic functions f_1, f_2 with $f_2(x_0) \neq 0$. Then, as $n \to \infty$,

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In our case, $\alpha = -3/2$.

Analysis of the system

$$
\begin{tikzcd}
G_{t+1} \arrow[r] & G_{t+1}^{(t)} \arrow[d] & & & & \\
& G_t^{(t)} \arrow[r] & G_t \arrow[r] & G_t^{(t-1)} \arrow[d] & & \\
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& & & \vdots \arrow[d] & & \\
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Analysis of the system

3/2 singularity

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G_{t+1} \arrow[r] & G_{t+1}^{(t)} \arrow[d] & & \text{3/2 singularity} \arrow[dl] & & & \\
& G_t^{(t)} \arrow[r] & G_t \arrow[r] & G_t^{(t-1)} \arrow[d] & 1/2 \arrow[l] & & \\
& & & G_{t-1}^{(t-1)} \arrow[d] & & & \\
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Lemma 1. We have singular expansions of the same type in any variable.

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Lemma 1. We have singular expansions of the same type in any variable.

Lemma 2. Derivating a $3/2$-singularity yields a $1/2$-singularity and integrating a $1/2$-singularity yields a $3/2$-singularity.

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Lemma 3. The solution to the implicit equation has a $1/2$-singularity.

Analysis of the system

Theorem 2 follows from these singular expansions by an application of the so-called Quasi Power Theorem.

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Some values of the singularities $\rho_{t,k}$

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$t=1$	0.36788						
$t=2$	0.14665	0.18394					
$t=3$	0.07703	0.08421	0.12263				
$t=4$	0.04444	0.04662	0.05664	0.09197			
$t=5$	0.02657	0.02732	0.03092	0.04152	0.07358		
$t=6$	0.01608	0.01635	0.01773	0.02184	0.03214	0.06131	
$t=7$	0.00974	0.00984	0.01038	0.01204	0.01614	0.02583	0.05255

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Theorem. ([Bender, Richmond, Wormald '85])
Almost all chordal graphs are split.

Therefore, the number of chordal graphs grows like $2^{n^2/4}$ and $\rho_t \to 0$ as $t \to \infty$.

Open questions

1. At what rate does ρ_t go to 0?

$$c_1 \frac{1}{t2^t} < \rho_t < c_2 \frac{1}{t}.$$

tw at most t

t-trees

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2. What happens if we let $t = t(n)$ grow with n?

If $t = (1+\varepsilon)\log n$, then the class is large.

Consider split graphs with a clique of size t. There are $2^{(1+\varepsilon)\log n(n-(1+\varepsilon)\log n)}$ such graphs. This number grows faster that $c^n n!$ for any c.

At which point between $t = O(1)$ and $t = \log n$ the class ceases to be small?