Enumeration of unlabelled chordal graphs with bounded tree-width

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Work in collaboration with Michael Drmota and Clément Requilé

CENTRE DE RECERCA MATEMÀTICA

EXCELENCIA MARÍA DE MAEZTU 2022 - 2025

VII Congreso de Jóvenes Investigadores de la RSME - Bilbo

Introduction

How to build a tree?

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Iteratively add a new vertex connected to an existing vertex.

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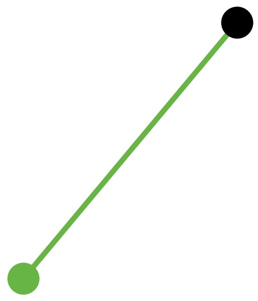

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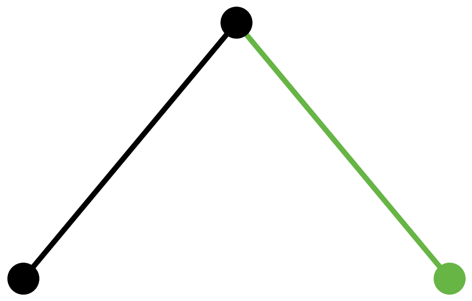

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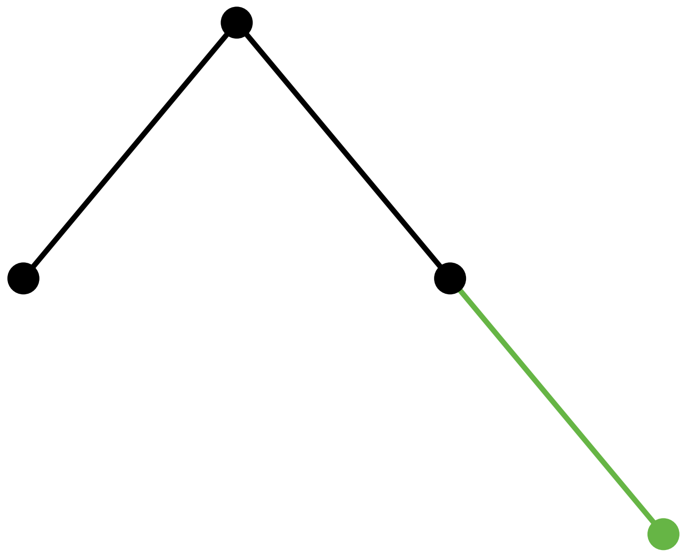

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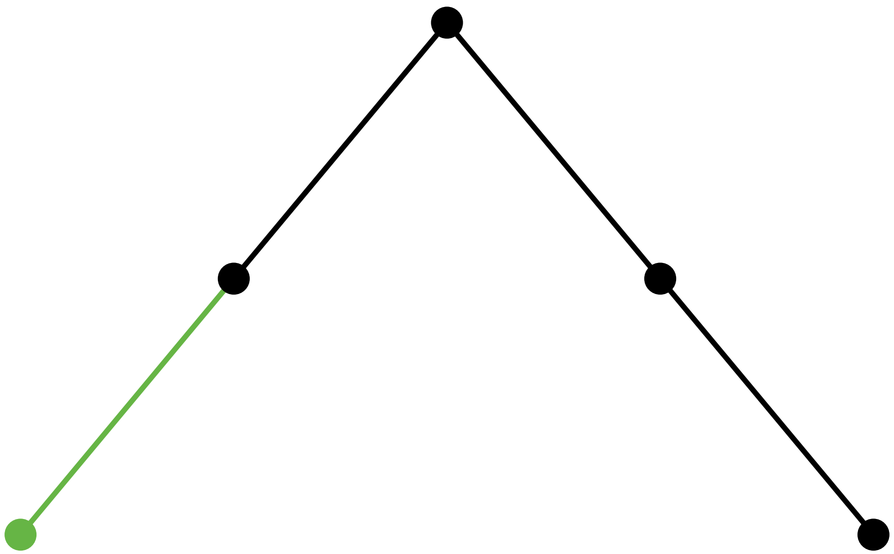

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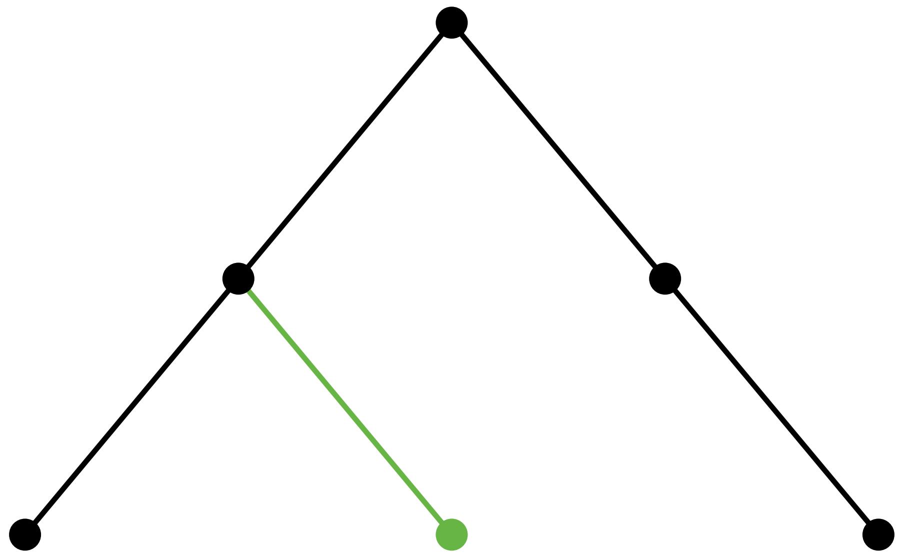

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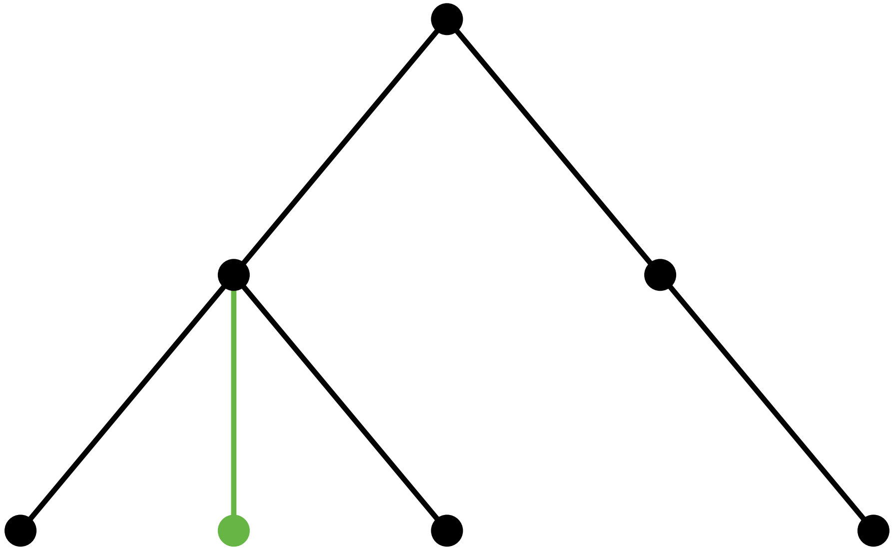

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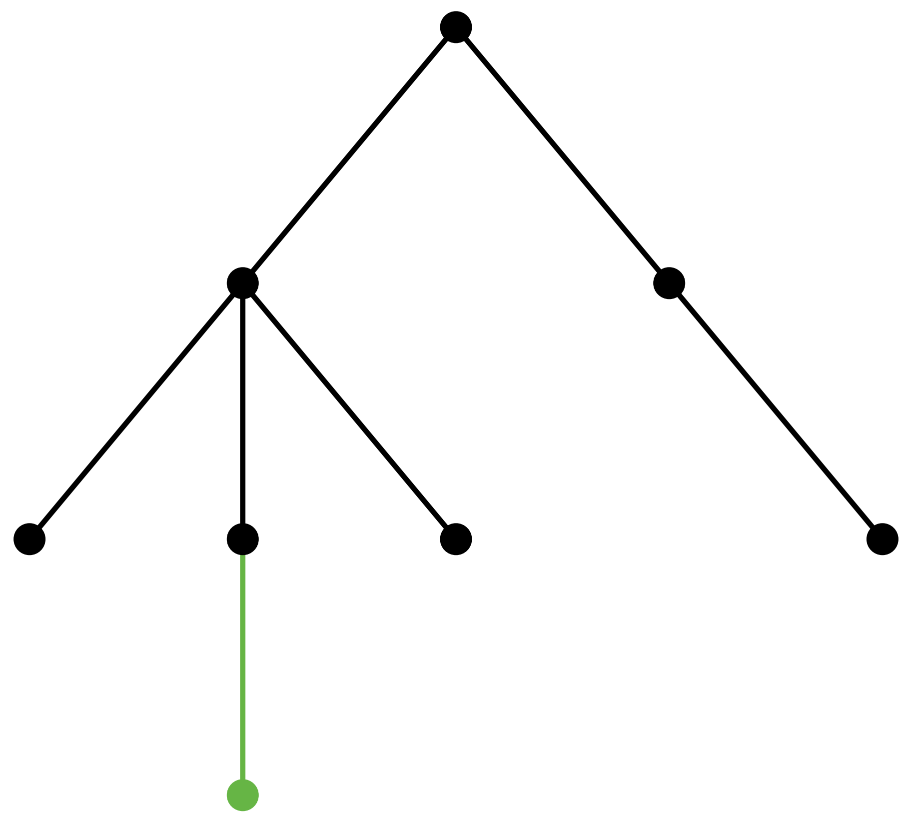

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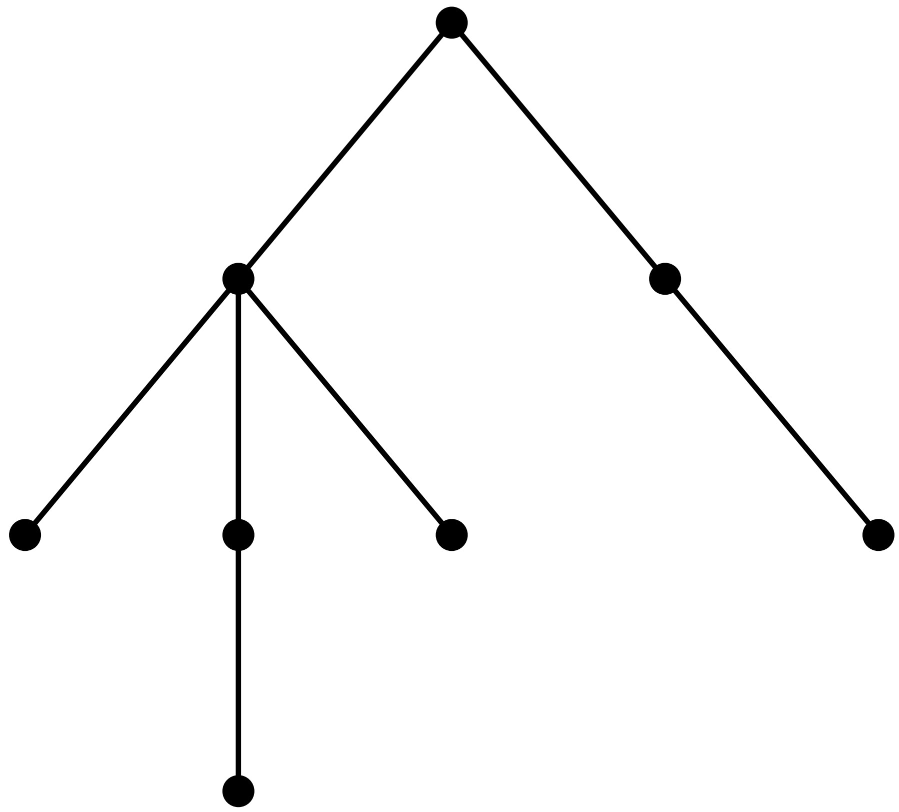

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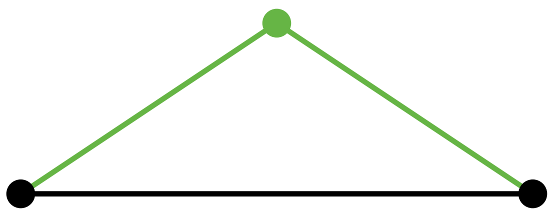

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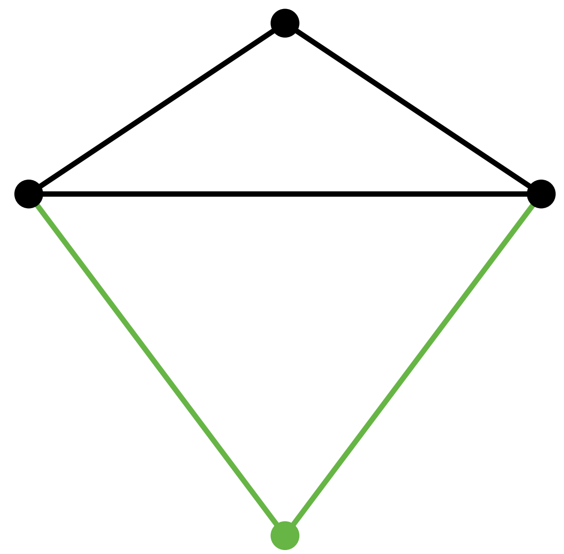

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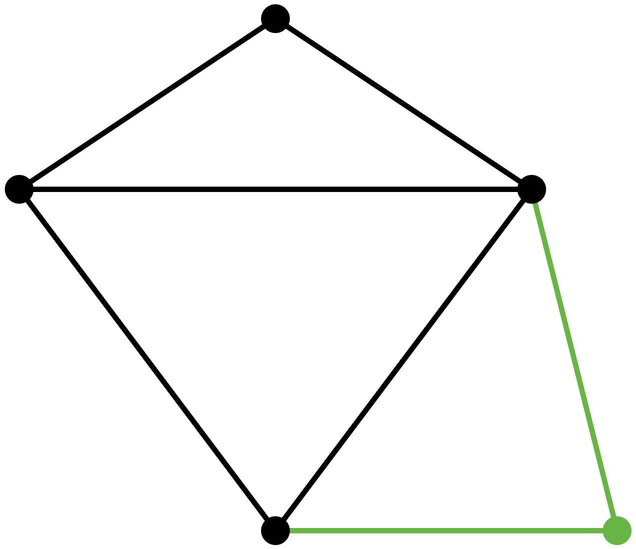

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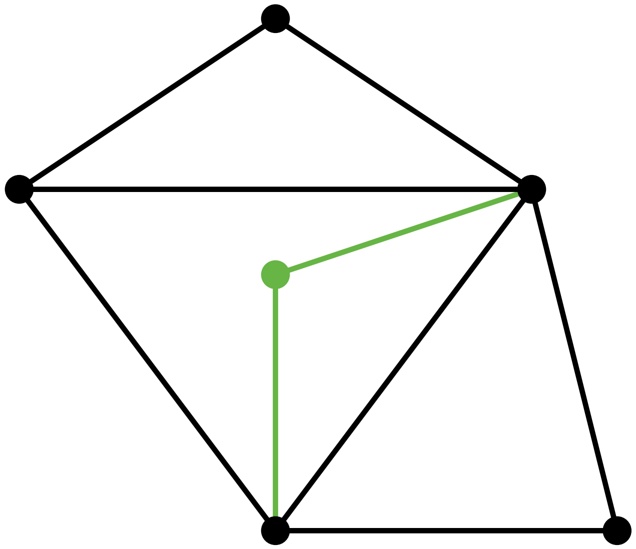

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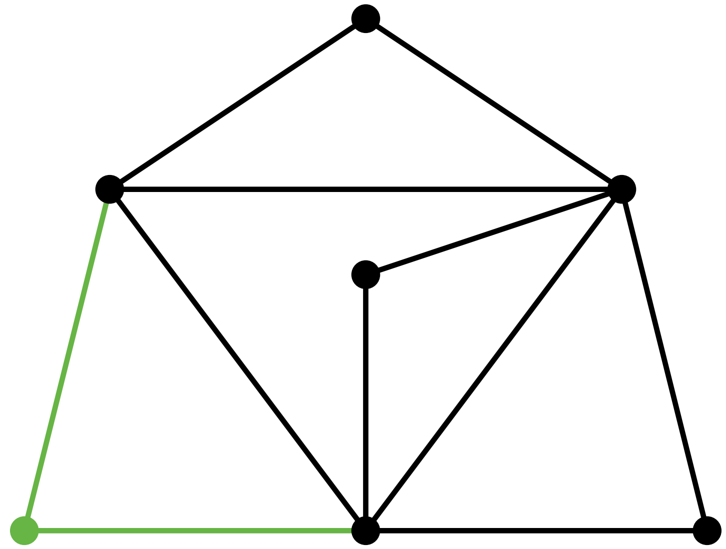

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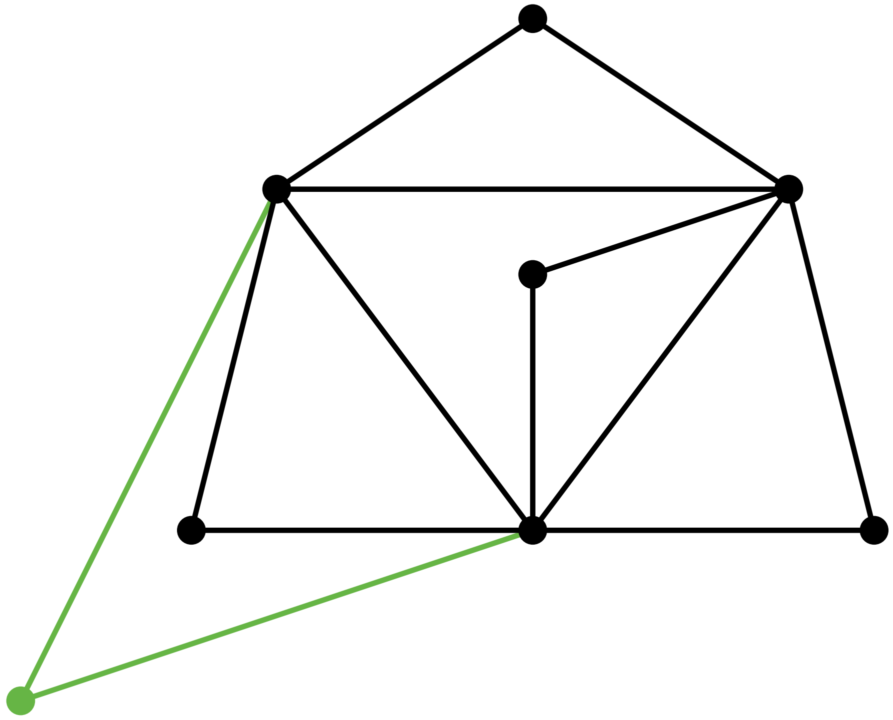

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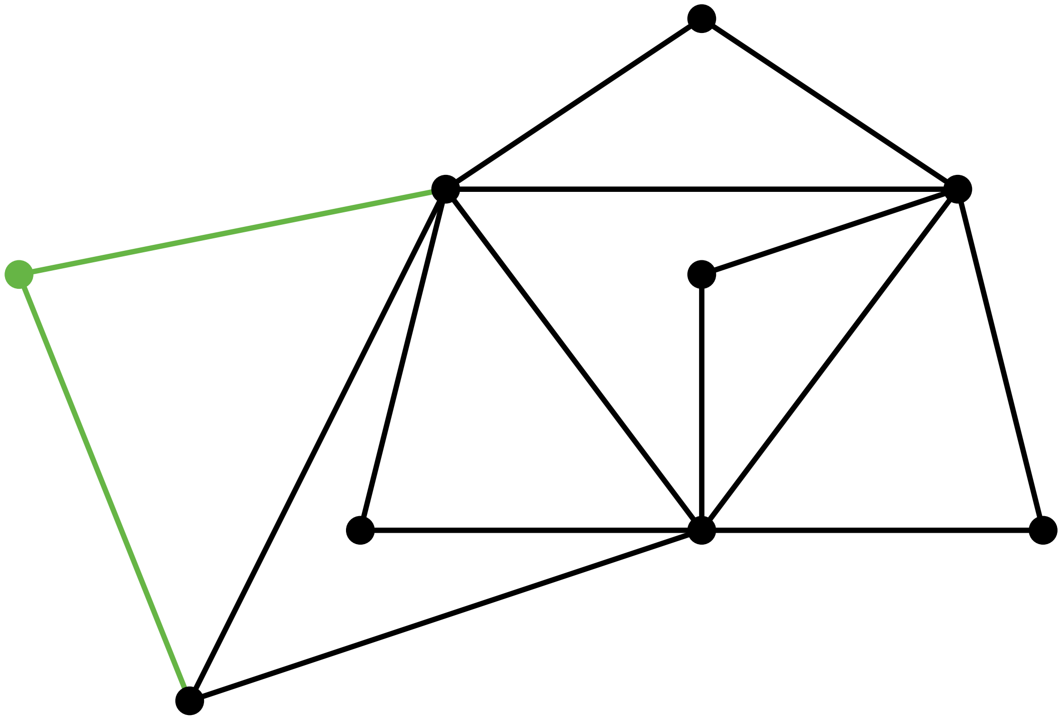

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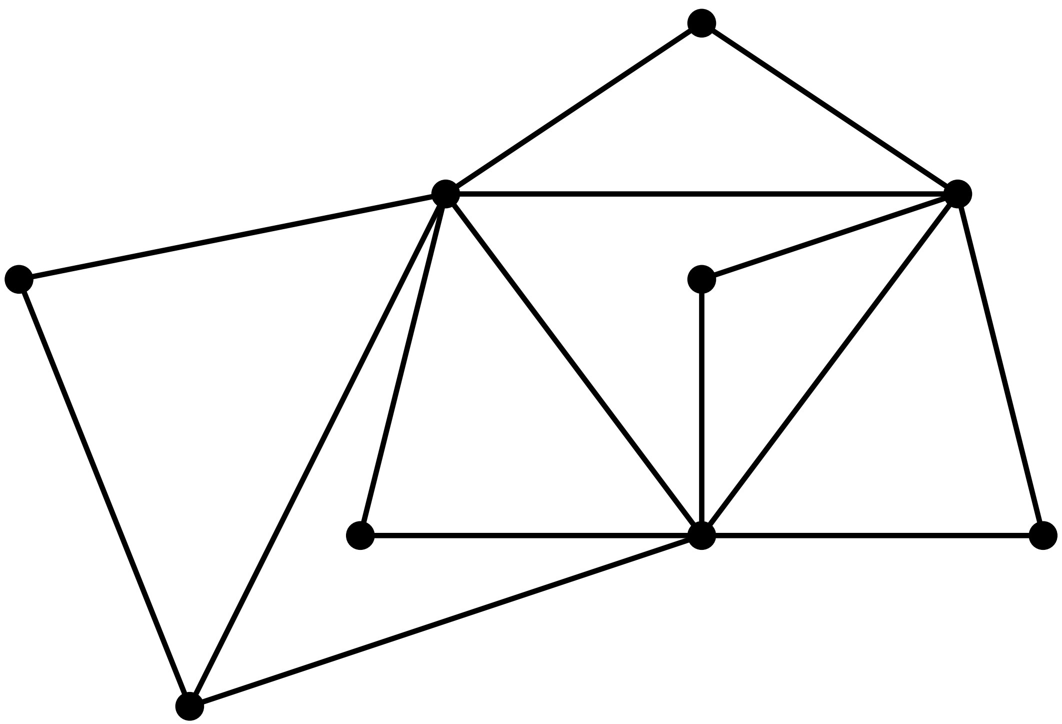

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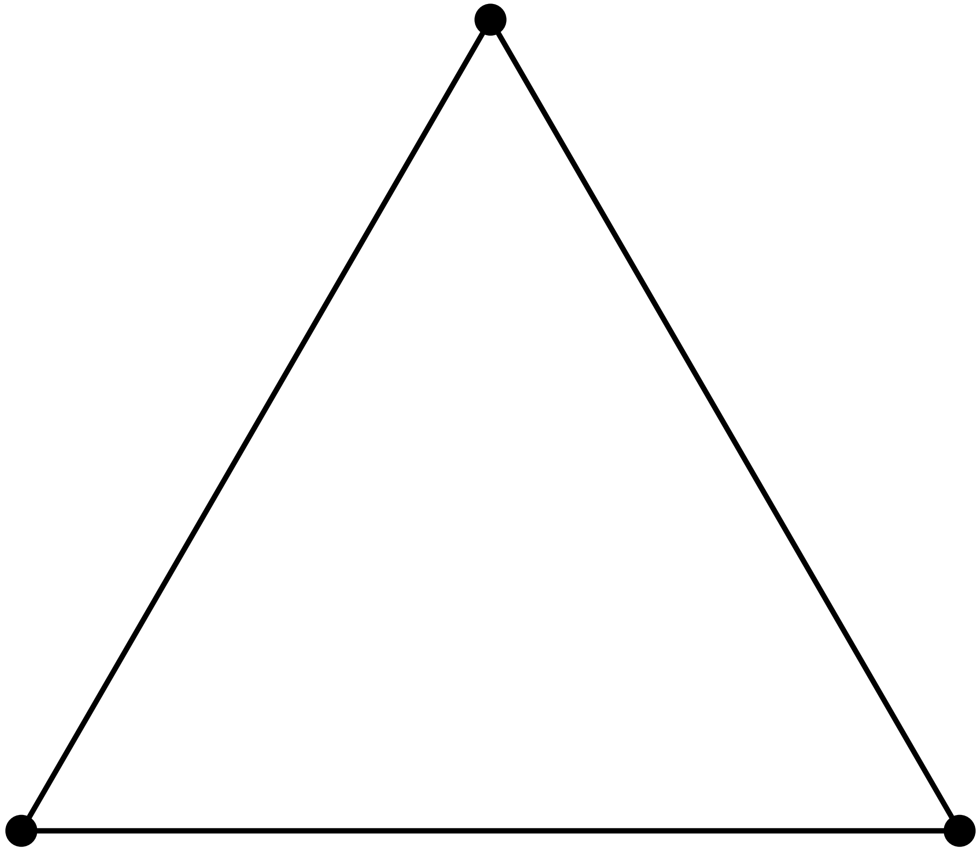

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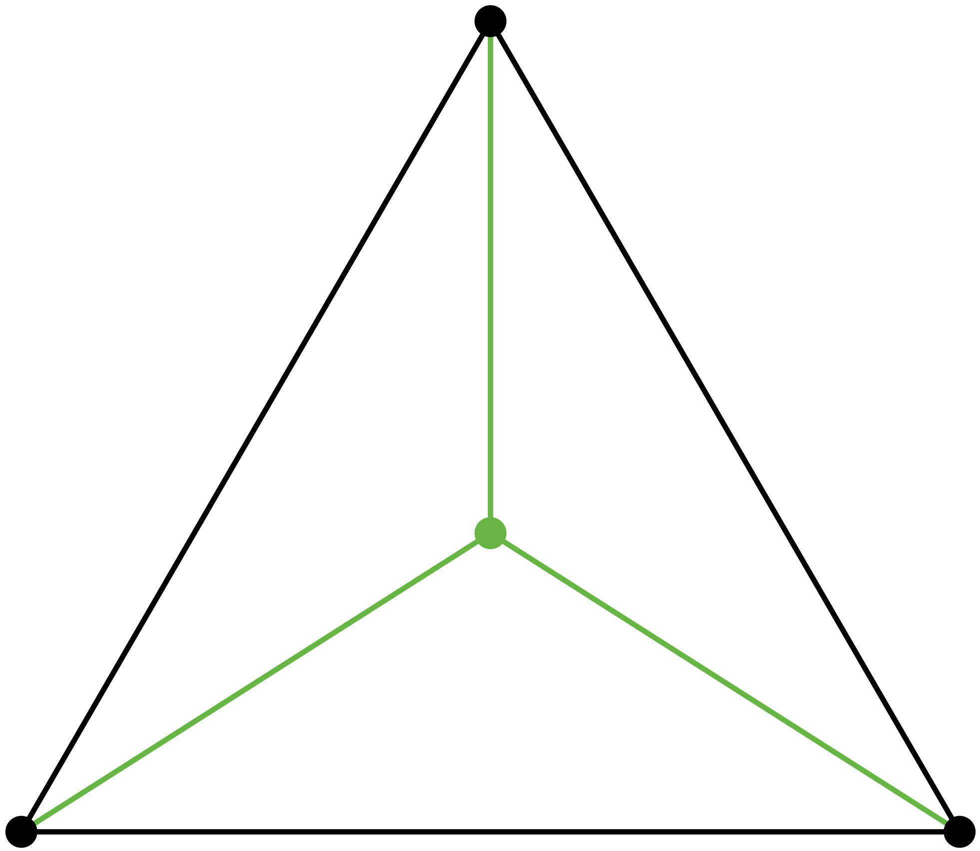

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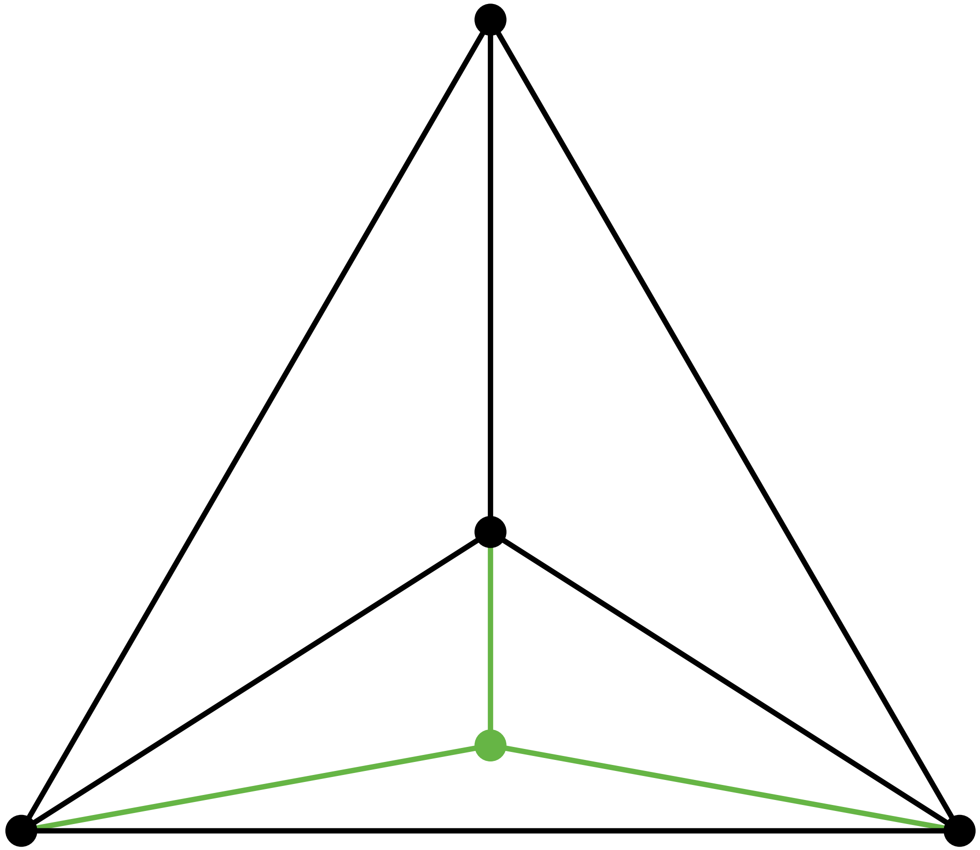

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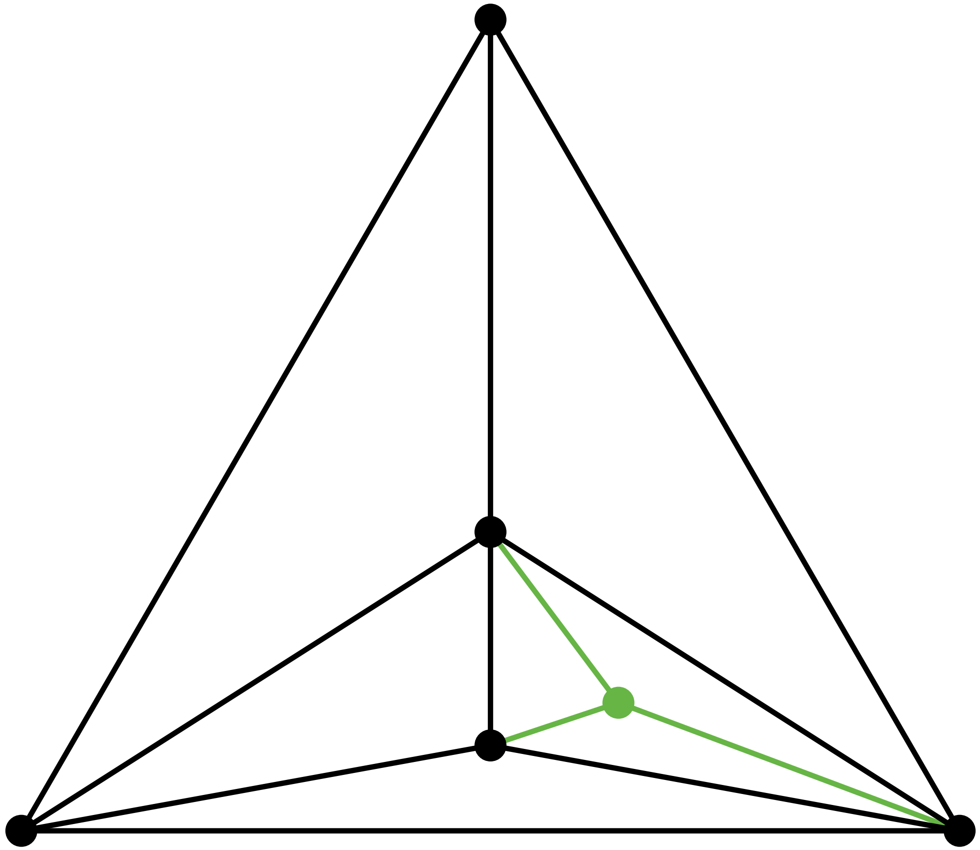

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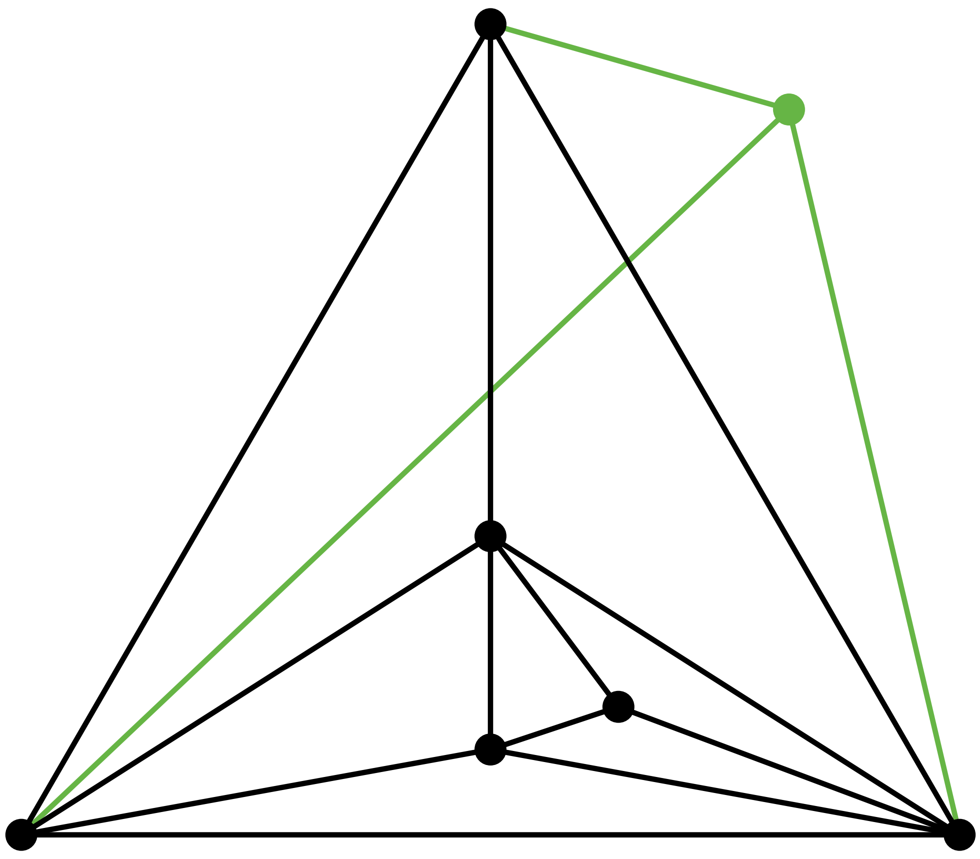

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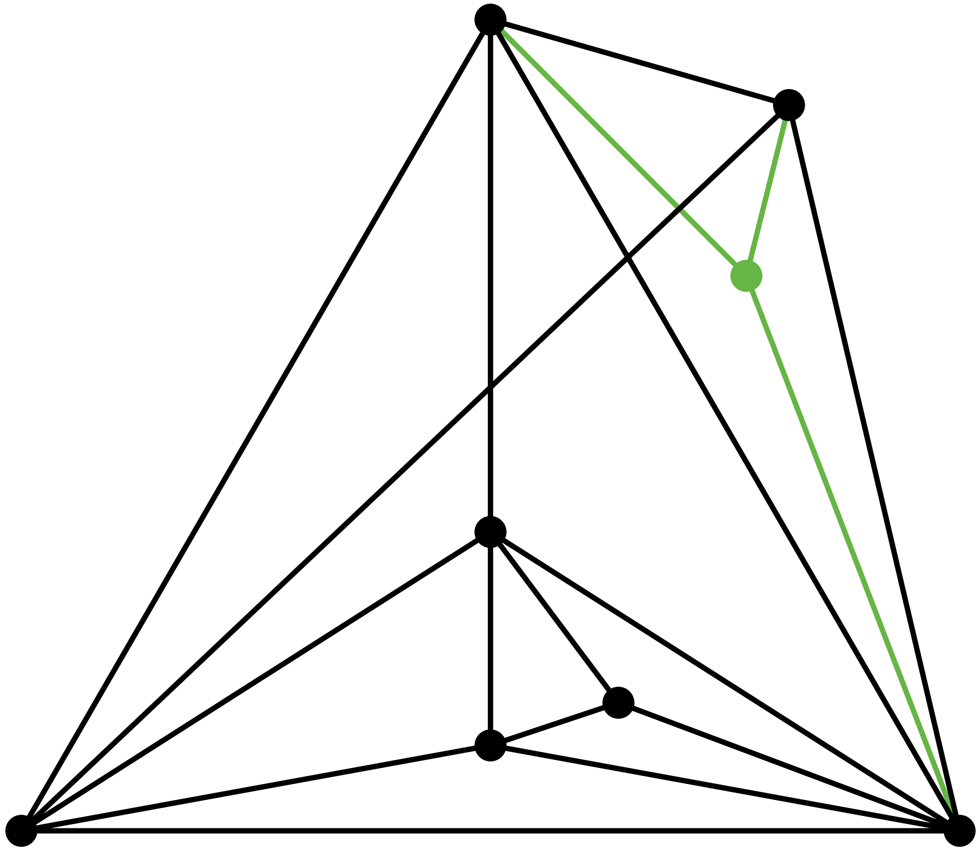

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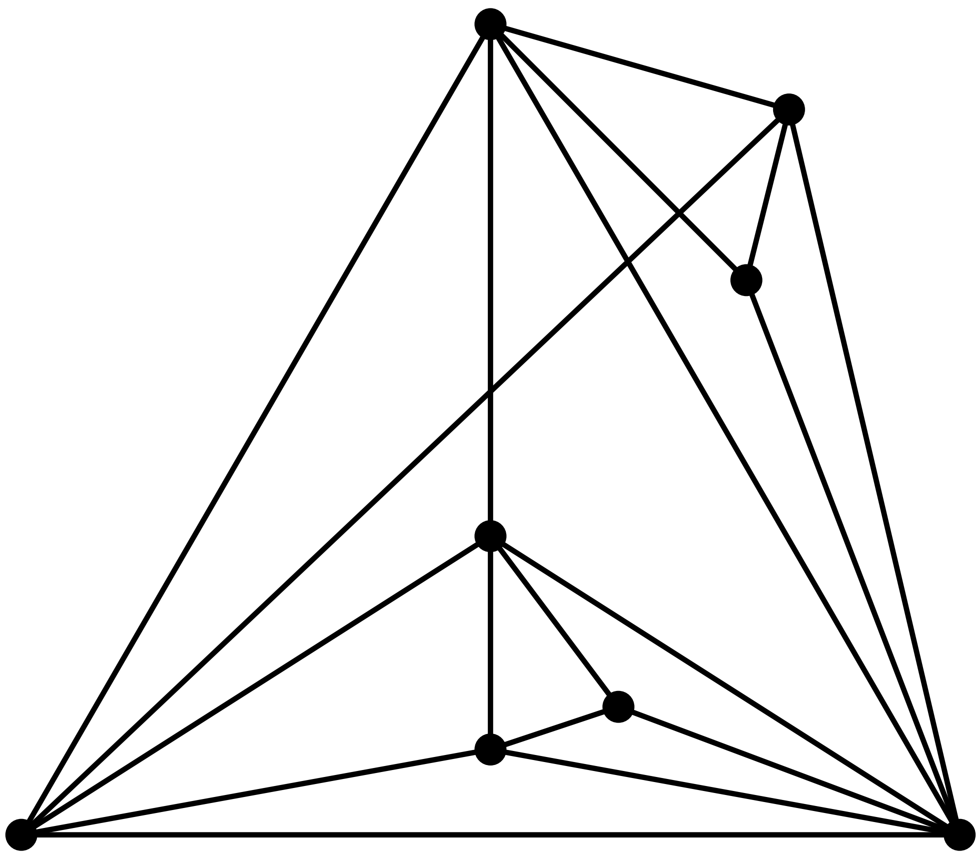

3-trees

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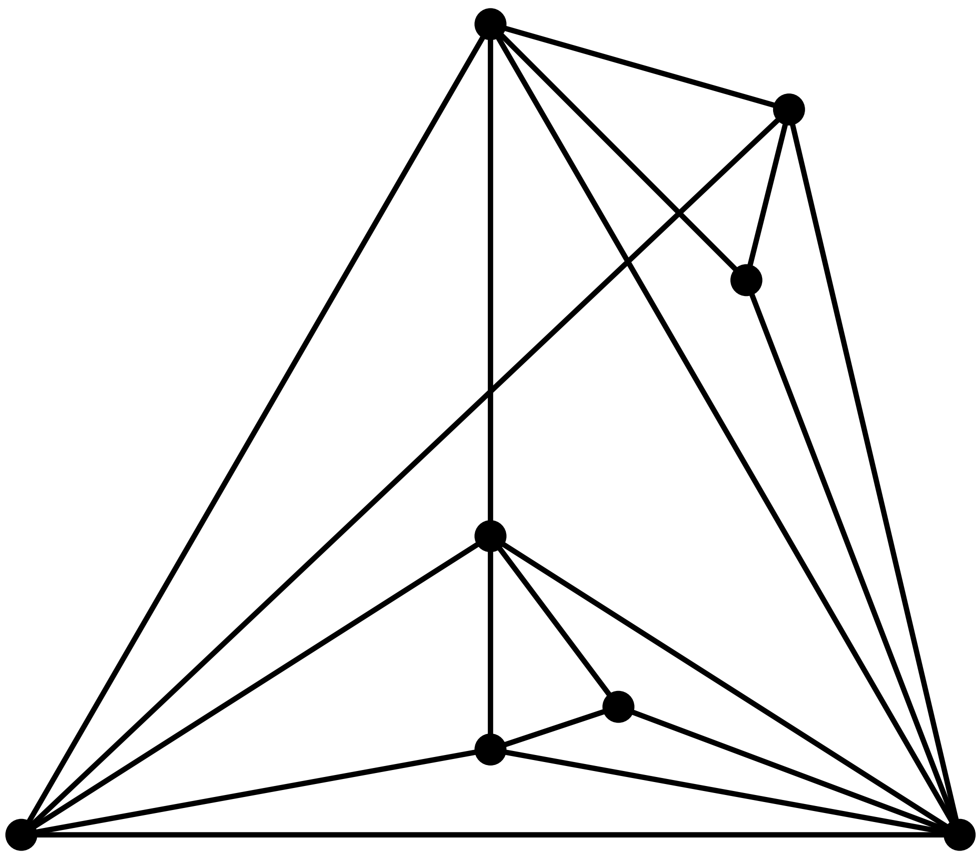

3-trees

Definition. A **k-tree** is a graph obtained from a $(k+1)$-clique by successively adding a new vertex connected to all vertices of an existing k-clique.

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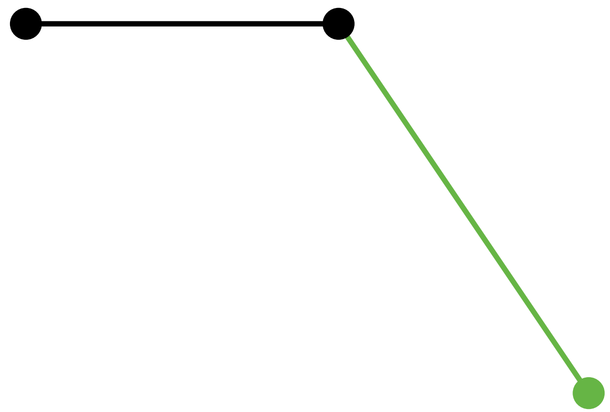

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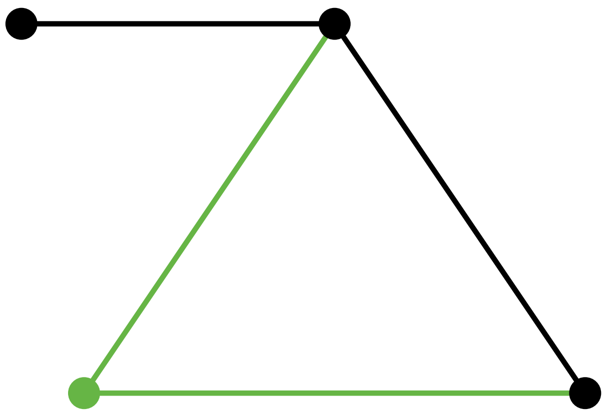

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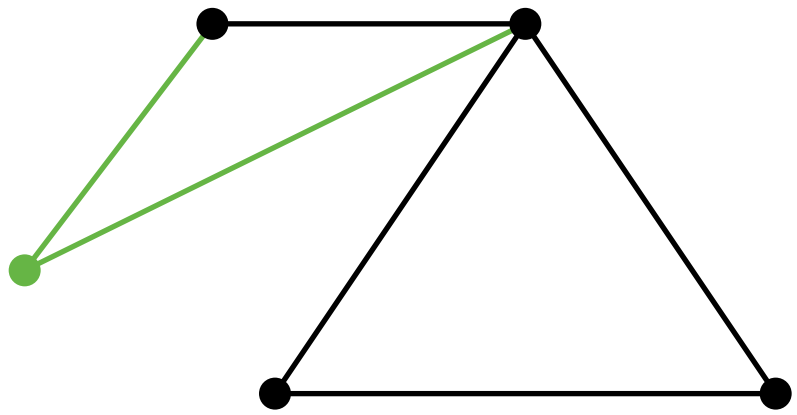

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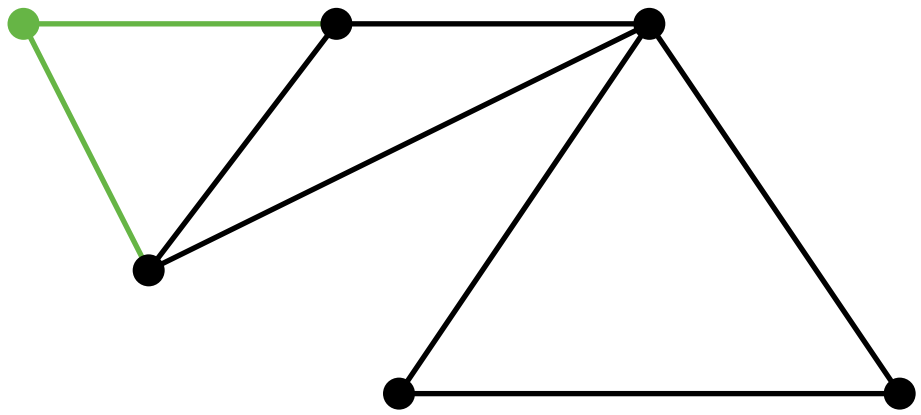

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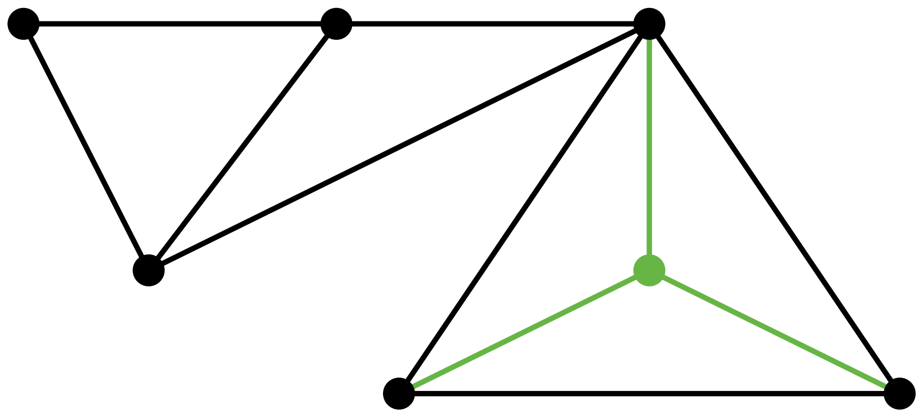

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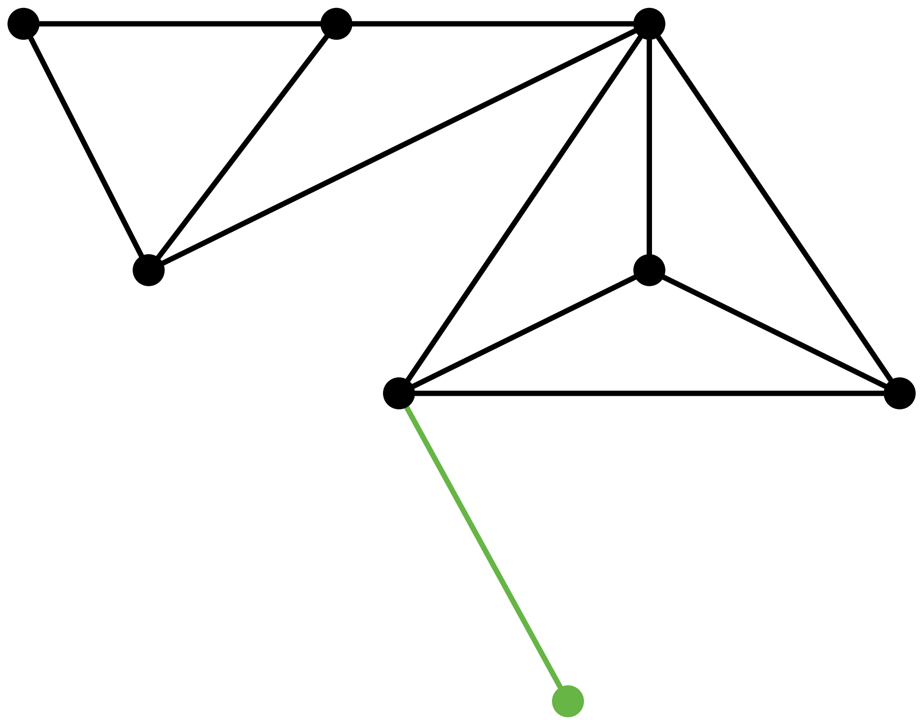

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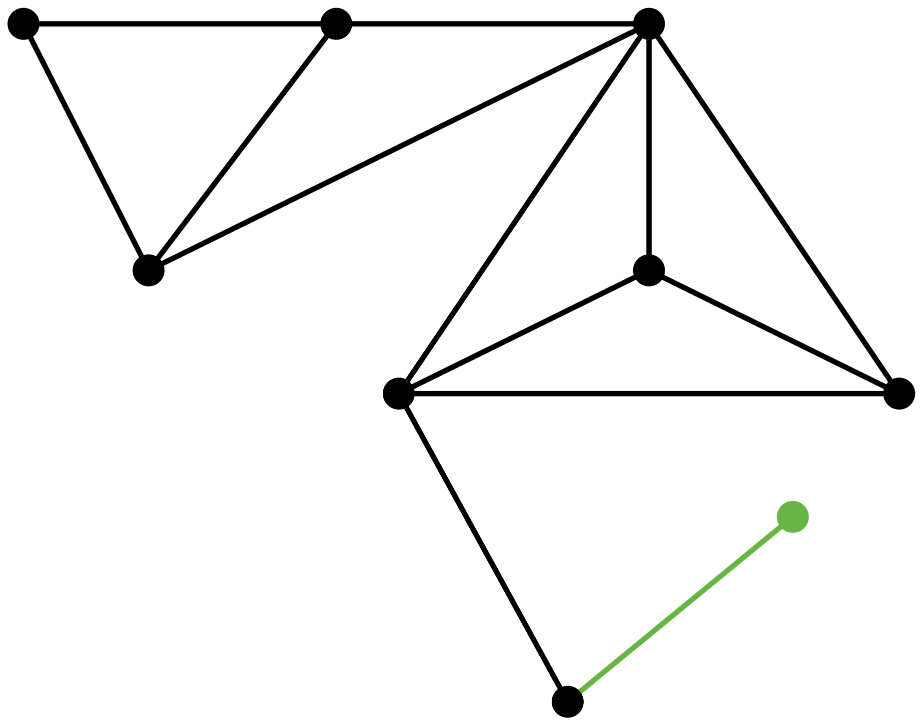

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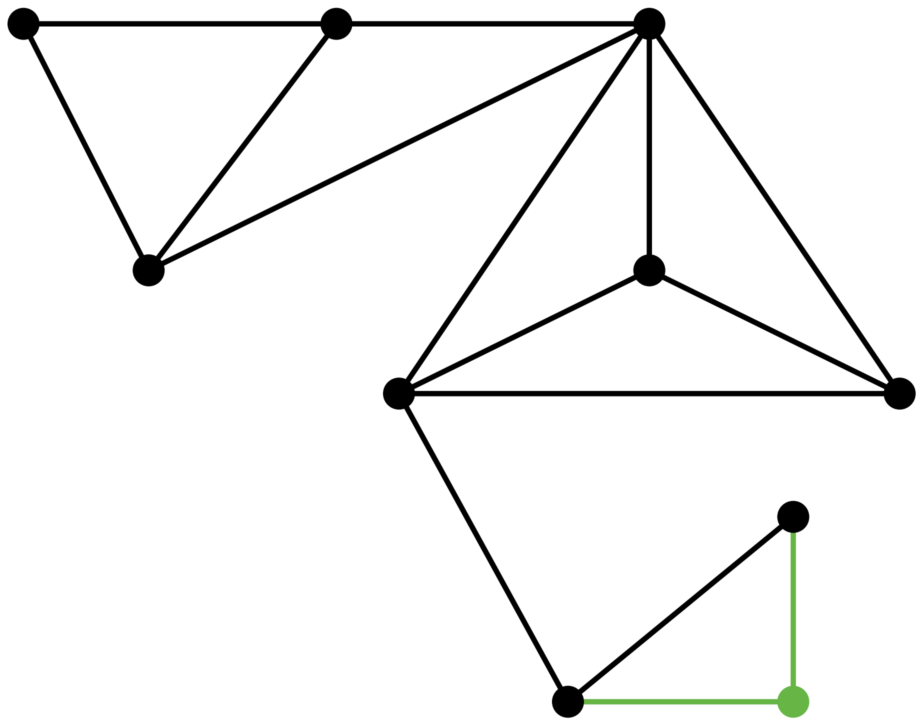

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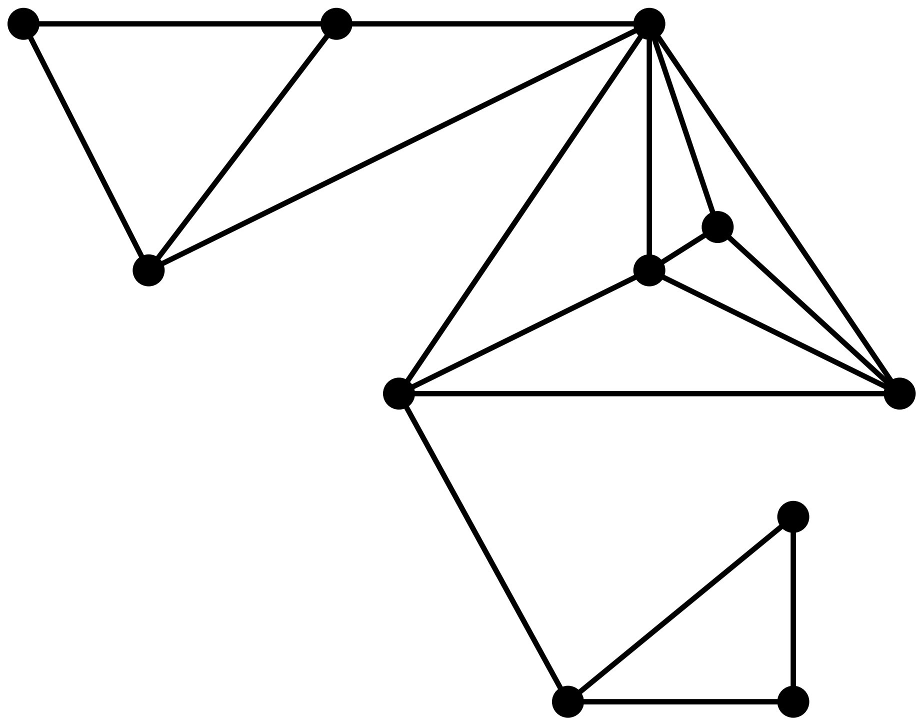

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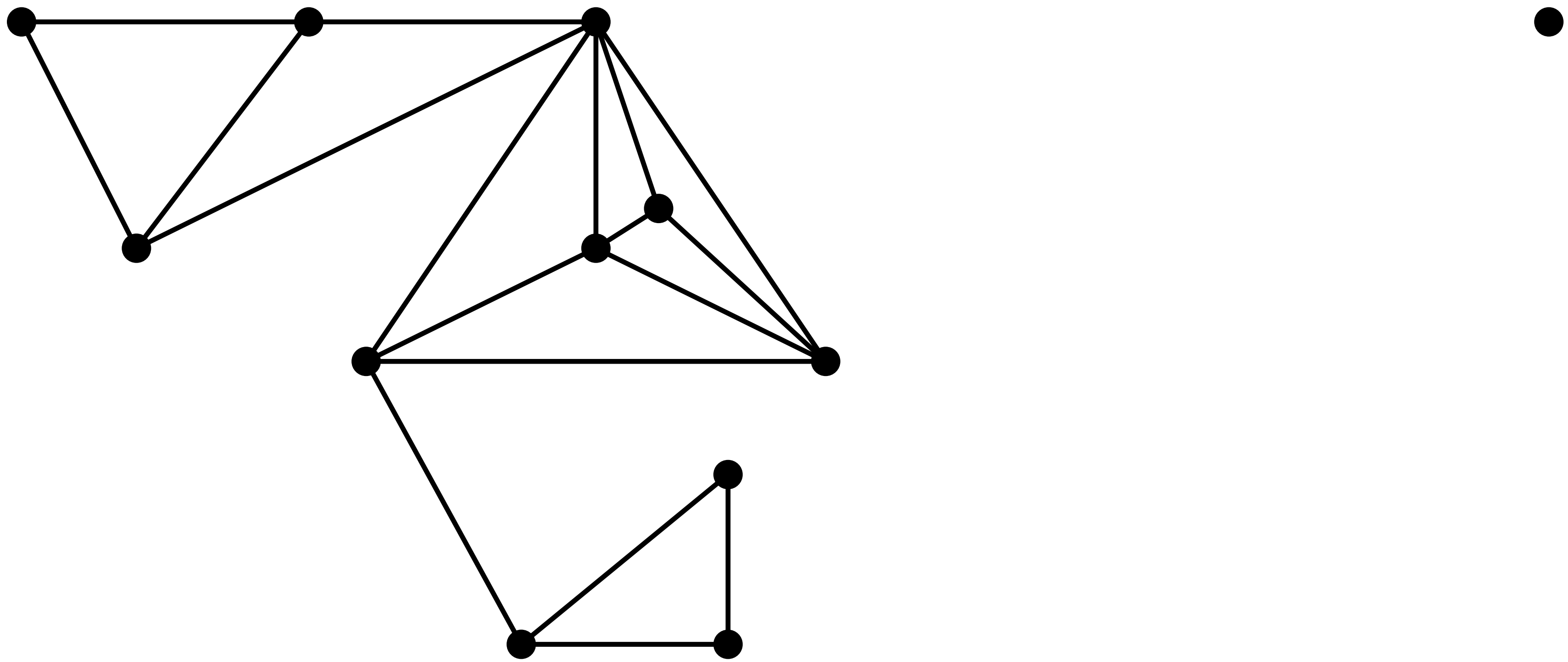

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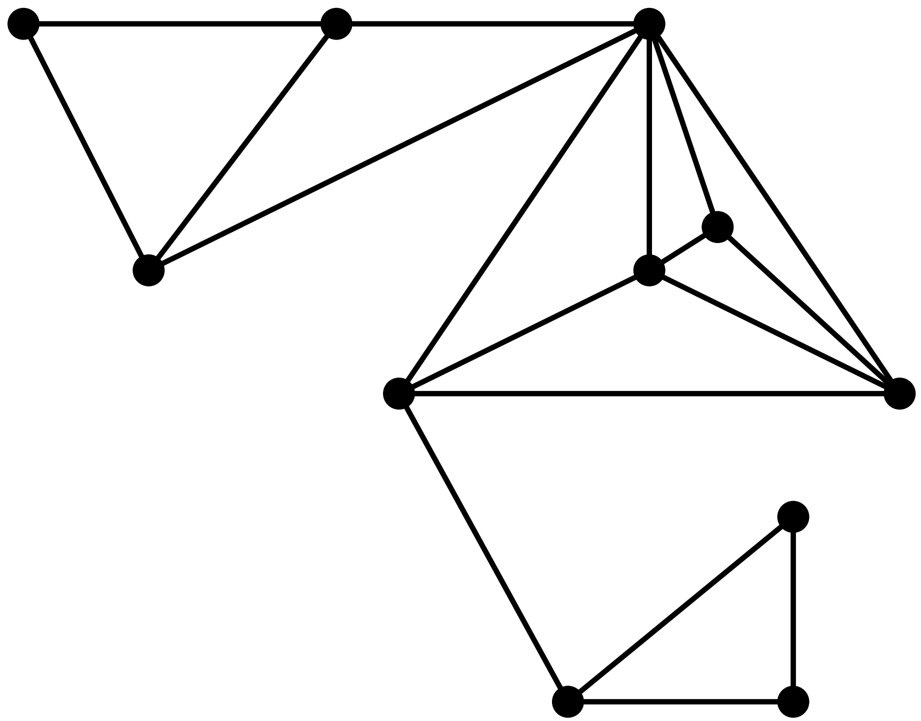

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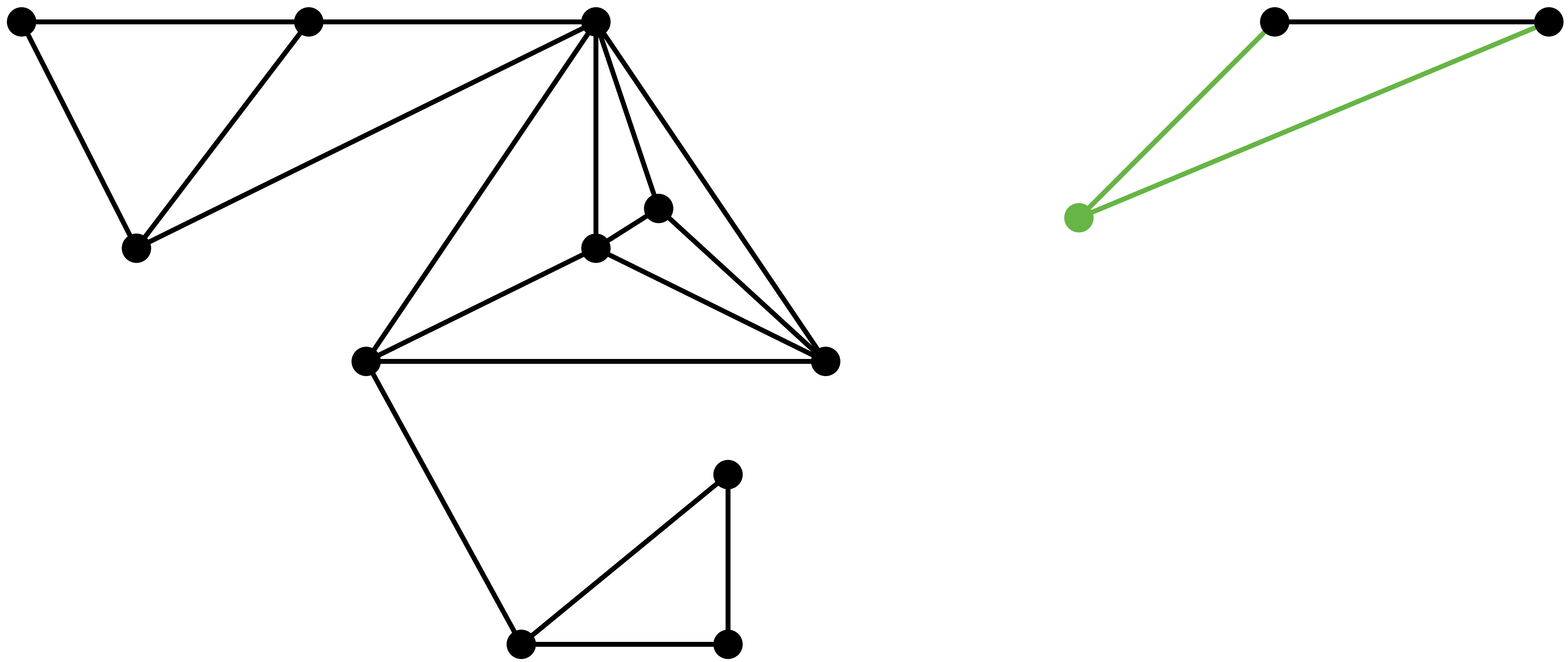

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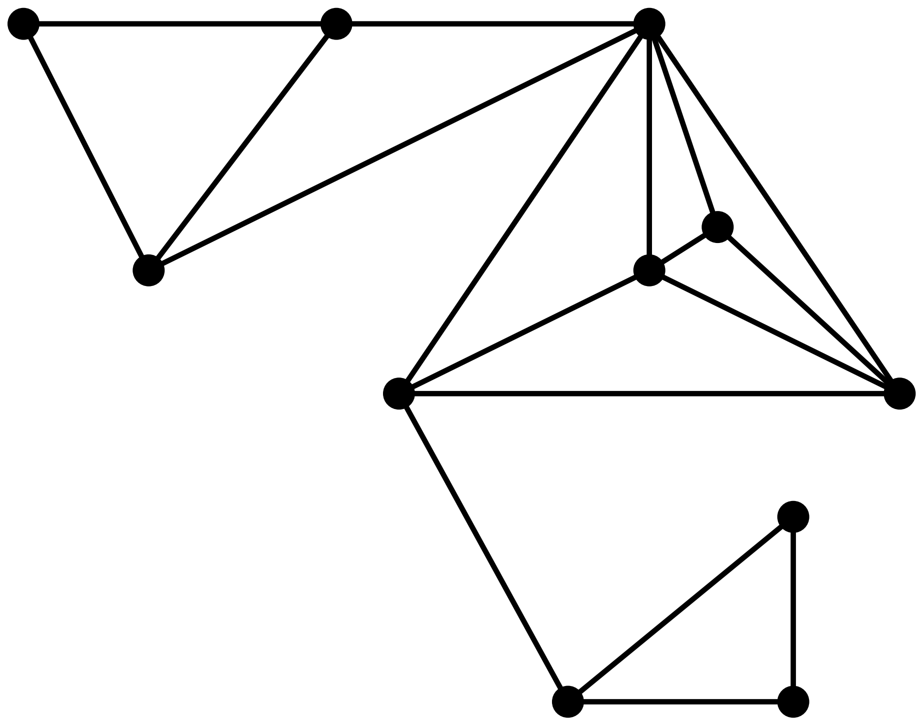

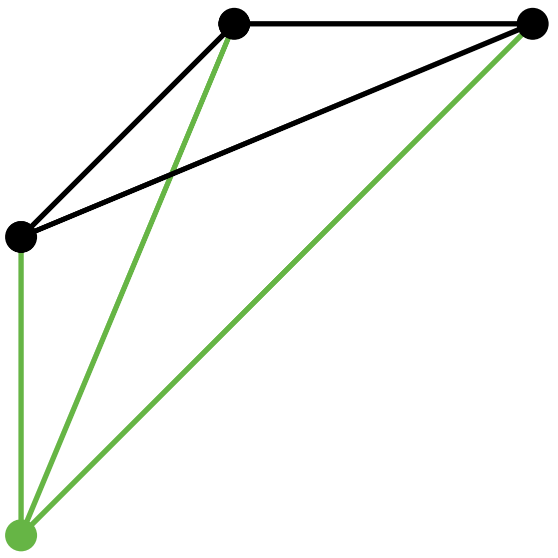

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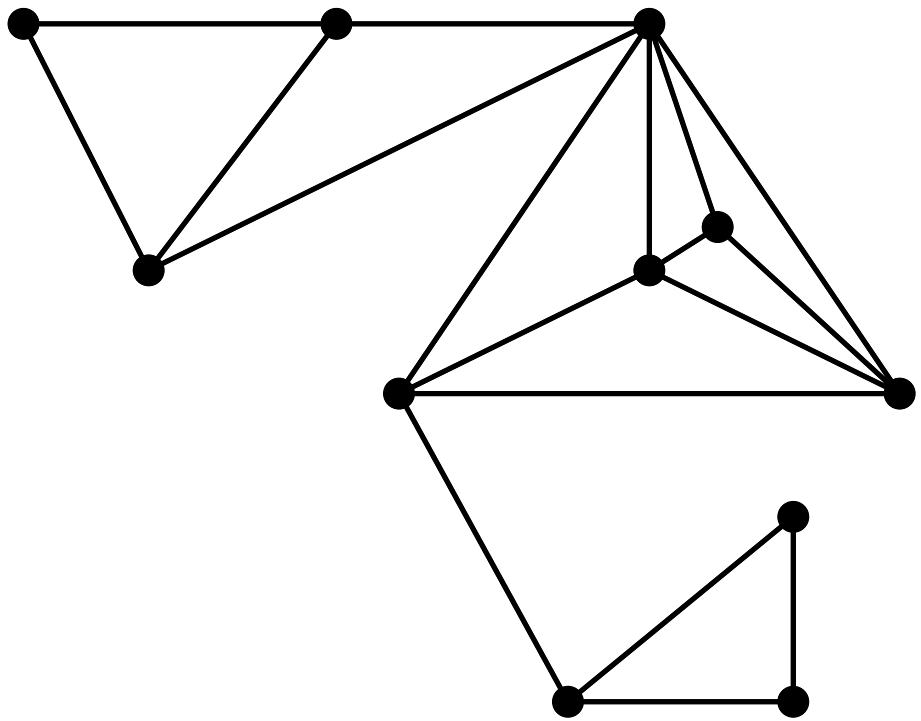

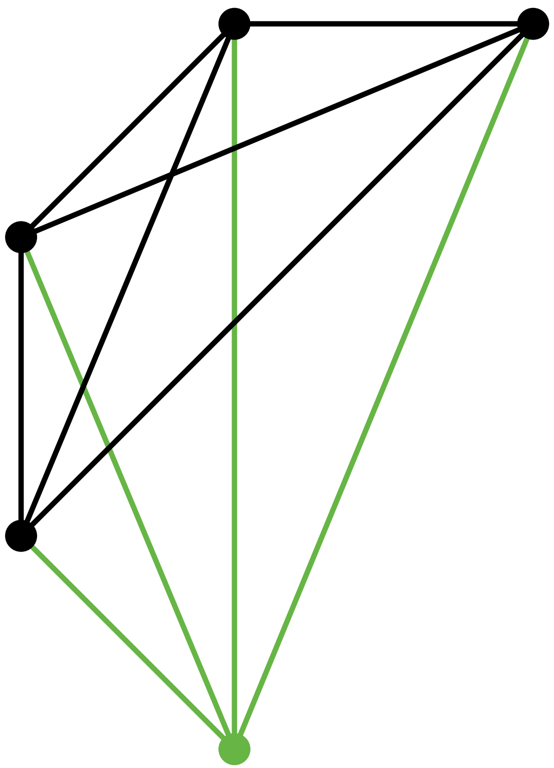

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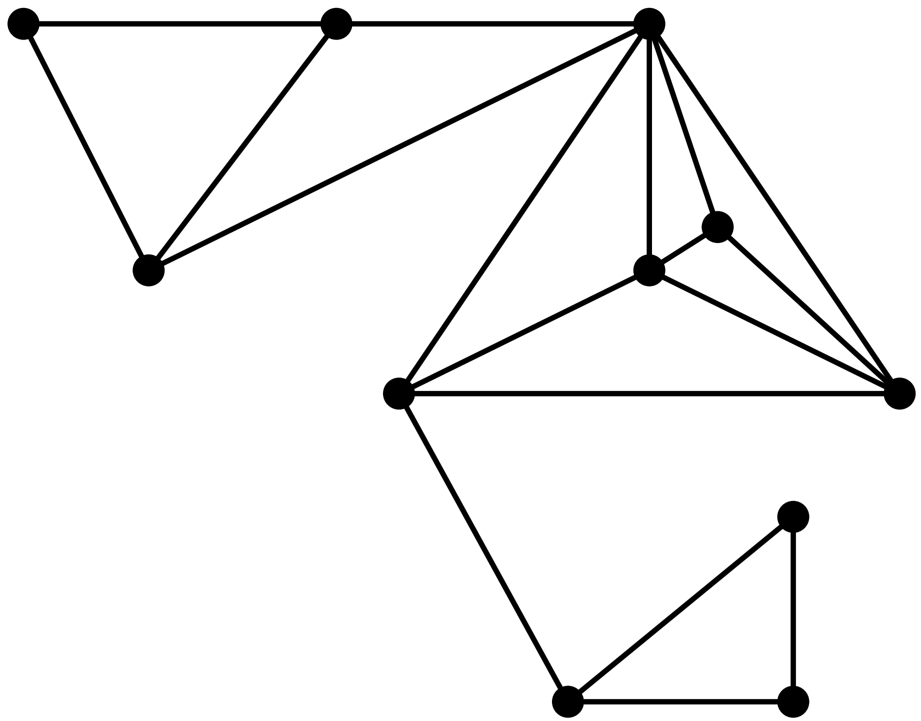

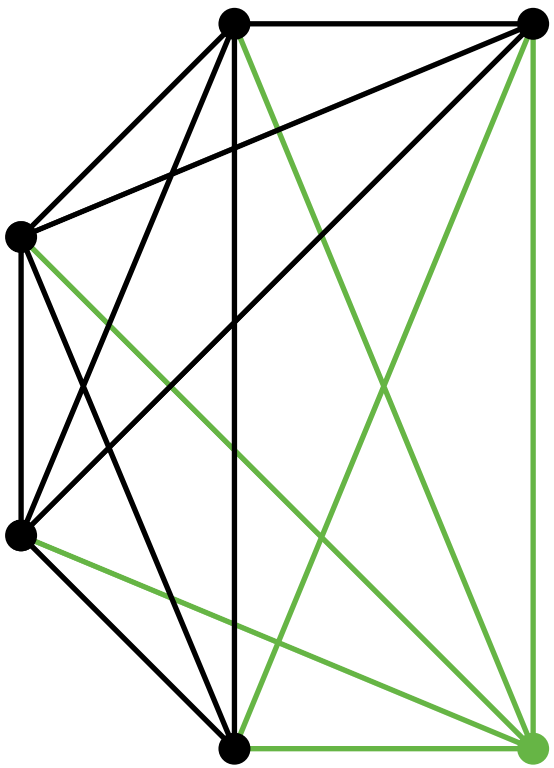

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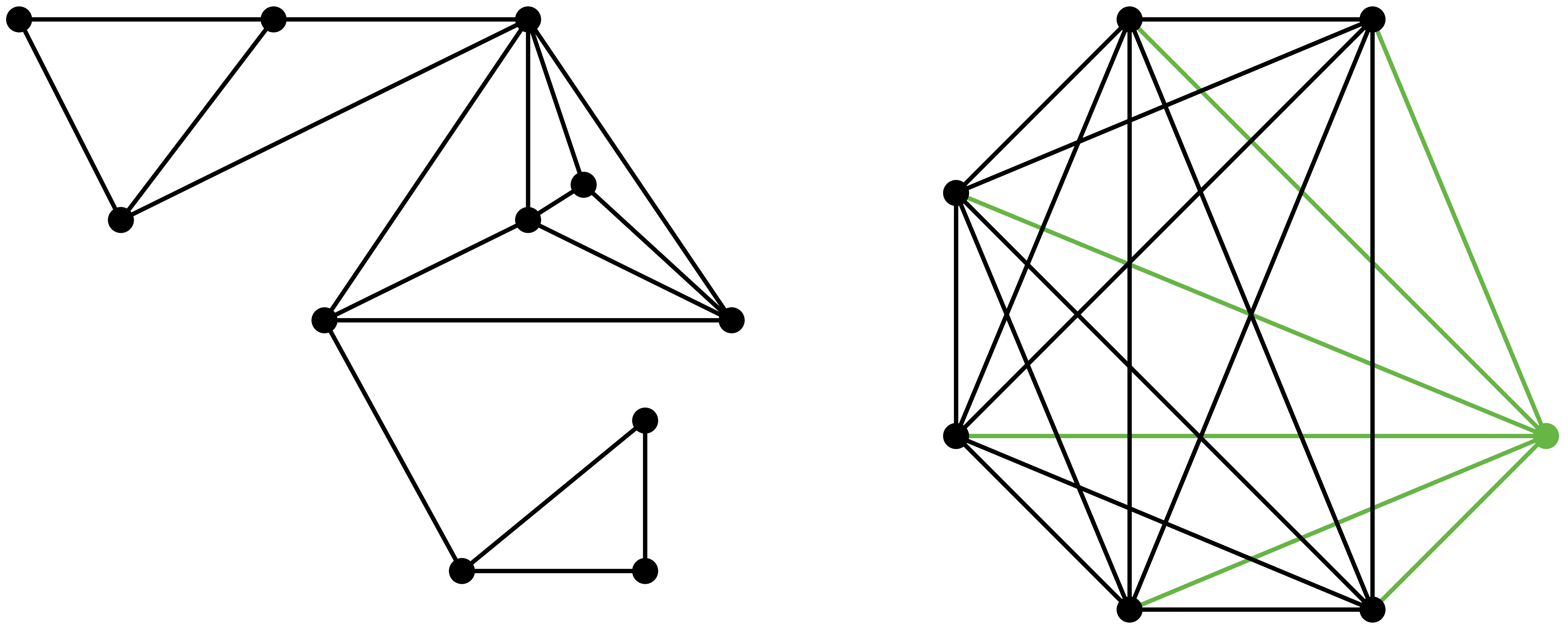

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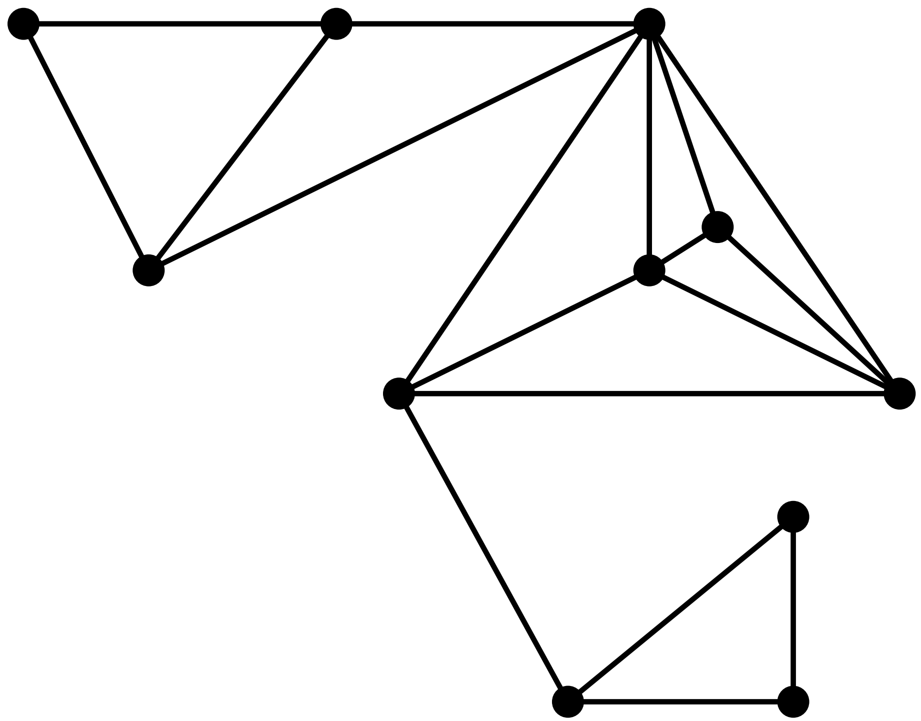

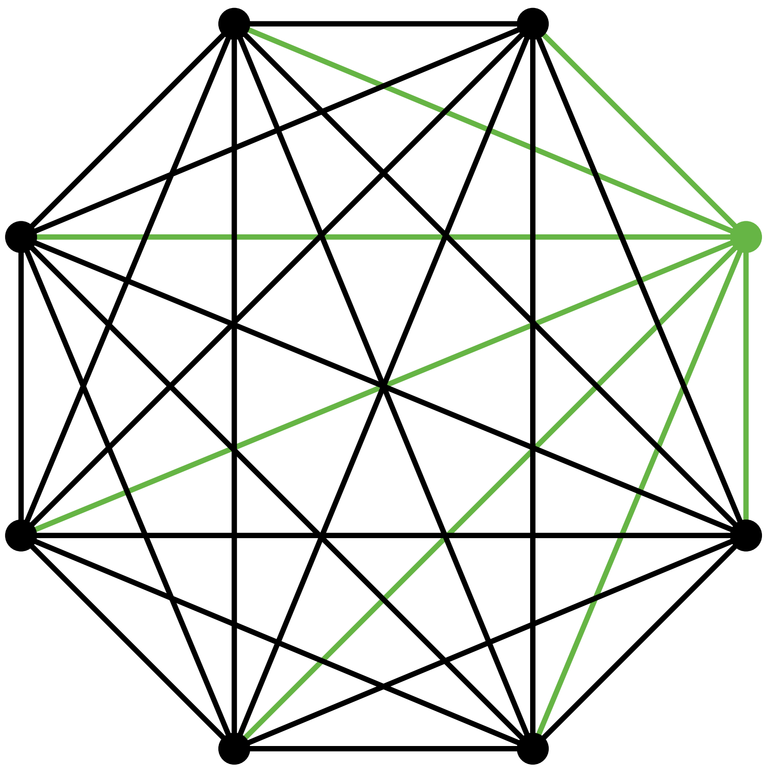

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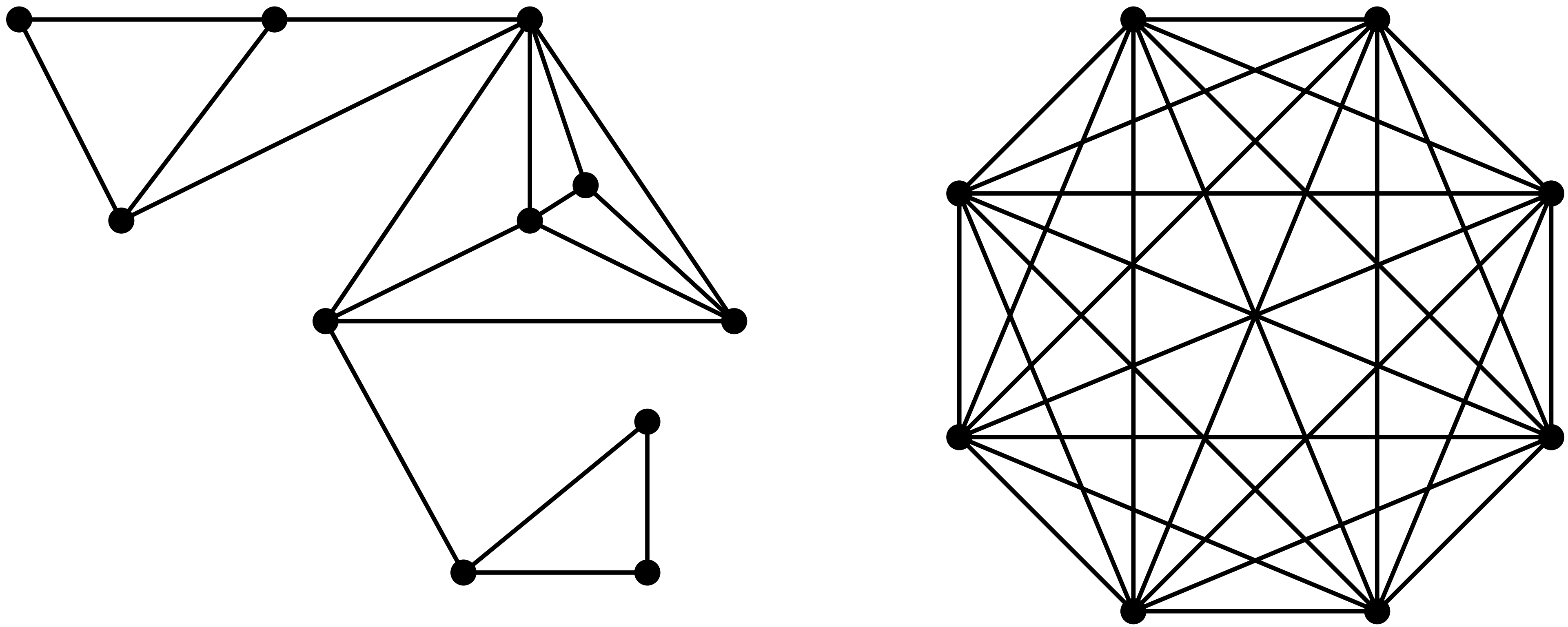

Chordal graphs

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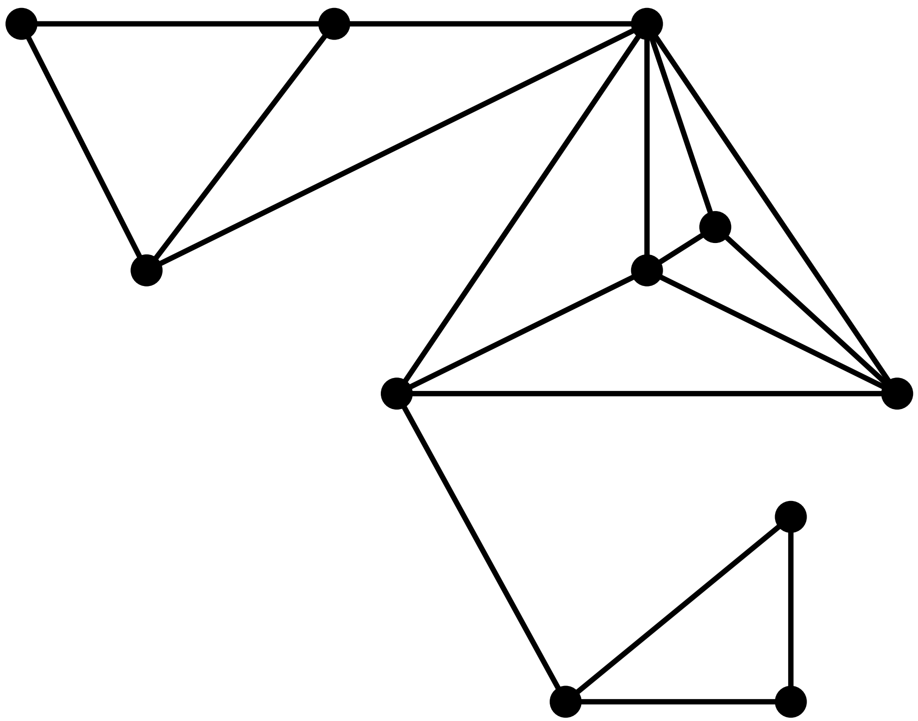

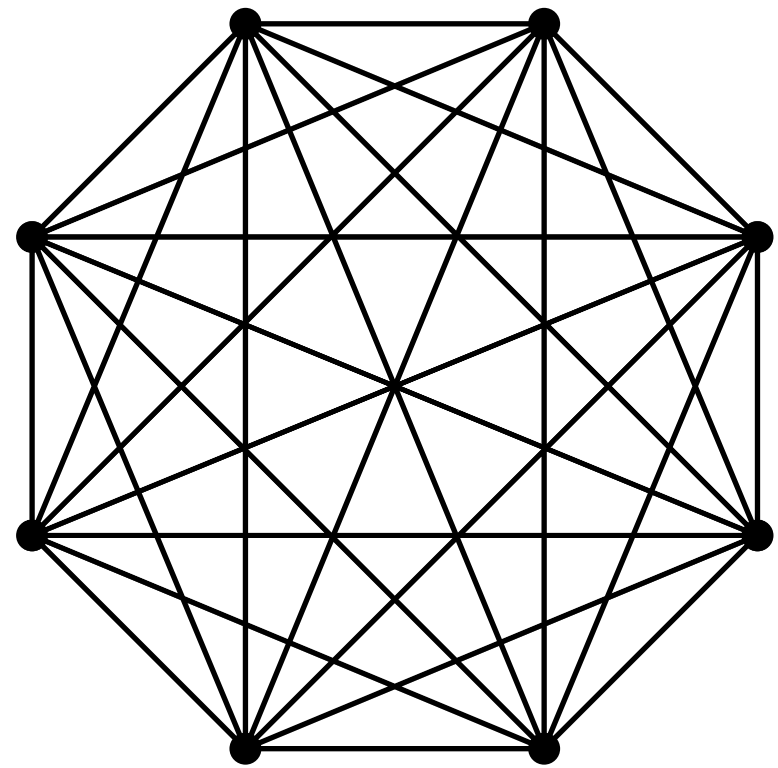

Chordal graphs

Definition. A graph is **chordal** if it has no induced cycle of lengh greater than 3.

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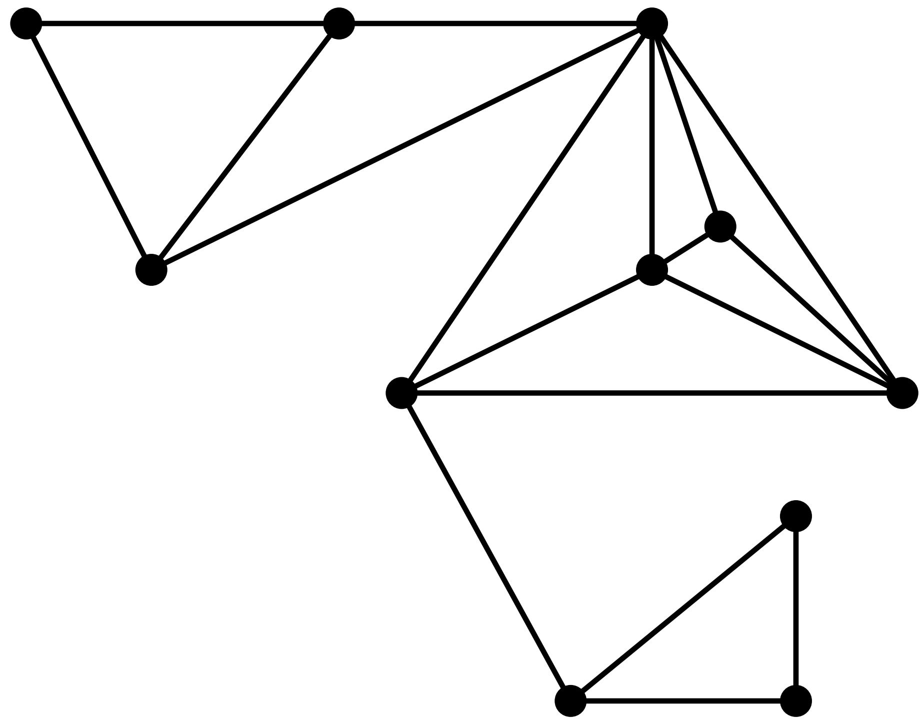

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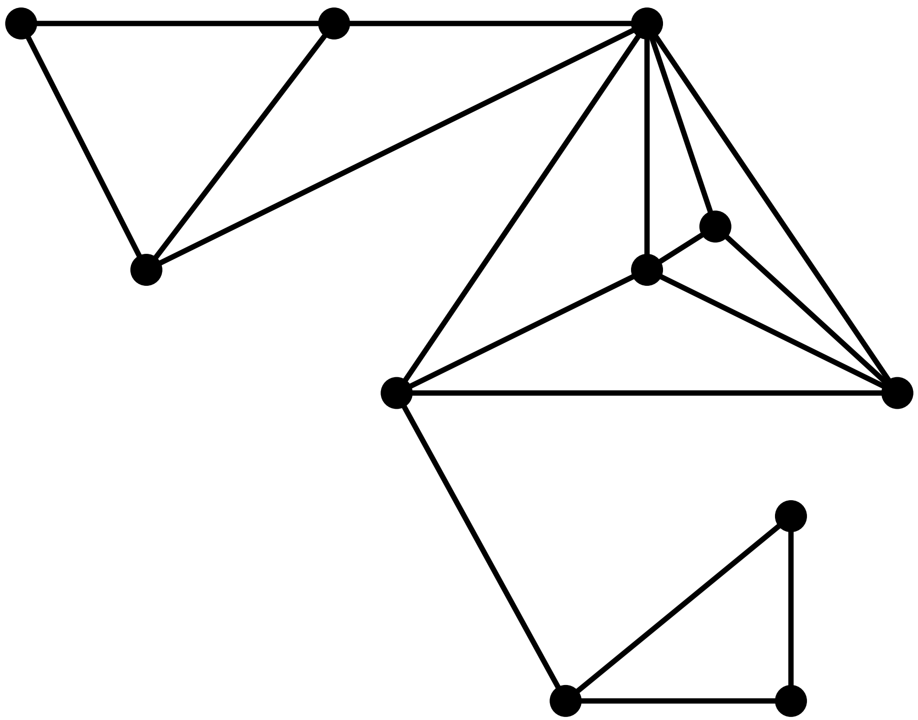

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Chordal graphs with tree-width at most t

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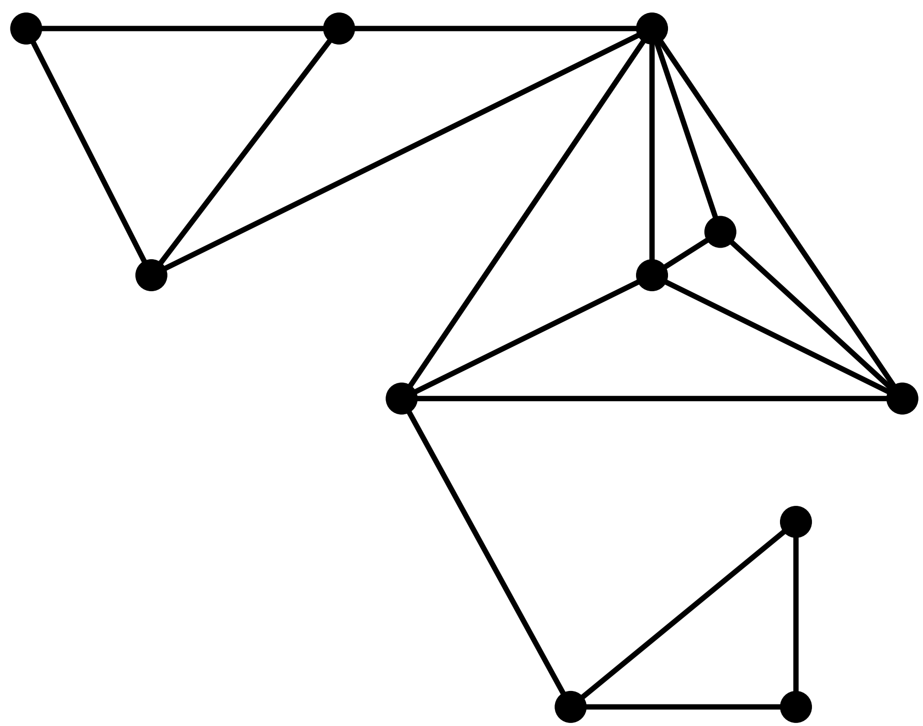

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Chordal graphs with tree-width at most t

Definition. The **tree-width** of a graph G is the minimum k such that G is the subgraph of a k-tree.

Labelled vs unlabelled

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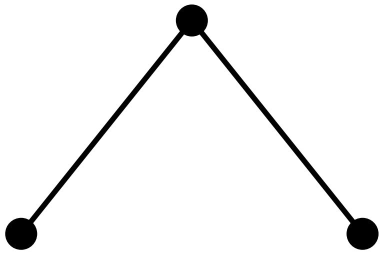

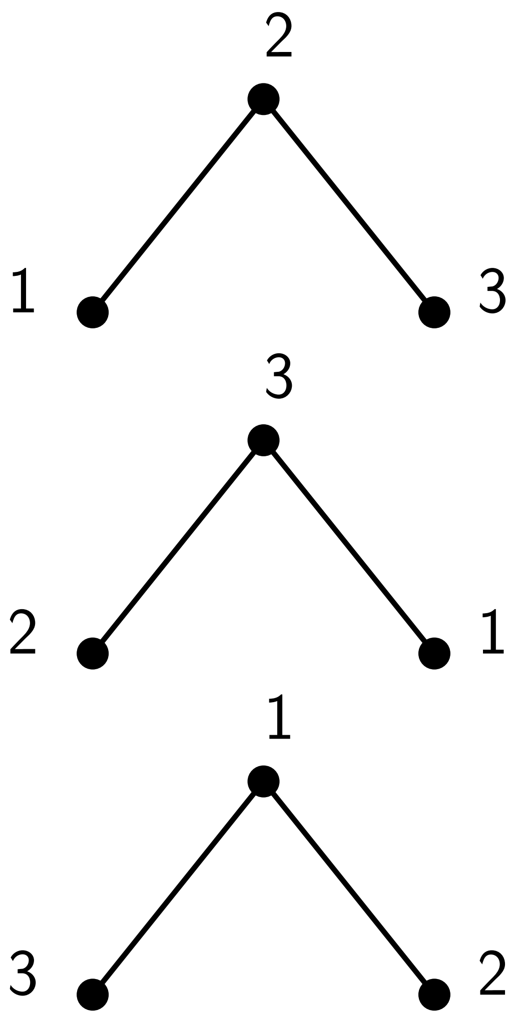

The symbolic method

Our goal is to determine the number of graphs in the family with size n.

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Definition. A **combinatorial class** is a pair $(\mathcal{A}, |\cdot|)$ where

- $\mathcal{A}$ is a family of combinatorial objects,
- $|\cdot| : \mathcal{A} \to \mathbb{N}$ is a size function,
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$$A(x) = \sum_{n \geq 0} a_n x^n.$$

Suitable for unlabelled classes.

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Operations between **classes** translate into relations involving their **generating functions**. **The goal** is to obtain (a system of) equations that determine the GF of our class.

Labelled trees

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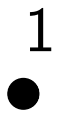

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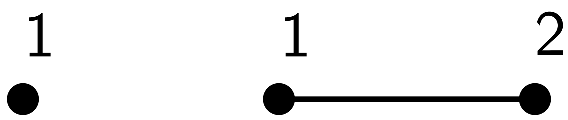
$$T(x) = \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{3}{3!}x^3 + \frac{16}{4!}x^4 \cdots .$$



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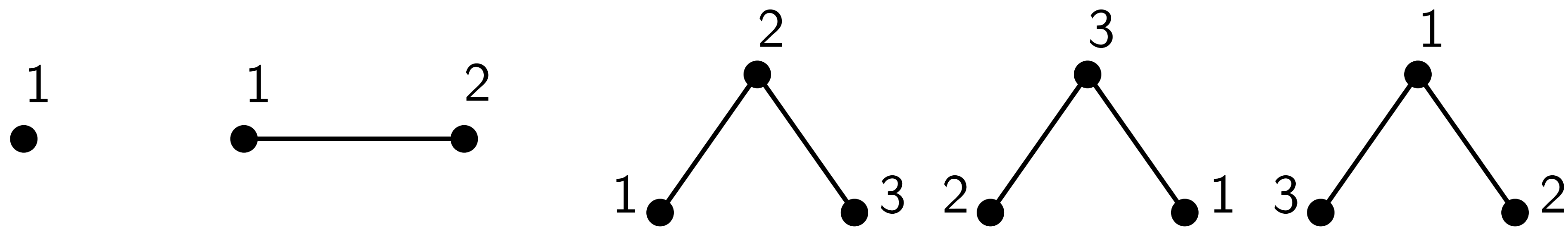

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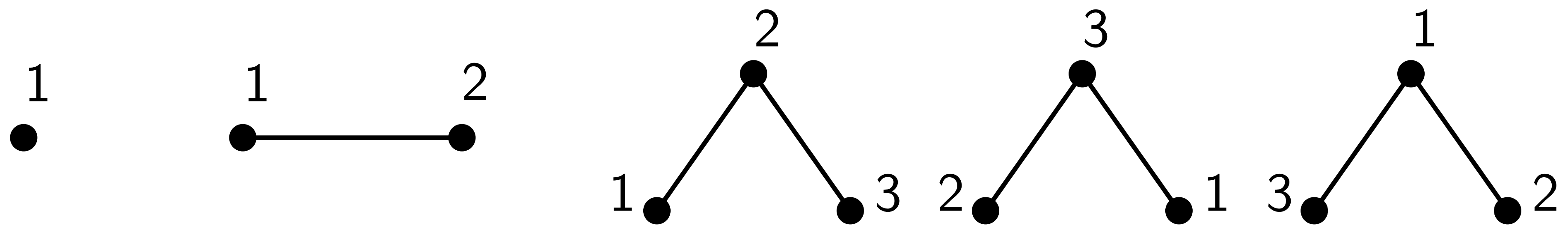

Rooting. Let $\mathcal{T}^\bullet$ be the class of rooted labelled trees.
Since all vertices are distinguishable, there are n ways to root a tree with n vertices. Thus,

$$T^\bullet(x) = \sum_{n \geq 0} n \frac{t_n}{n!} x^n = xT'(x).$$

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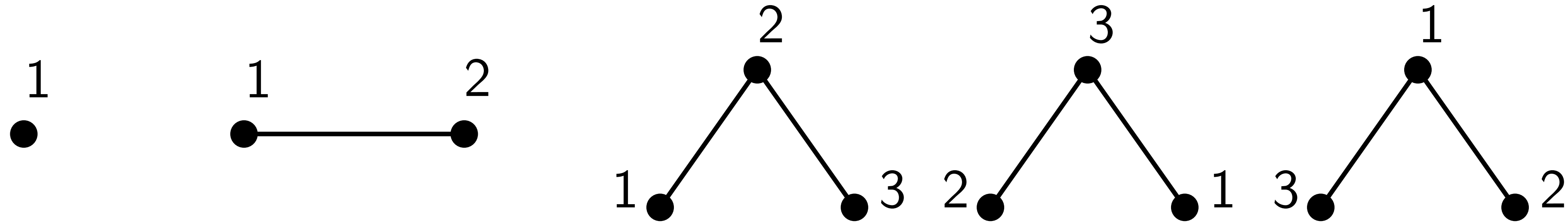

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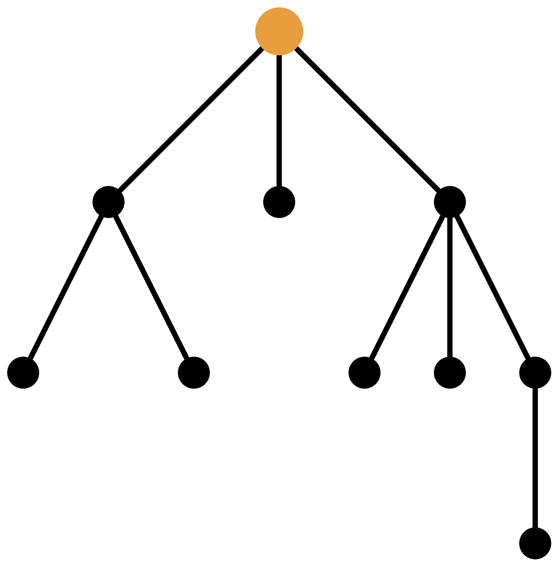
Unrooting. To do the inverse operation, we can simply integrate:

$$T(x) = \int T^\bullet(x)/x \; dx.$$

Labelled trees



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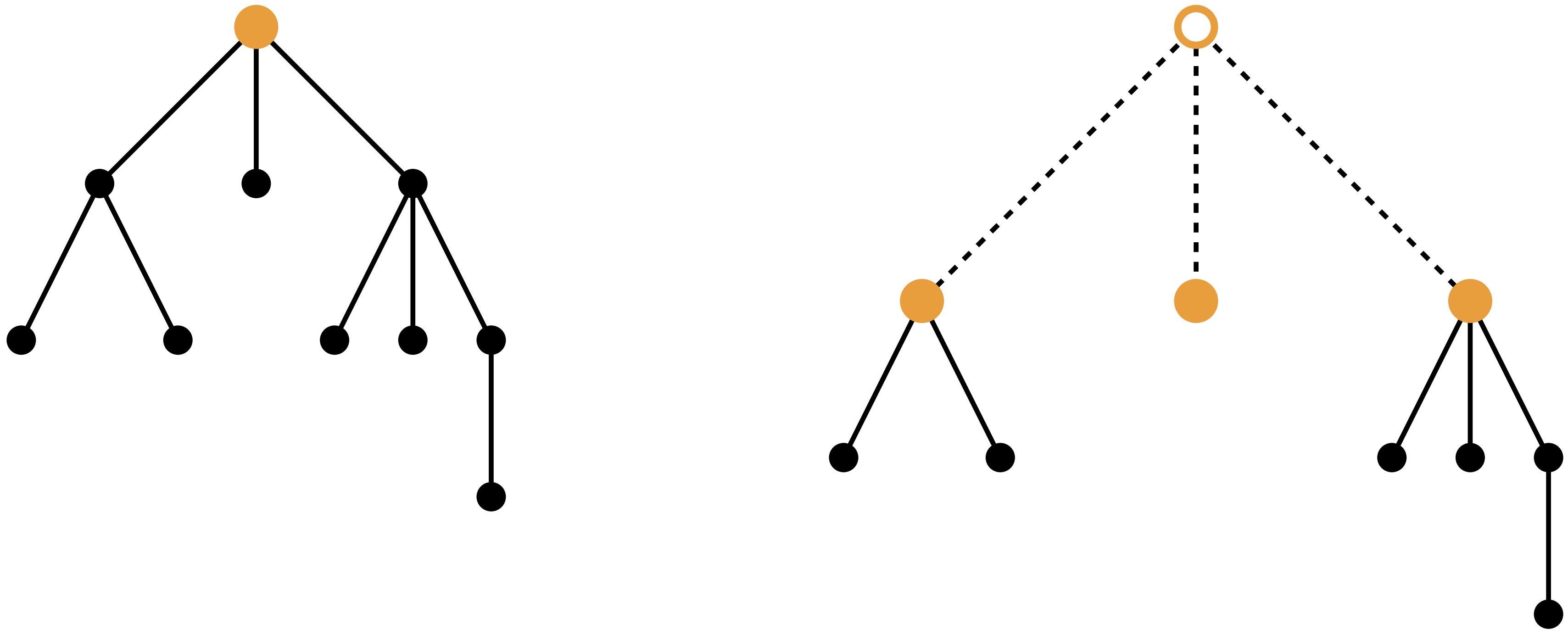

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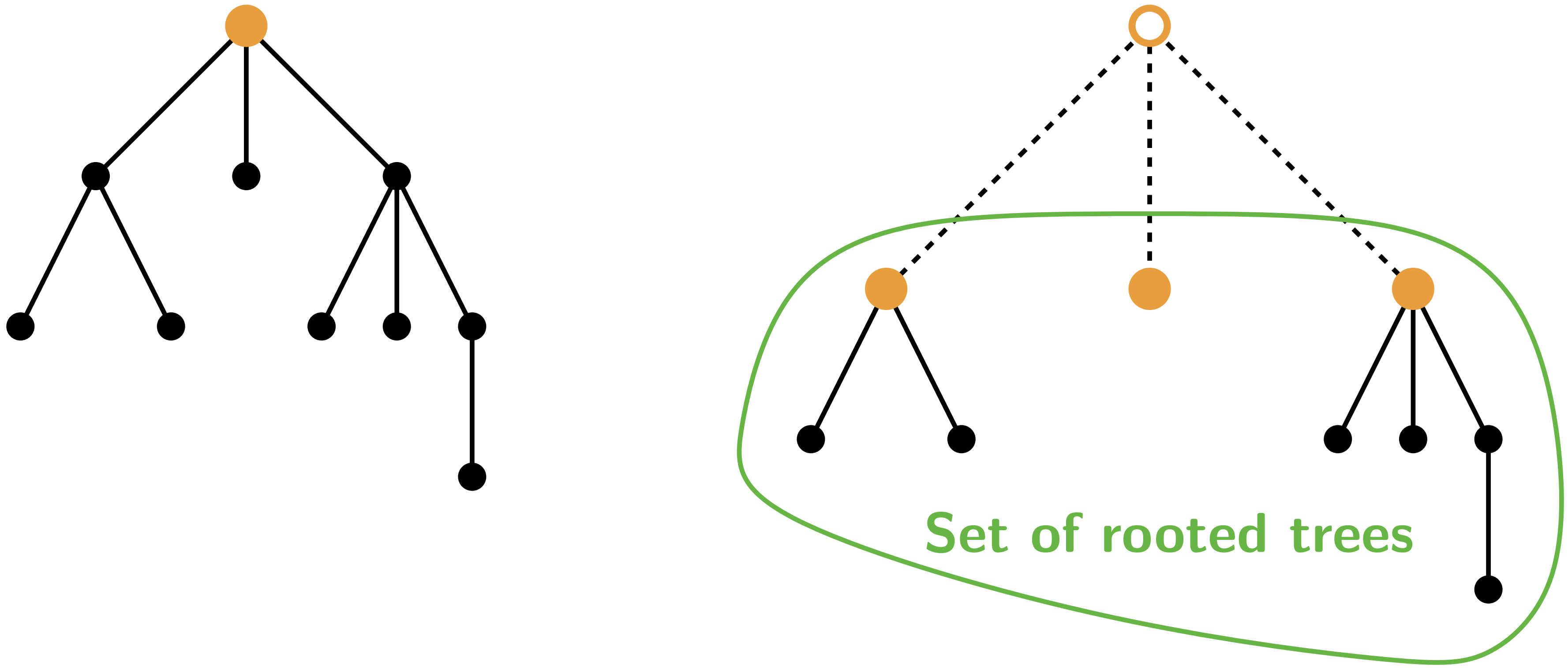

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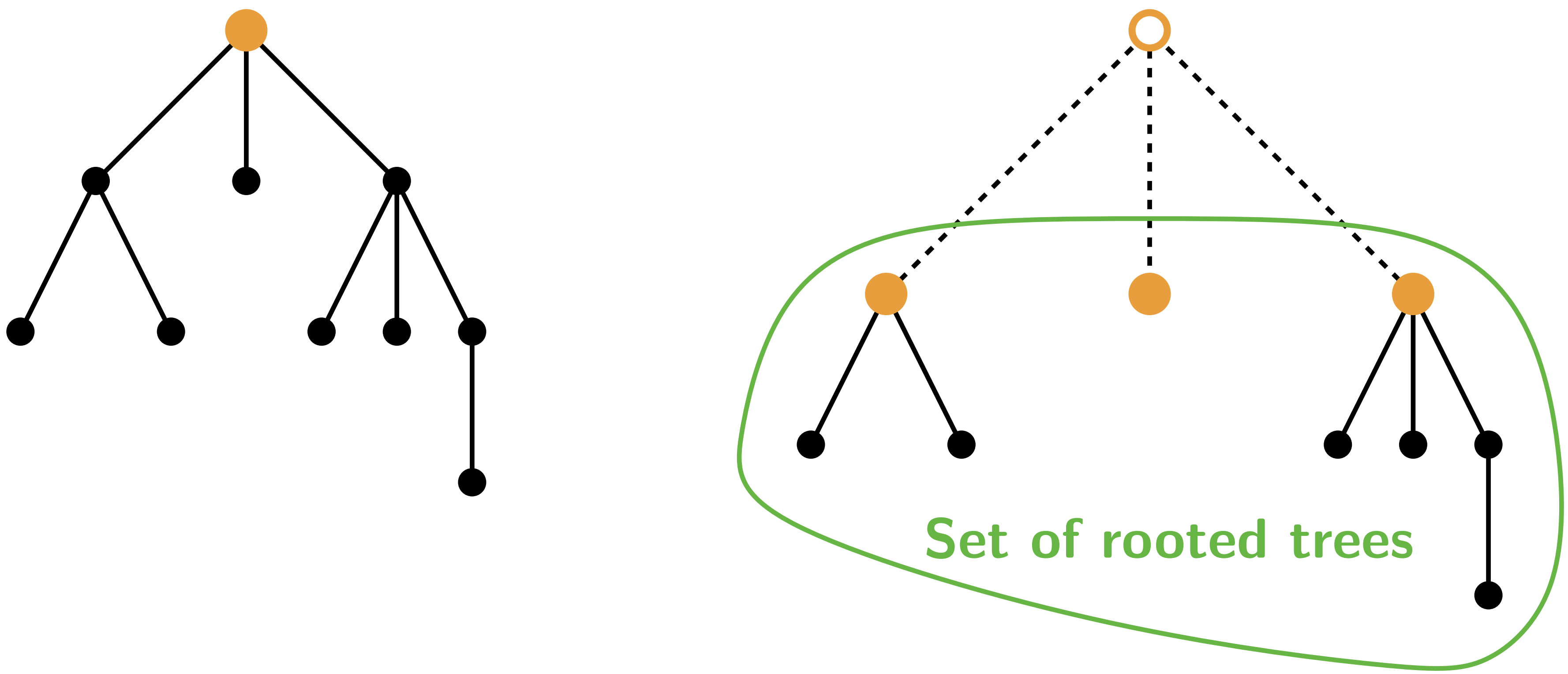
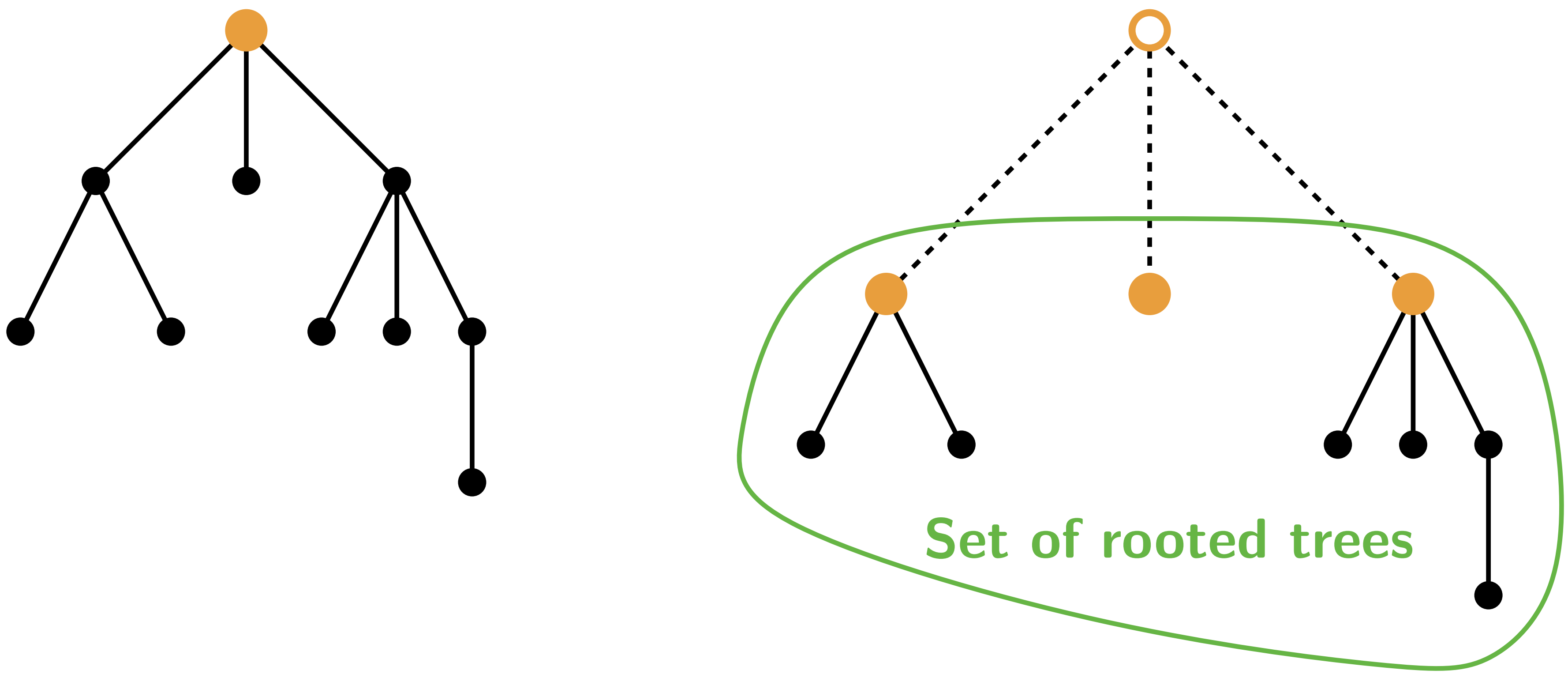


Implicit equation:

$$T^{\bullet}(x) = x \exp\left(T^{\bullet}(x)\right) = x + x^2 + \frac{3x^3}{2} + \cdots . \qquad (2)$$

Labelled trees



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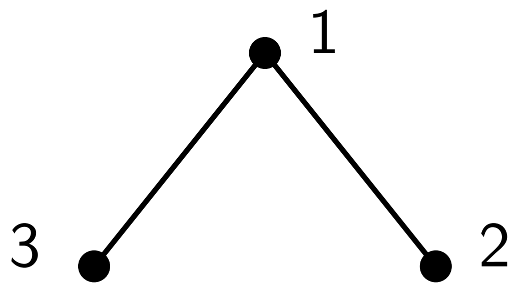
By using the **Lagrange inversion formula** we obtain:

$$|\mathcal{T}_n^\bullet| = n![x^n]T^\bullet(x) = n^{n-1} \implies |\mathcal{T}_n| = |\mathcal{T}_n^\bullet|/n = n^{n-2}.$$

Pólya theory



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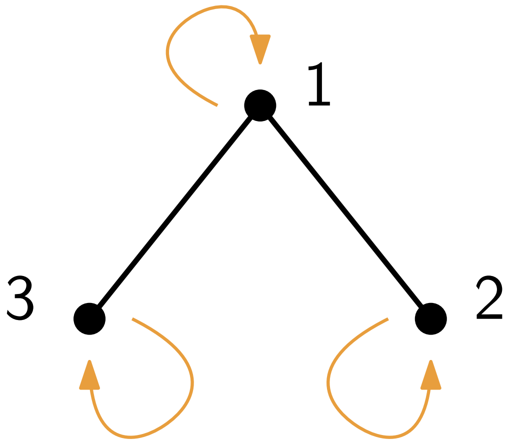

$(1)(2)(3) \longrightarrow s_1^3$

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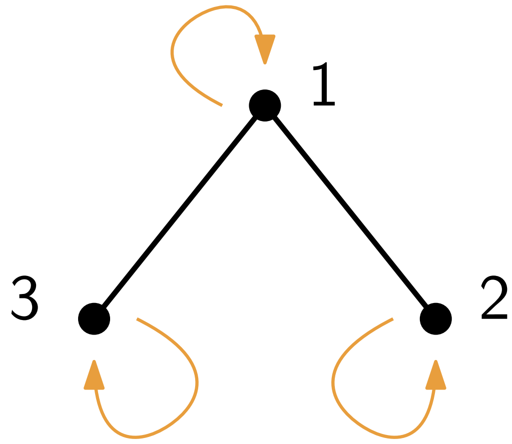

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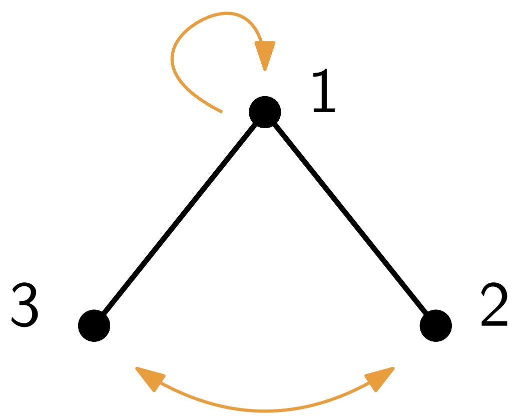

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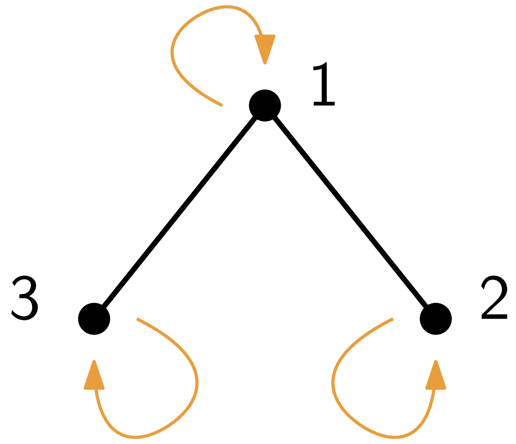

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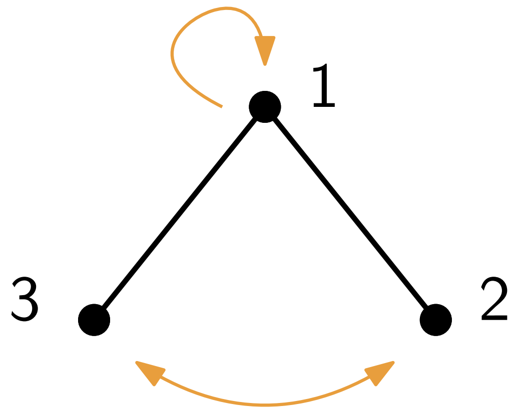

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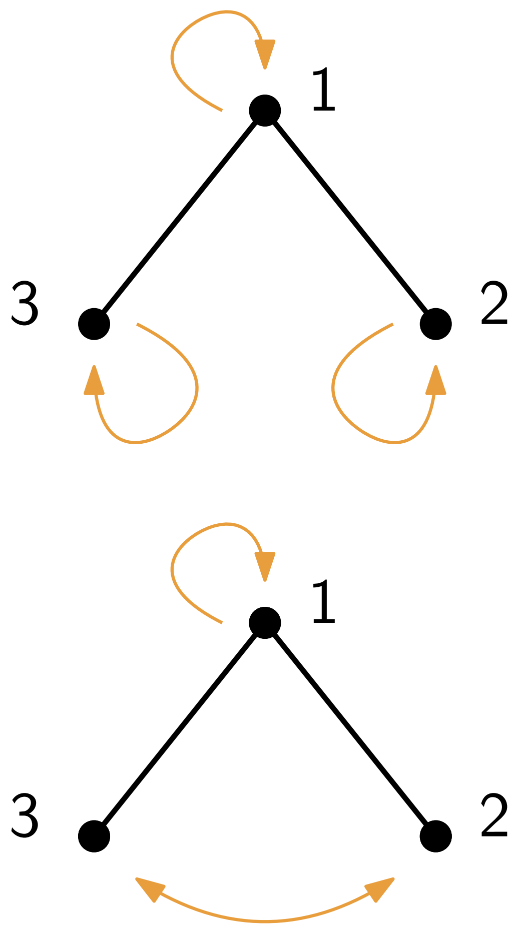

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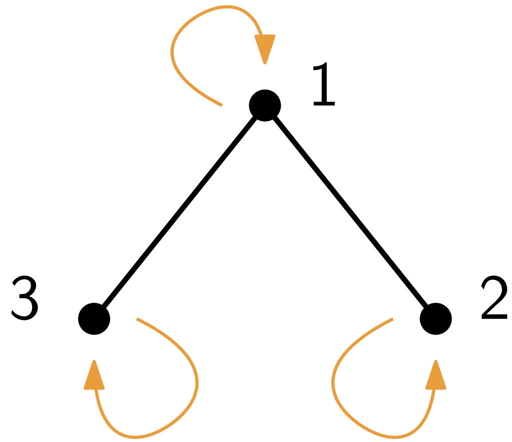

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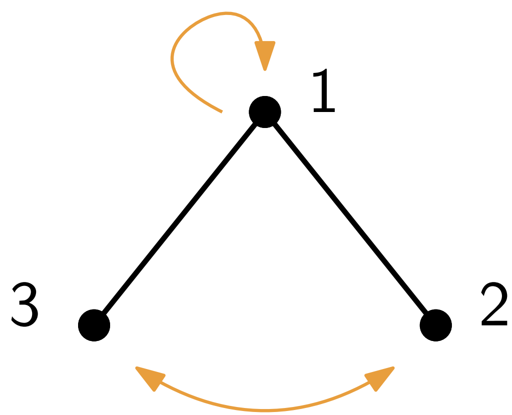

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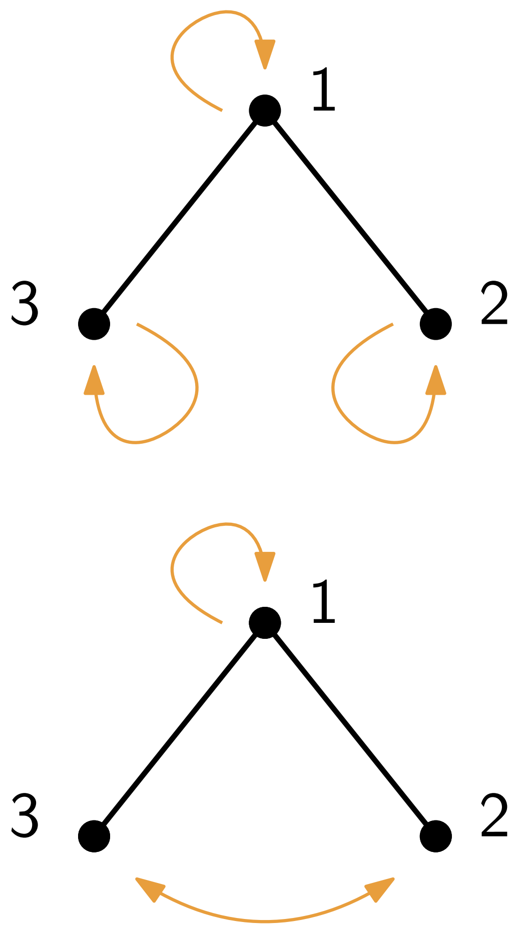

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The OGF of the unlabelled class $\tilde{\mathcal{G}}$ is given by

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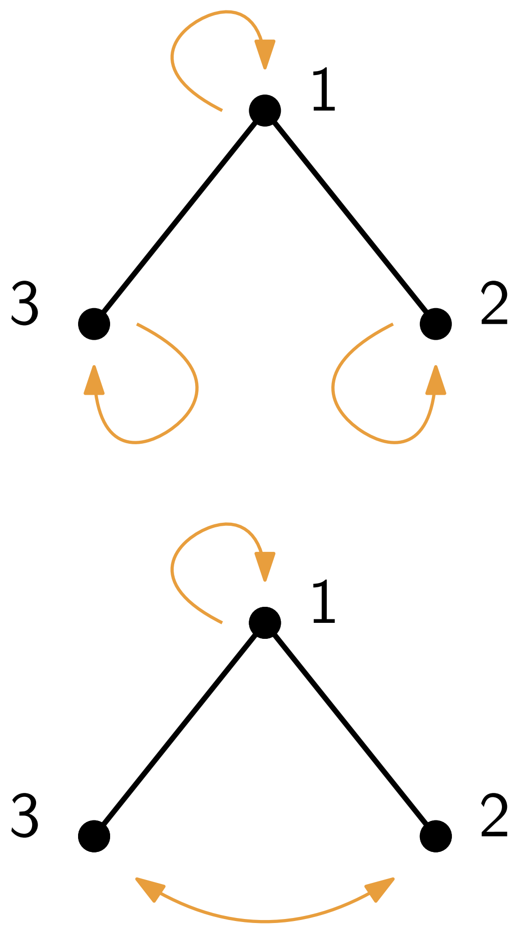

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In our case,

$$G(x) = \frac{3}{3!}(x^3 + x \cdot x^2) = x^3$$

Unlabelled trees

Pólya trees: rooted, unlabelled trees.

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Theorem. [Pólya, 1937]
The OGF $P(x)$ of Pólya trees is given by

$$P(x) = x \exp(P(x) + \frac{P(x^2)}{2} + \frac{P(x^3)}{3} + \dots).$$

As $n \to \infty$ we have

$$[x^n]P(x) \sim \frac{b\sqrt{\rho}}{2\sqrt{\pi}} \cdot n^{-3/2} \cdot \rho^{-n},$$

with $b \approx 2.681127$ and $\rho \approx 0.338219$.

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What about unrooted unlabelled trees?

Unlabelled trees

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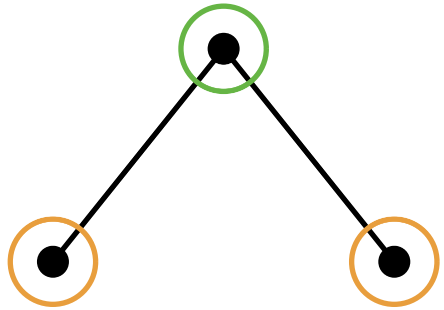
Rooting is biased in unlabelled graphs.
Not every unlabelled graph of size n gives rise to n rooted graphs.



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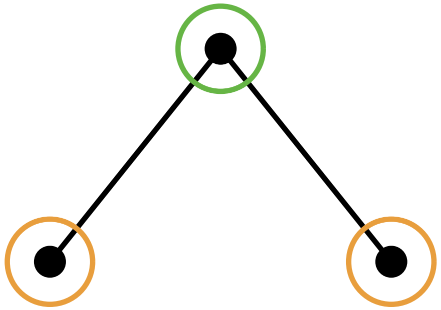

Theorem. [Otter, 1948]
The OGF $U(x)$ of unlabelled trees is given by

$$U(x) = P(x) + \frac{1}{2}(P(x^2) - P(x)^2).$$

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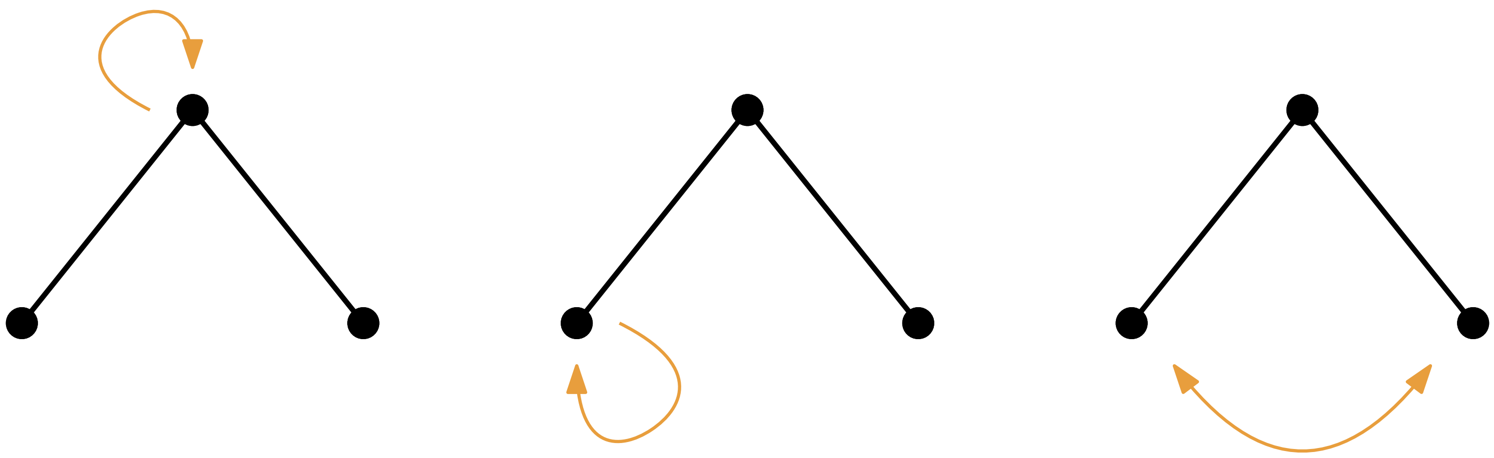
Proof. Using the **dissymmetry theorem**.

Cycle-pointing



Definition. A **cycle-pointed graph** is a pair (G, c) where $G \in \mathcal{G}$ is an unlabelled graph and c is a cycle of some automorphism of G.

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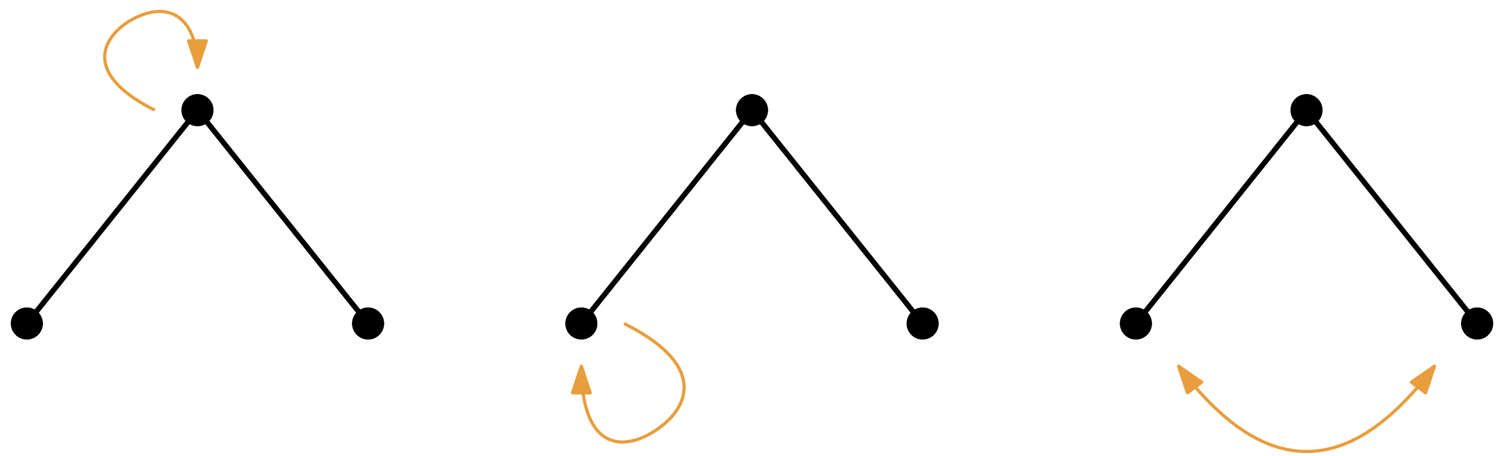

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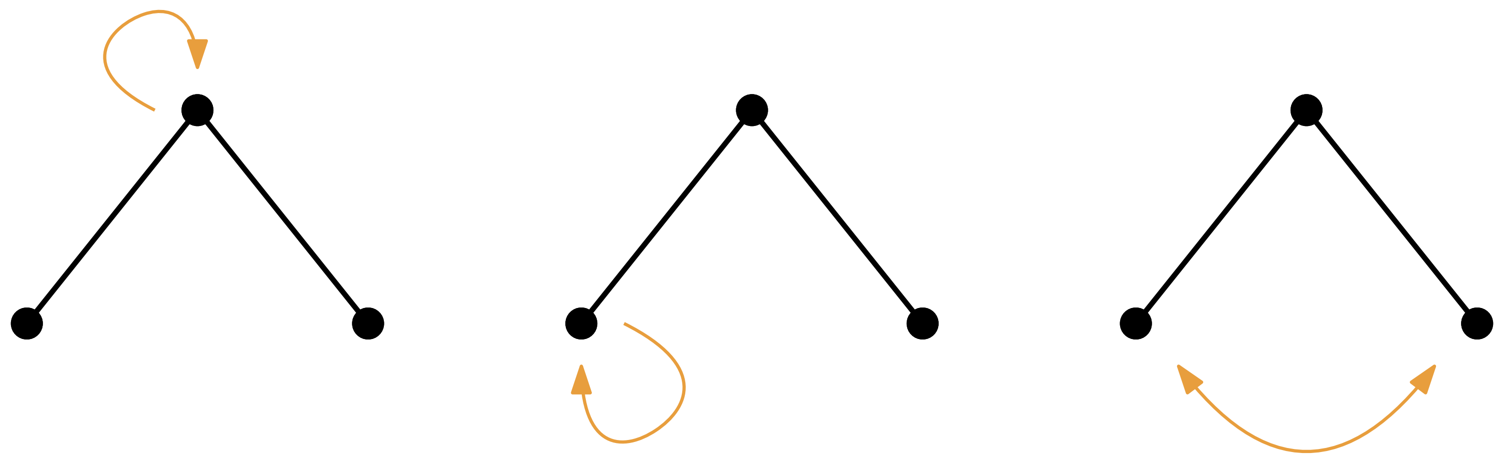

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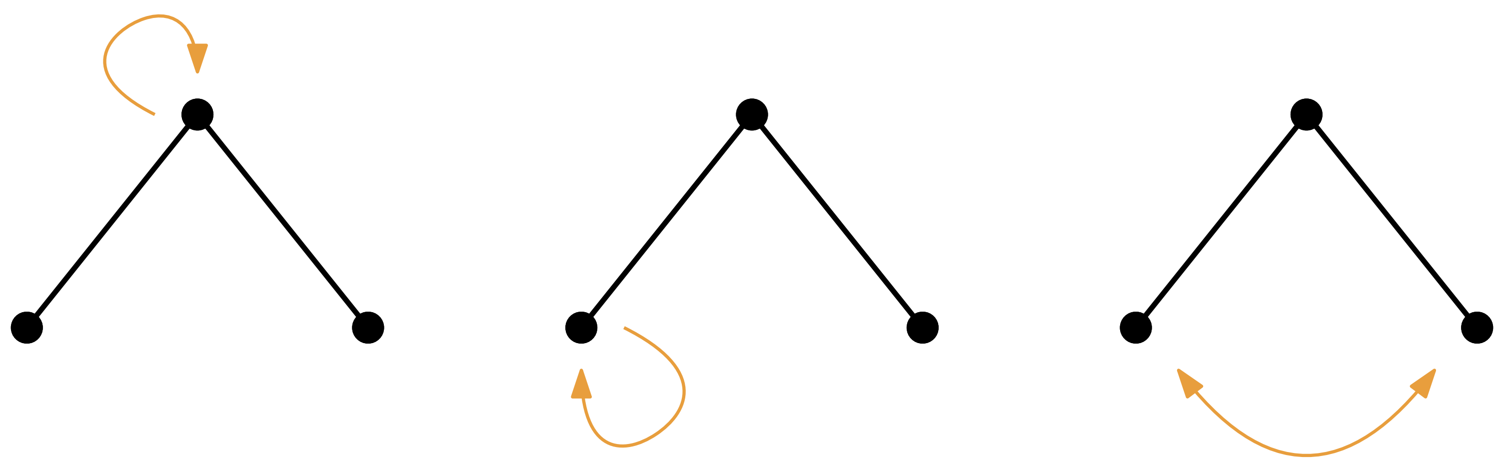

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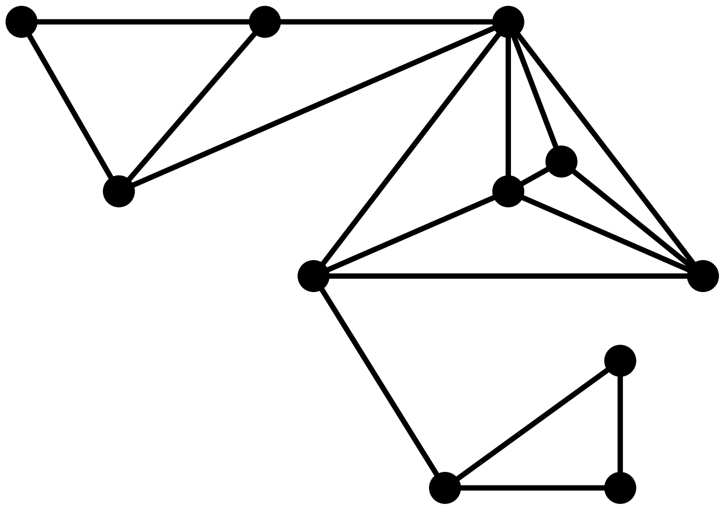
They extend Pólya theory to cycle-pointed graphs. In particular, they manage to unroot Pólya trees via cycle-pointing and they recover Otter's formula.

Our class of graphs



Chordal graphs with tree-width at most t

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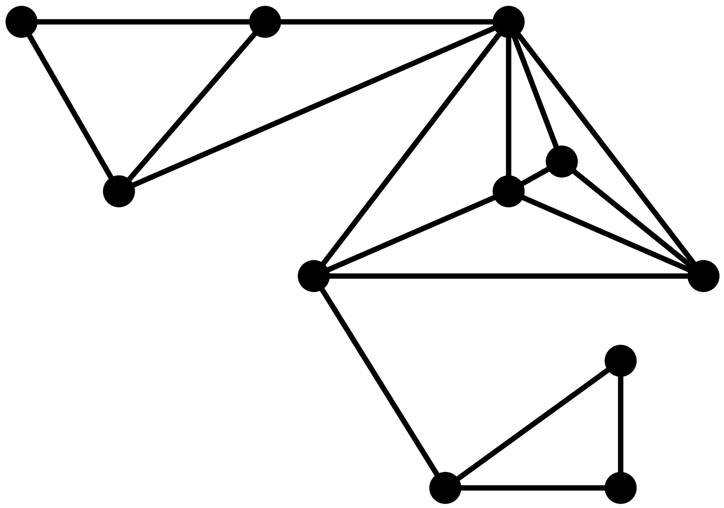

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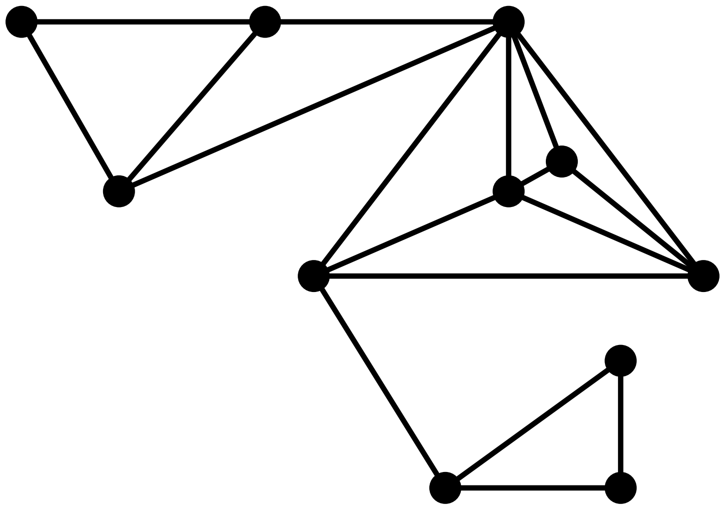

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[C., Drmota, Noy & Requilé, 2023]: assymptotic enumeration of the labelled class.

$$|\mathcal{G}_{t,n}| \sim c_t \cdot n^{-5/2} \cdot \gamma_t^n \cdot n! \qquad \text{as } n \to \infty,$$

for some $c_t > 0$ and $\gamma_t > 1$

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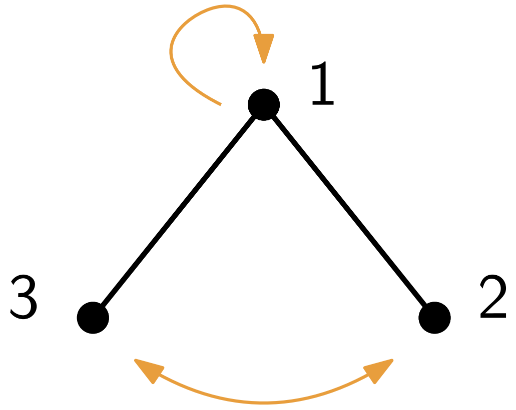

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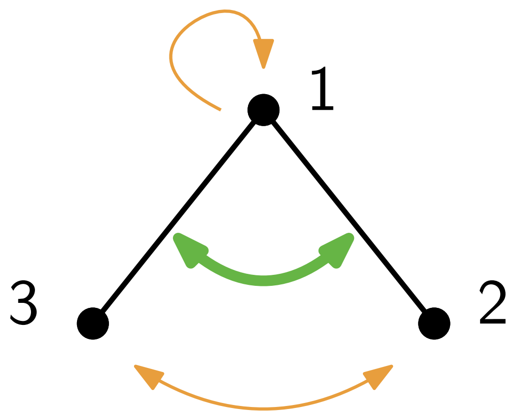

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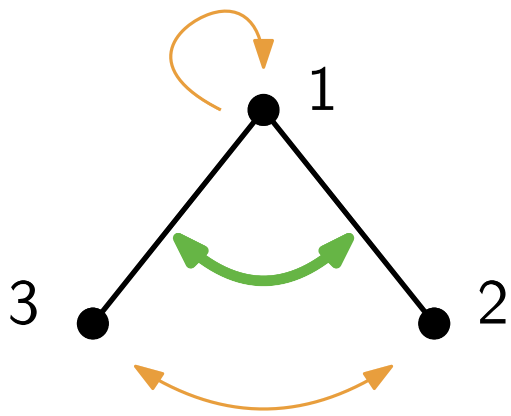

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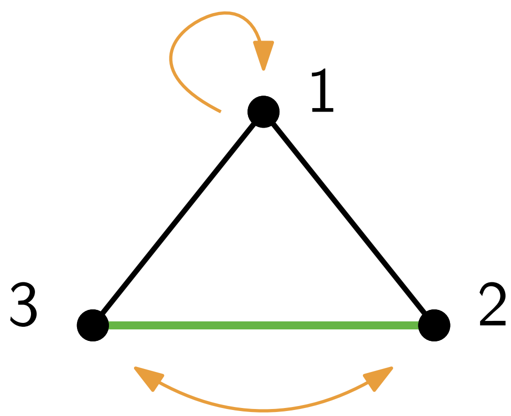

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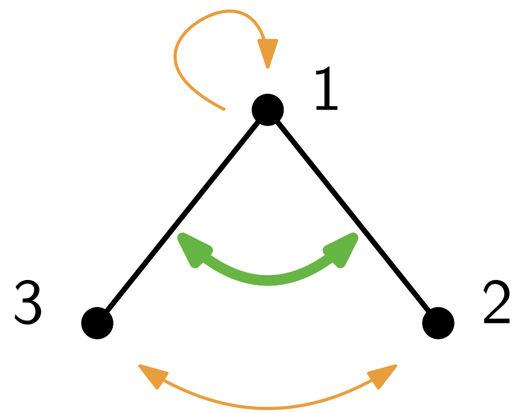

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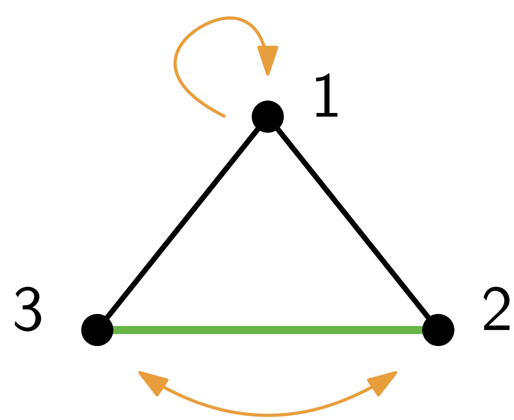

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What we do:

- Refinement of cycle index sums to encode cycles of cliques.
- Extend cycle-pointing to cycles of cliques.

Results

- [Harary & Palmer (1968)], [Fowler, Gessel, Labelle & Leroux (2002)]: asymptotic enumeration of unlabelled (rooted and unrooted) **2-trees**

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This talk: generalisation of previous results.

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This talk: generalisation of previous results.

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Future:

- [C.,Drmota & Requilé (soon?)]: asymptotic enumeration of unlabelled chordal graphs with bounded tree-width.

The system

$$
\begin{cases}
X^{\lambda}_{\mathcal{G}^{(k)}_{t,k+1}} = \frac{k!}{\alpha(\lambda)\kappa(\lambda)} \frac{\partial}{\partial s_{\lambda,1}} X_{\mathcal{G}_{t,k+1}}, \\
X^{\lambda}_{\mathcal{G}^{(k)}_{t,k}} = Z_{\mathrm{SET}}(s_j \to (X^{\lambda^j}_{\mathcal{G}^{(k)}_{t,k+1} \circ_k \mathcal{G}^{(k)}_{t,k}})^{[j]})_{j\geq 1}, \\
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X_{\mathcal{G}^{\bullet_k}_{t,k}} = \sum_{\mu\vdash k} \frac{\alpha(\mu)\kappa(\mu)}{k!} t_{\mu,1} X^{\mu}_{\mathcal{G}^{(k)}_{t,k}} + X_{(\mathcal{G}_{t,k})^{\bullet_k}_{\geq 2}}, \\
X_{(\mathcal{G}_{t,k})^{\bullet_k}_{\geq 2}} = X_{(\mathcal{G}_{t,k+1})^{\bullet_k}_{\geq 2}}(s_{\mu,j} \to (X^{\mu}_{\mathcal{G}^{(k)}_{t,k}})^{[j]}, t_{\mu,j} \to (X^{\mu}_{(\mathcal{G}^{(k)}_{t,k})^{\bullet_k}})^{[j]})_{\mu\vdash k, j\geq 1} \\
\qquad + \sum_{\mu\vdash k} \frac{\alpha(\mu)\kappa(\mu)}{k!} s_{\mu,1} Z_{\mathrm{SET}^{\bullet}_{\geq 2}}(s_j \to (X^{\mu^j}_{\mathcal{G}^{(k)}_{t,k+1} \circ_k \mathcal{G}^{(k)}_{t,k}})^{[j]}, \\
\qquad\qquad t_j \to (X^{\mu^j}_{(\mathcal{G}^{(k)}_{t,k+1} \circ_k \mathcal{G}^{(k)}_{t,k})^{\bullet_k}})^{[j]})_{j\geq 1}, \\
X_{\mathcal{G}_{t,k}} = \sum_{\lambda\vdash k} \sum_{1\leq j\leq \binom{t+1}{k}} \int \frac{1}{j t_{\lambda,j}} X_{\mathcal{G}^{\bullet_k}_{t,k}}(S(\lambda,j) \to 0, T(\lambda,j) \to 0) ds_{\lambda,j}
\end{cases}
$$