Enumeration of unlabelled chordal graphs with bounded tree-width

Jordi Castellví (CRM)

Work in collaboration with Michael Drmota
and Clément Requilé

CENTRE DE RECERCA MATEMÀTICA

EXCELENCIA MARÍA DE MAEZTU 2022 - 2025

RandNET Workshop - Wien

Introduction

How to build a tree?

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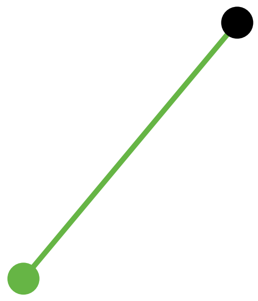

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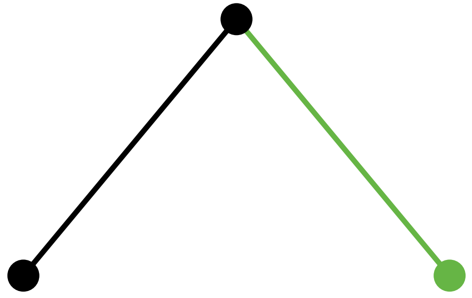

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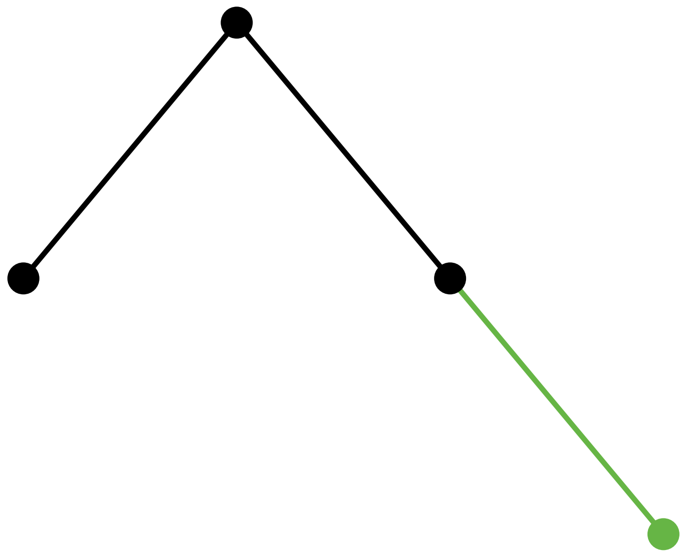

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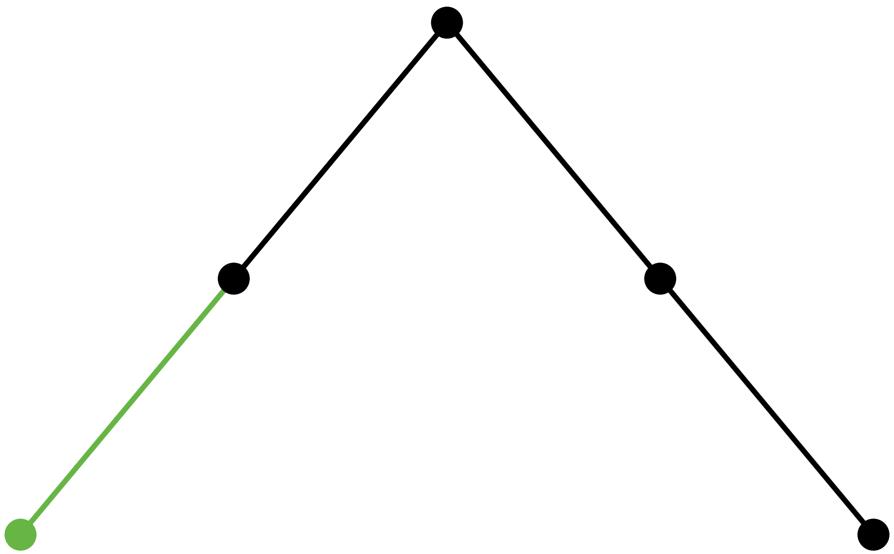

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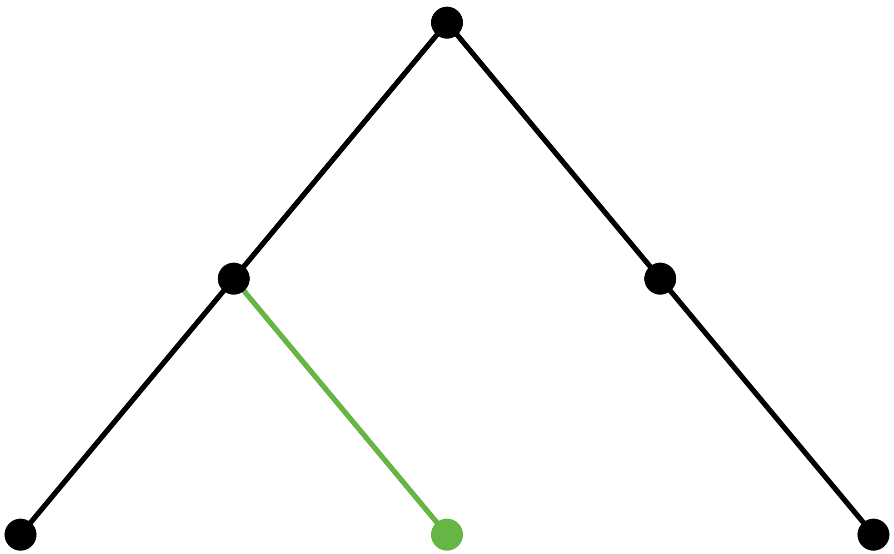

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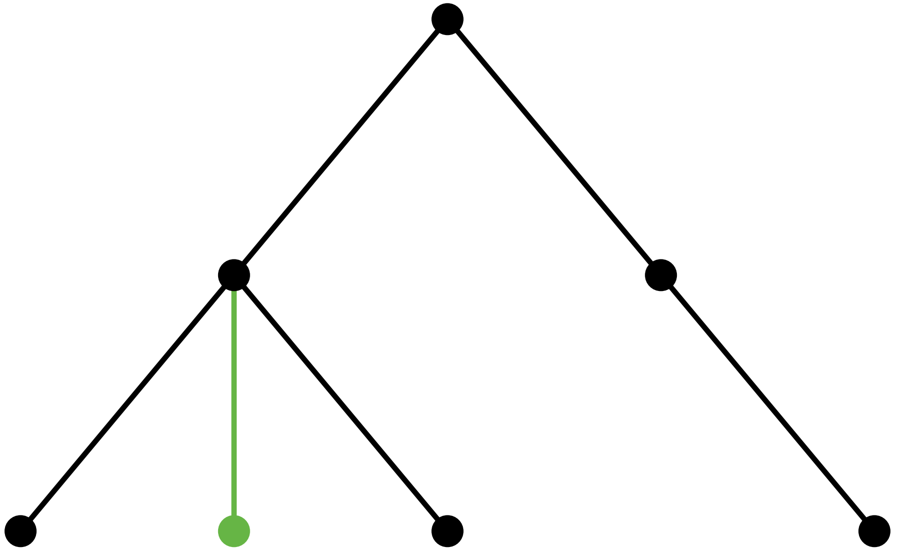

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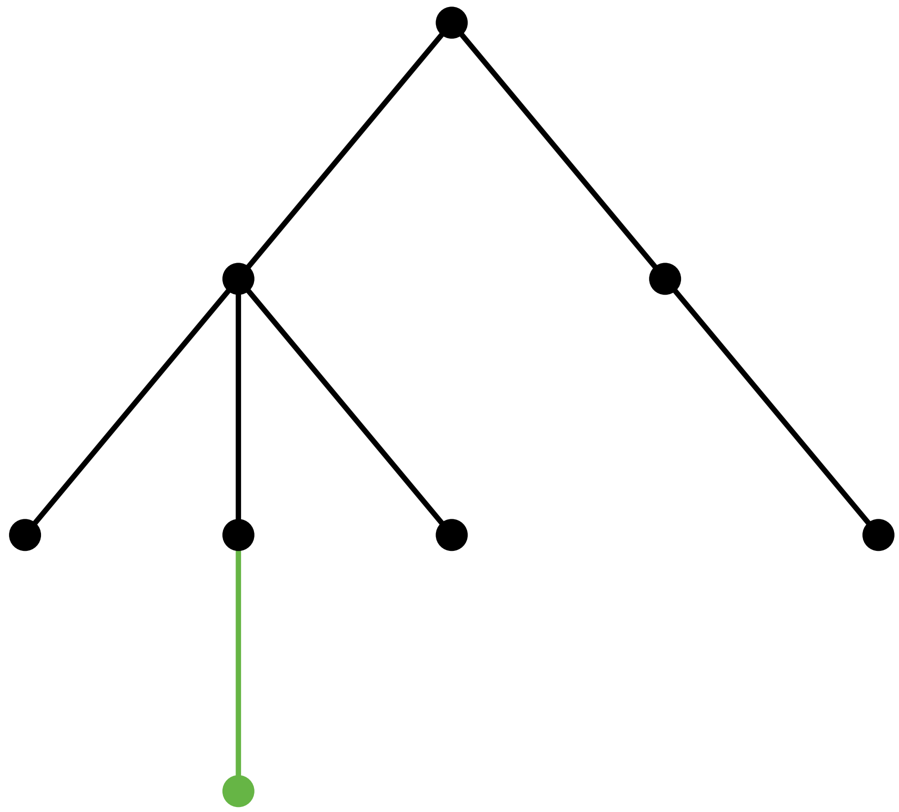

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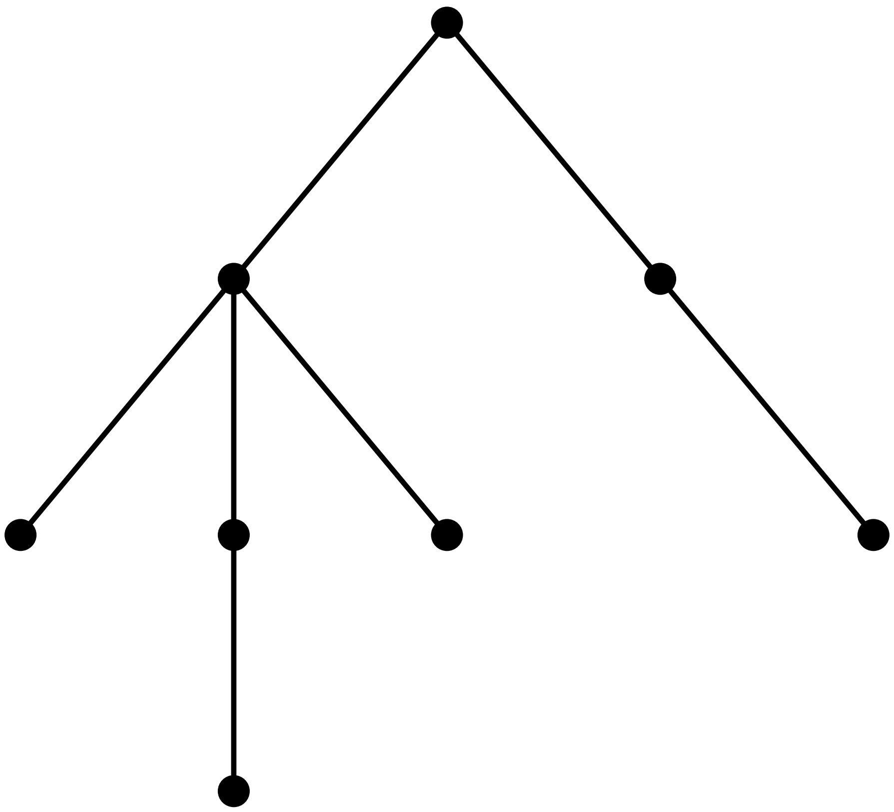

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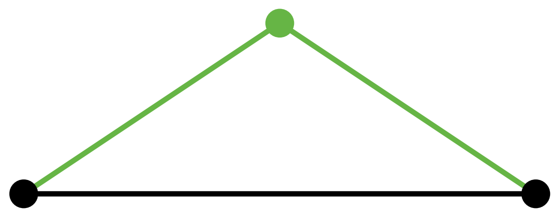

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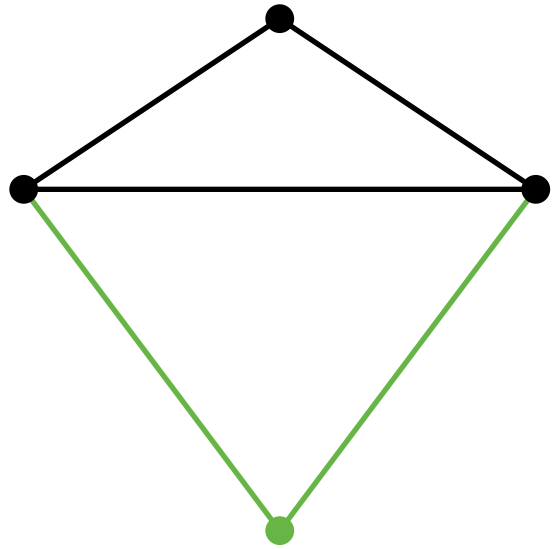

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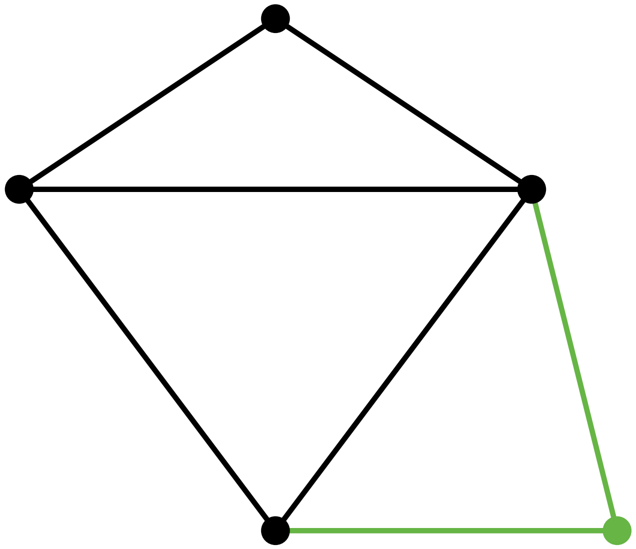

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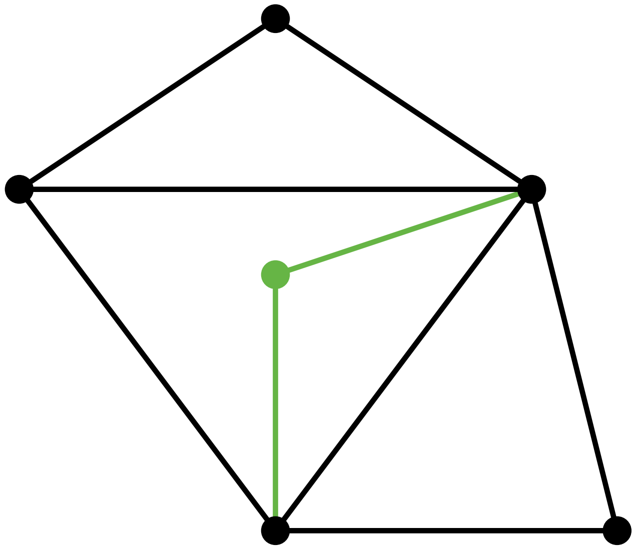

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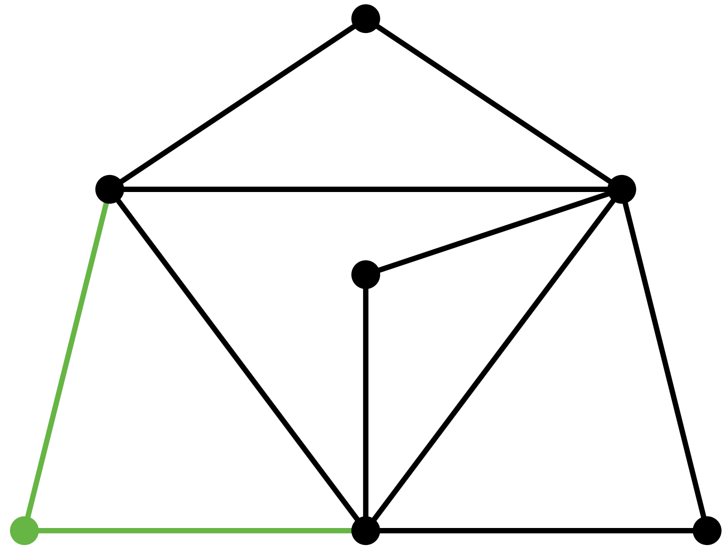

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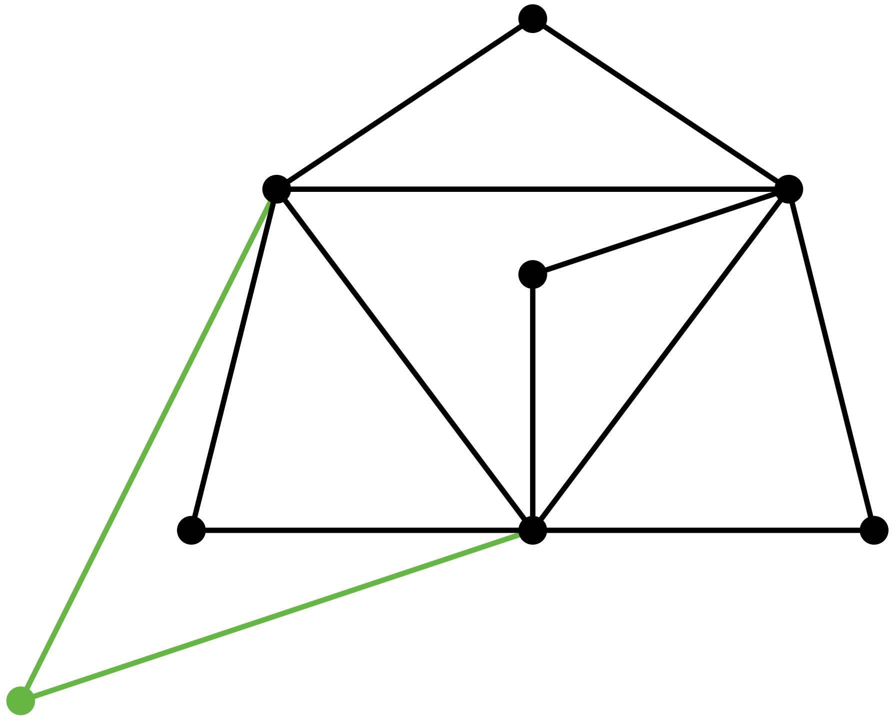

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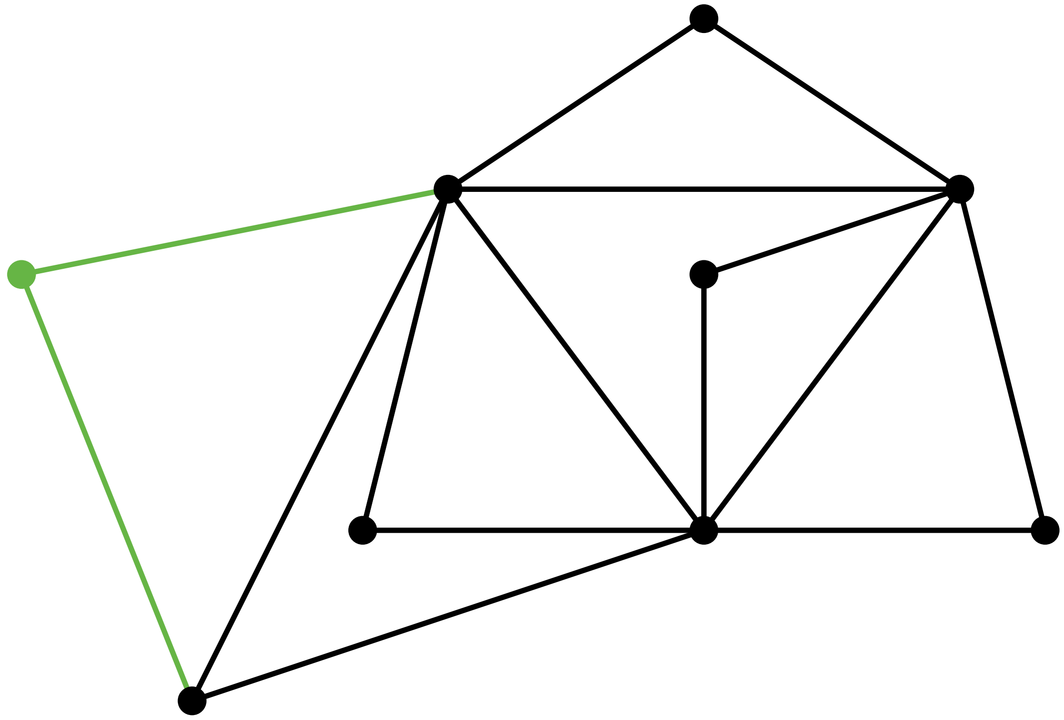

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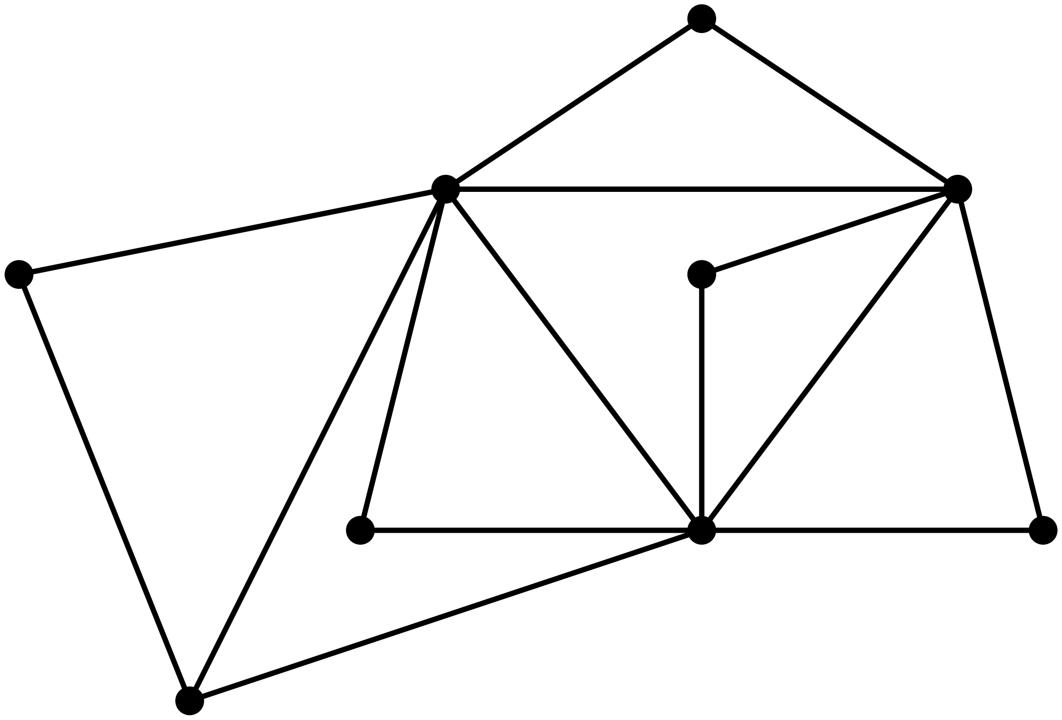

2-trees

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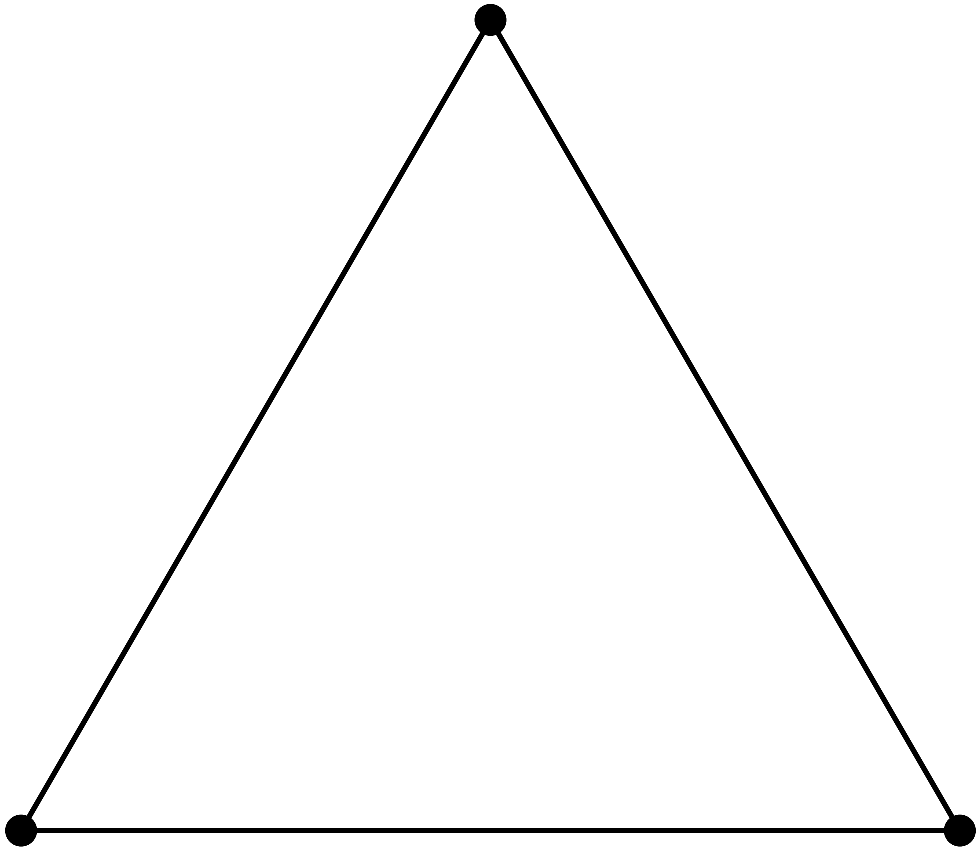

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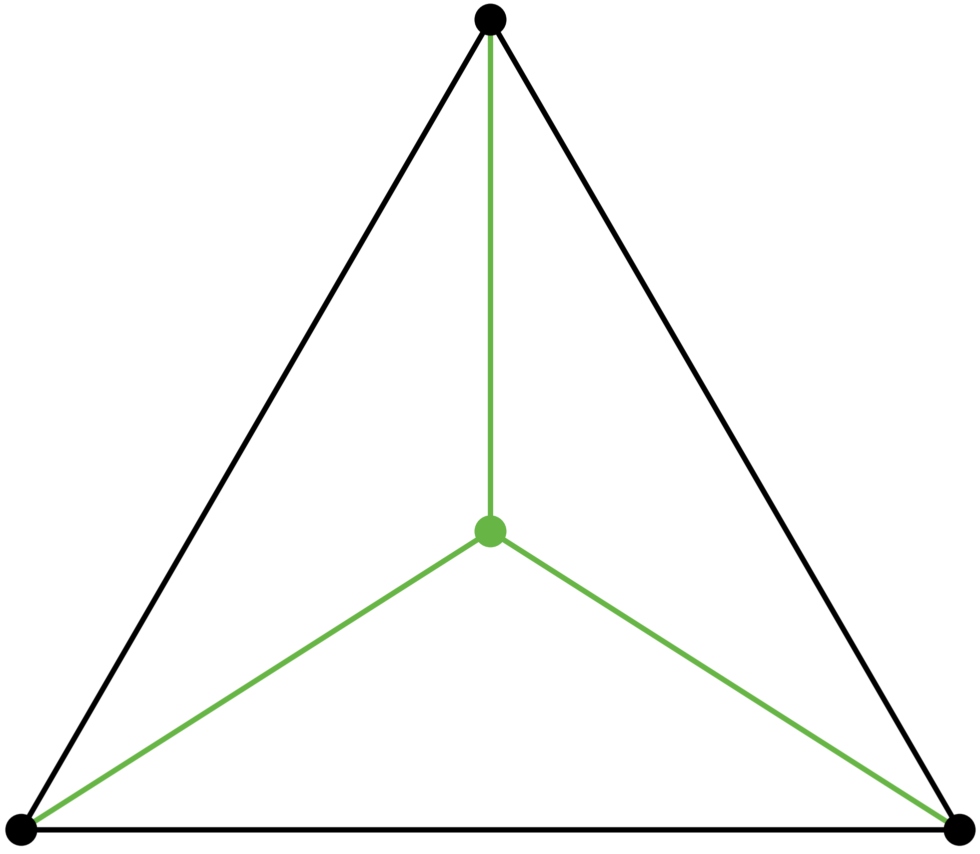

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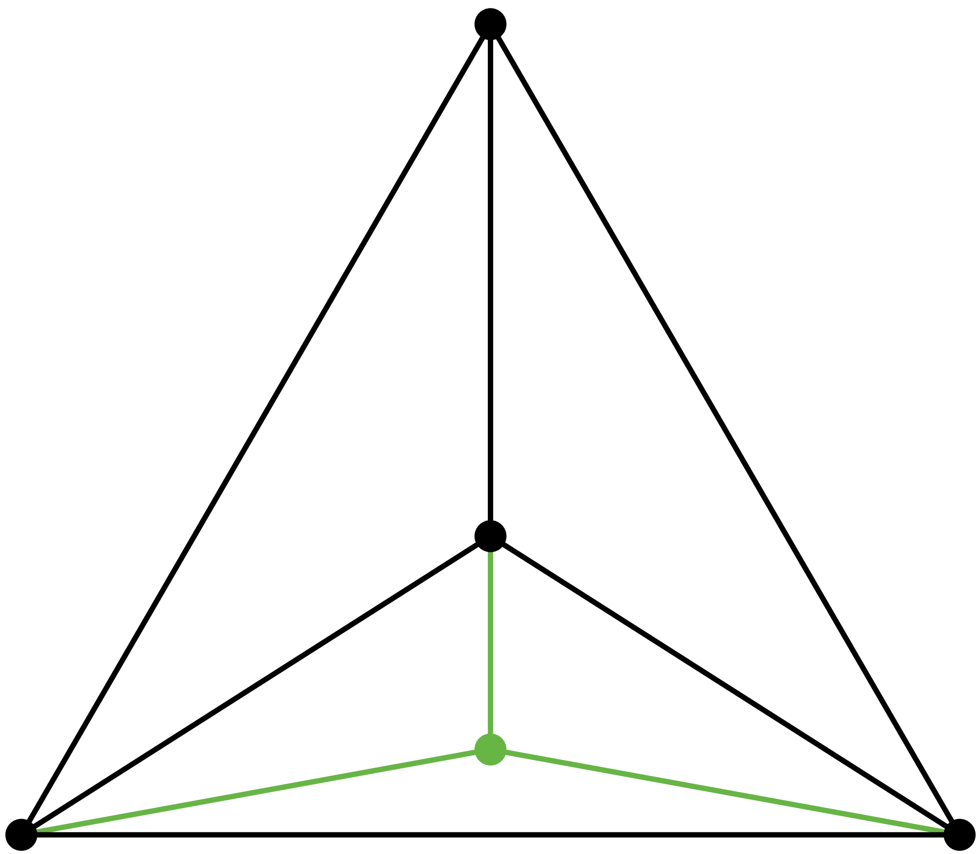

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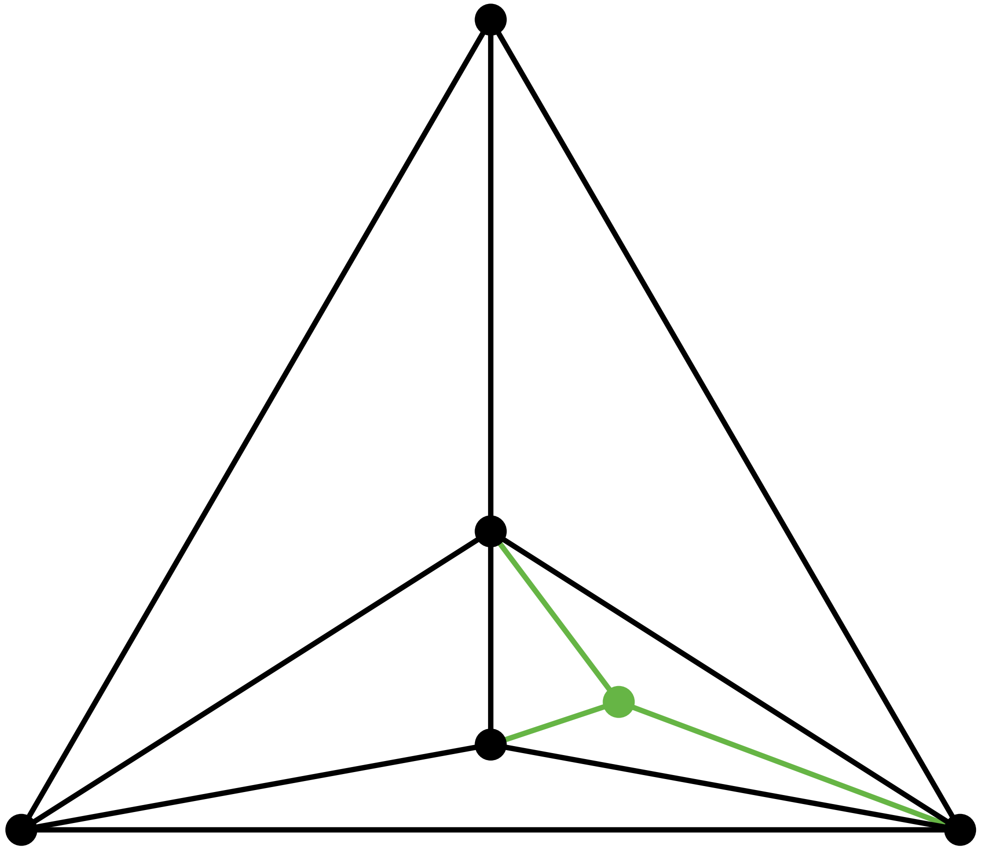

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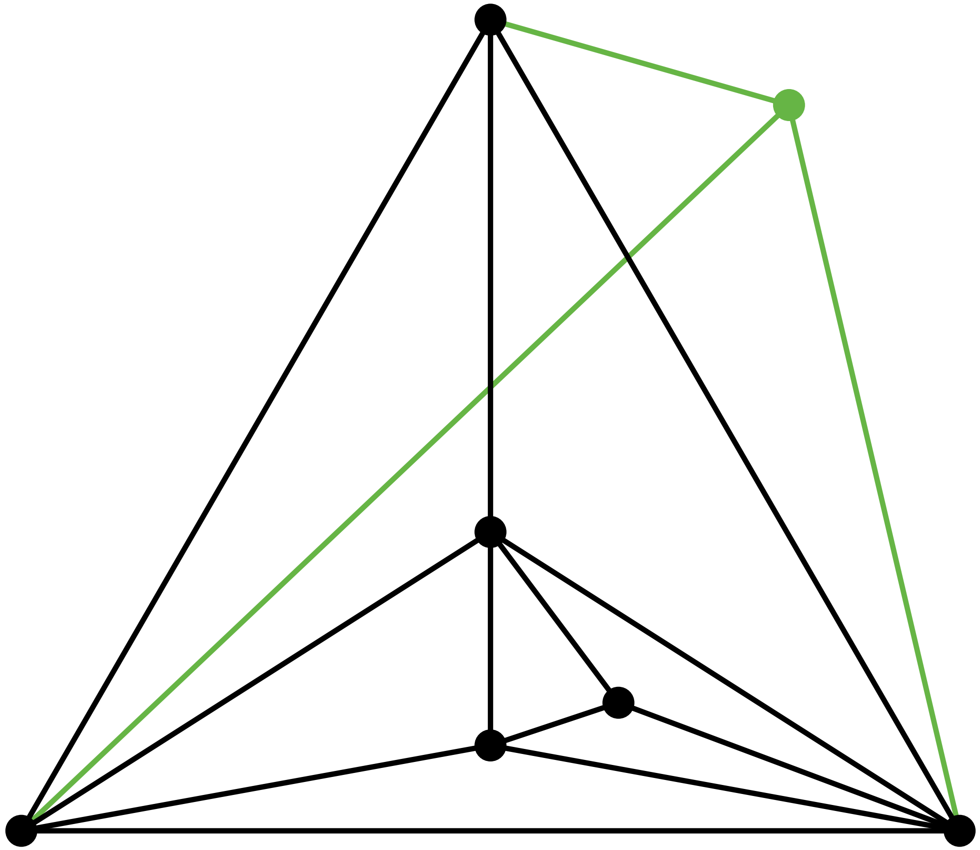

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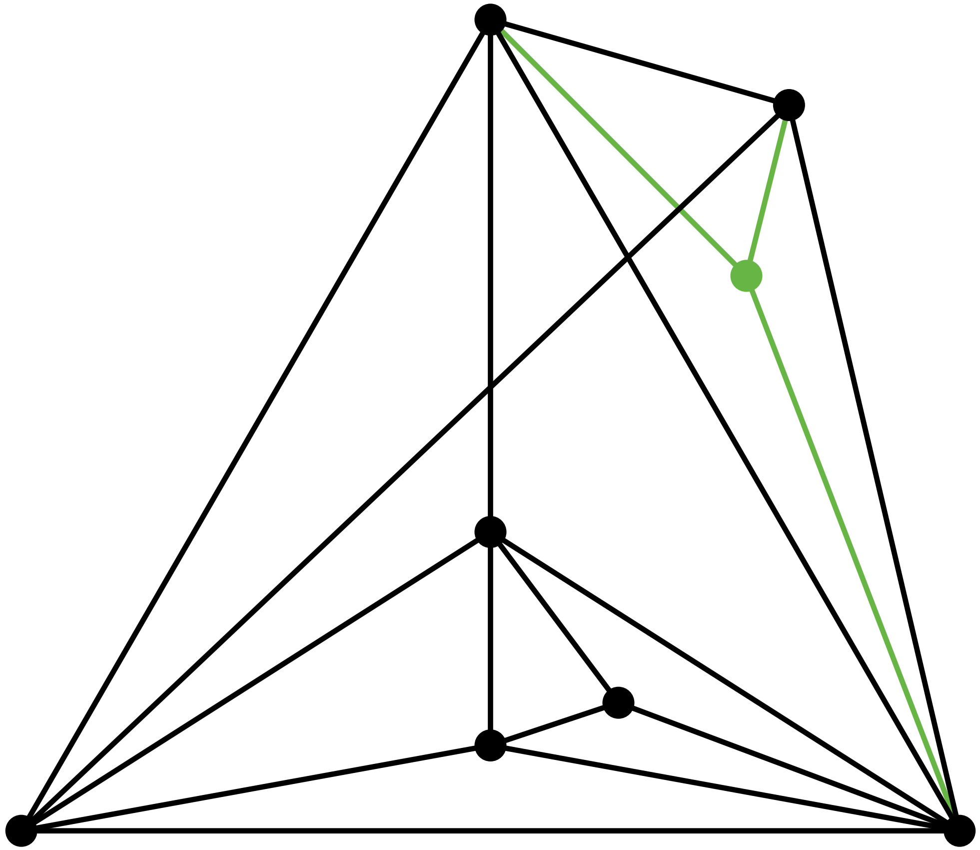

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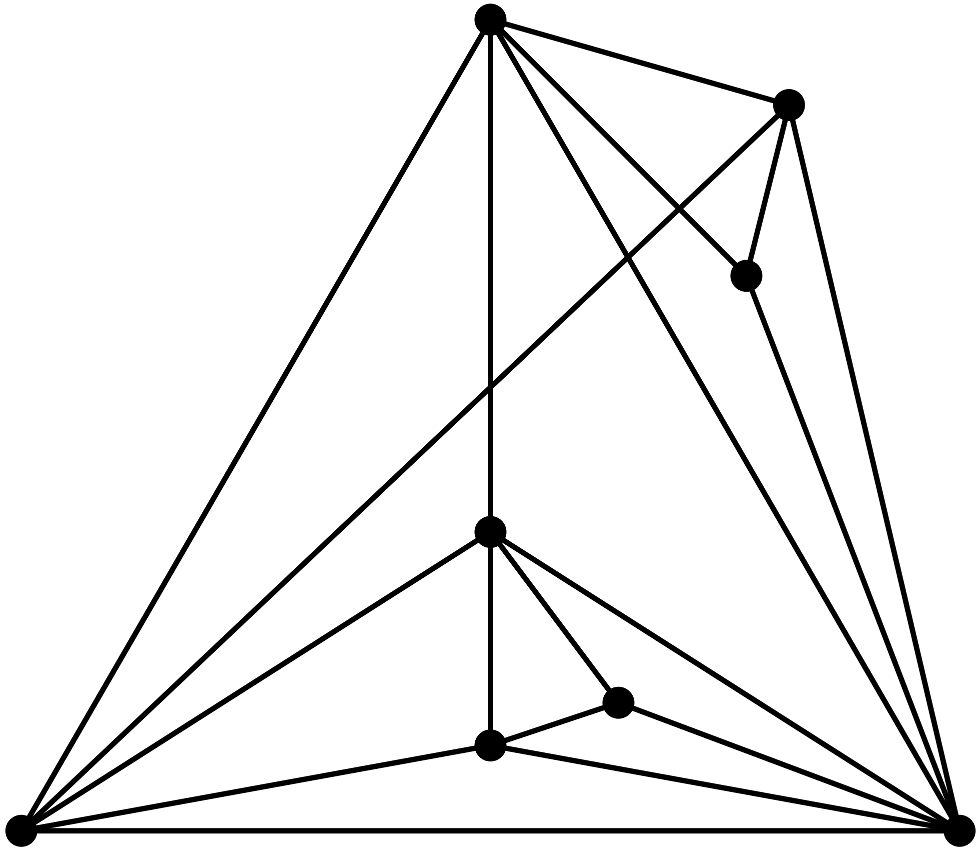

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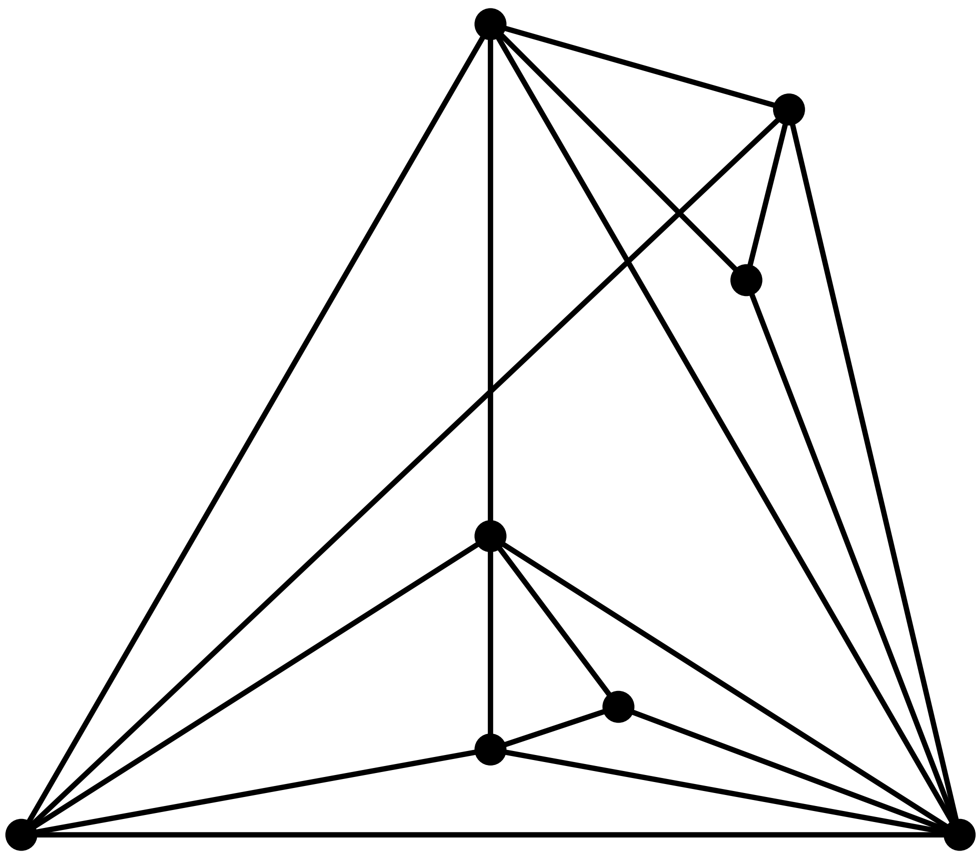

3-trees

Definition. A **k-tree** is a graph obtained from a $(k+1)$-clique by successively adding a new vertex connected to all vertices of an existing k-clique.

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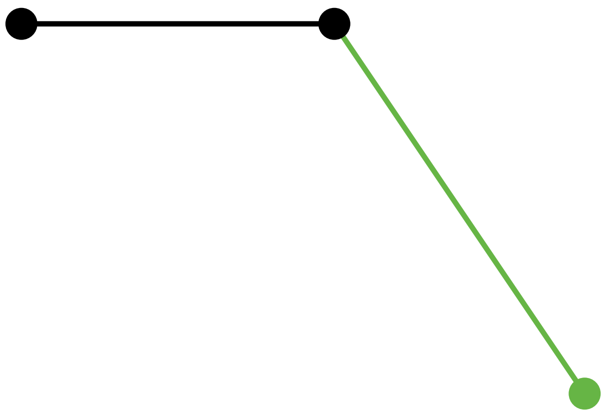

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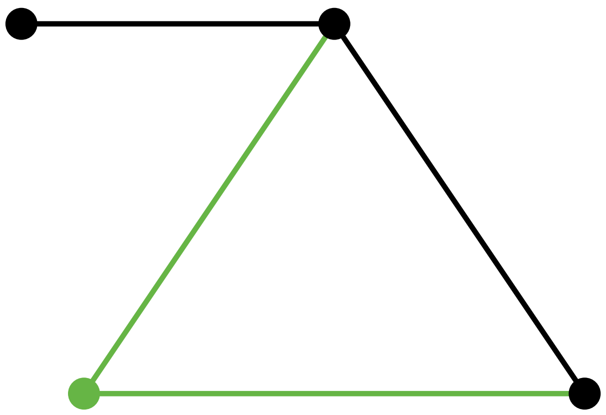

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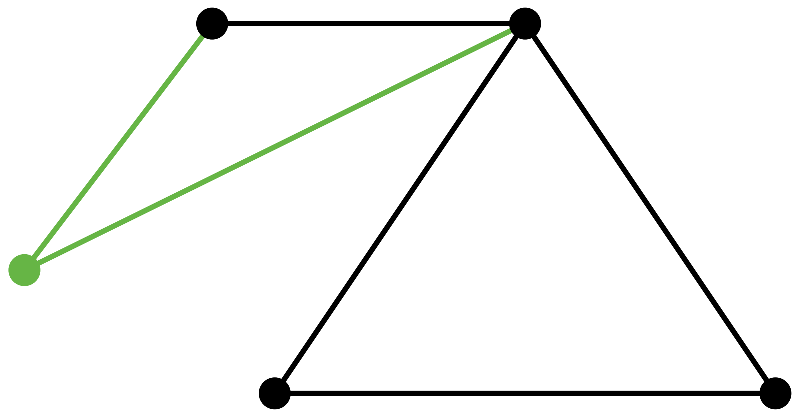

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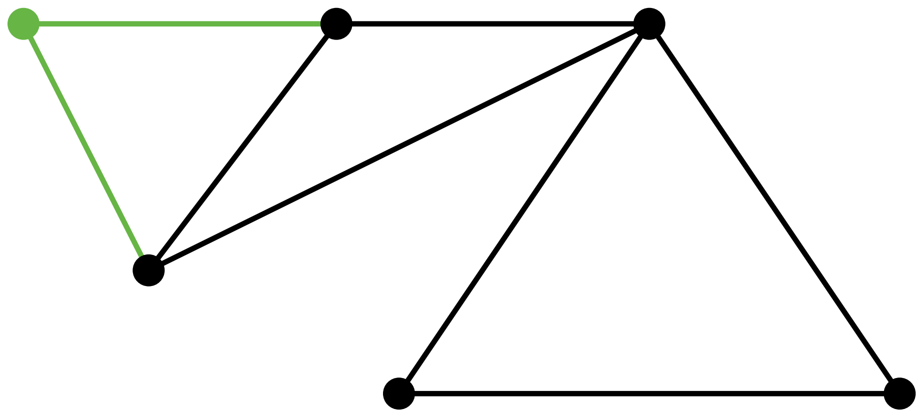

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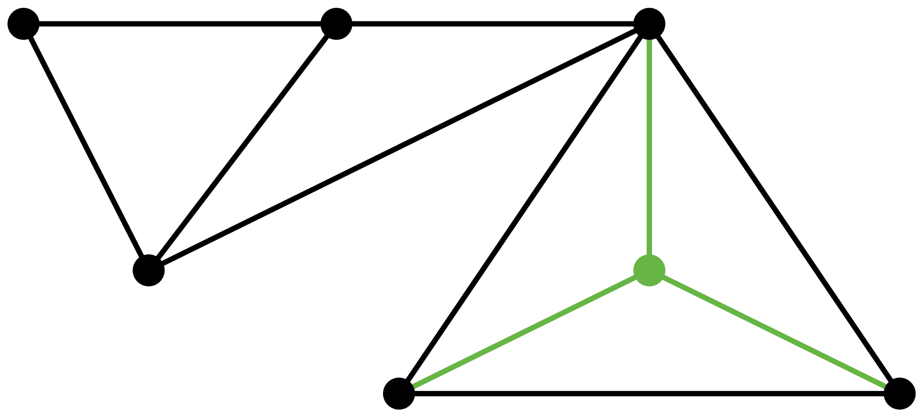

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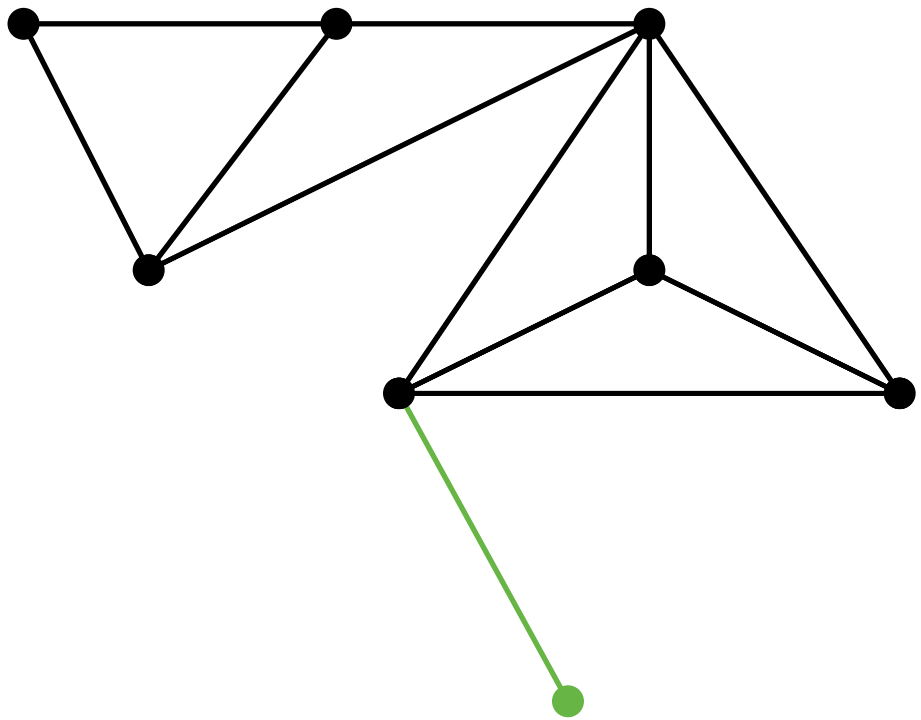

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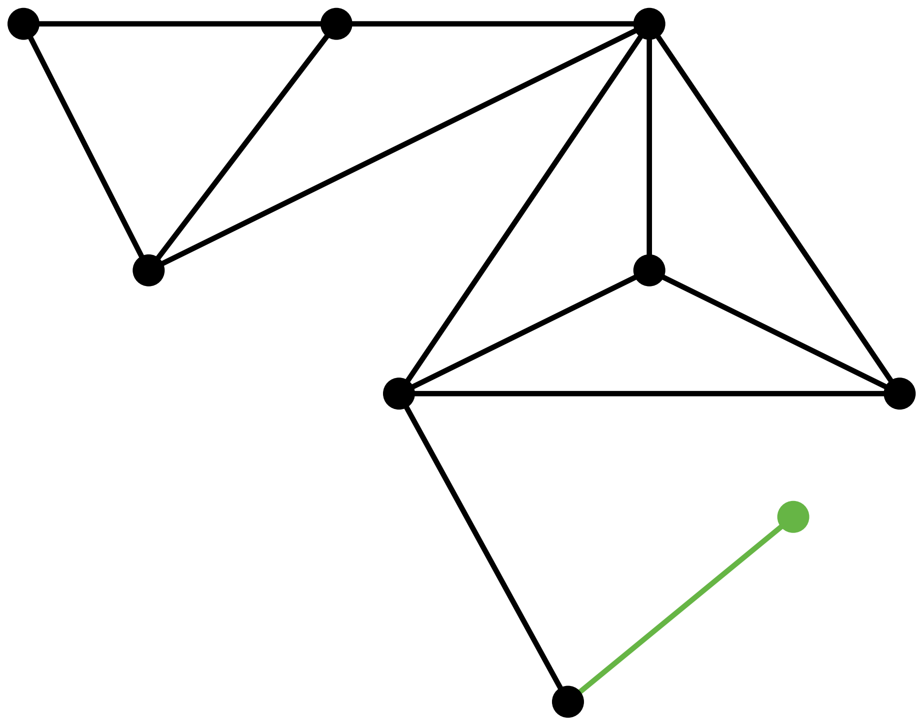

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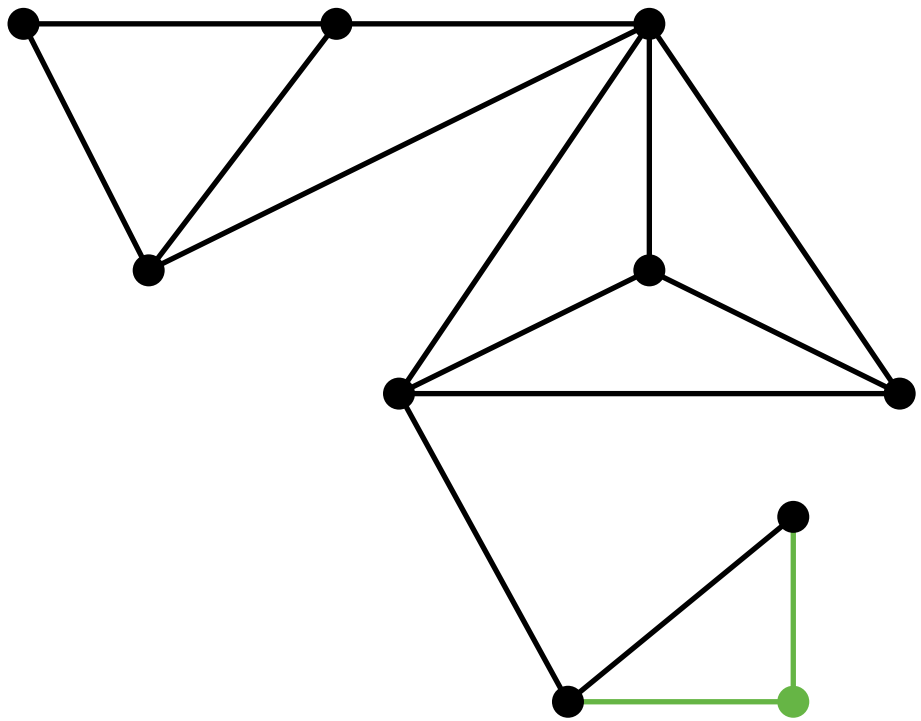

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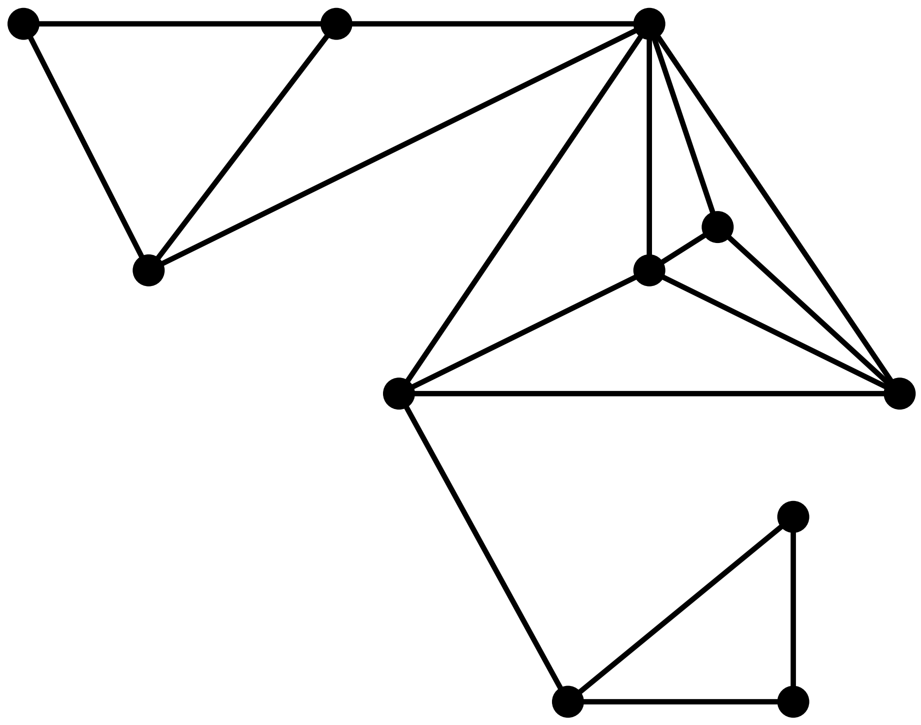

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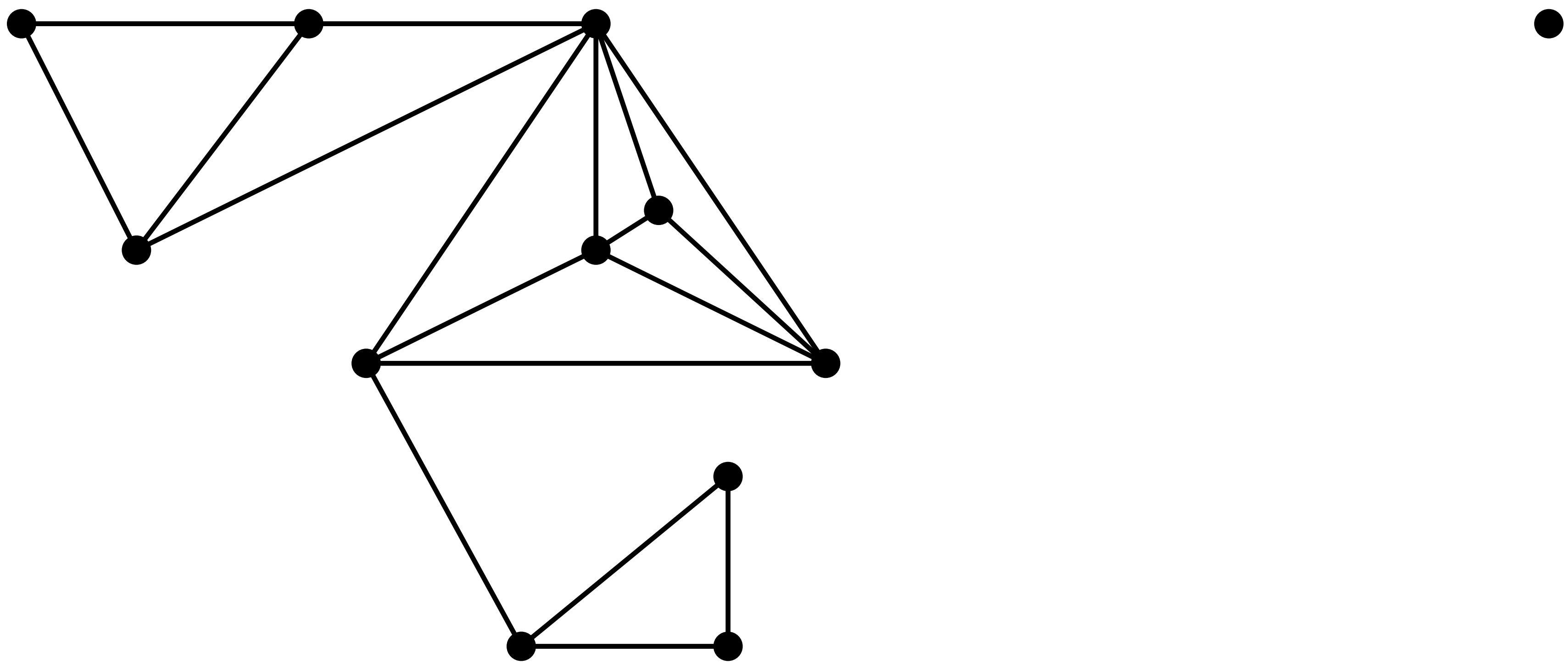

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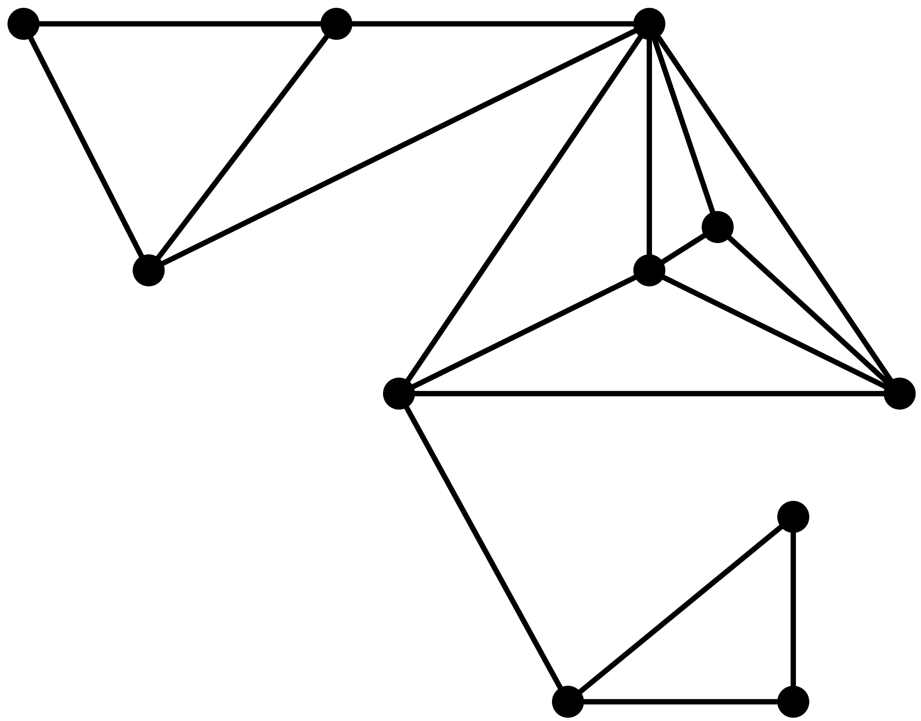

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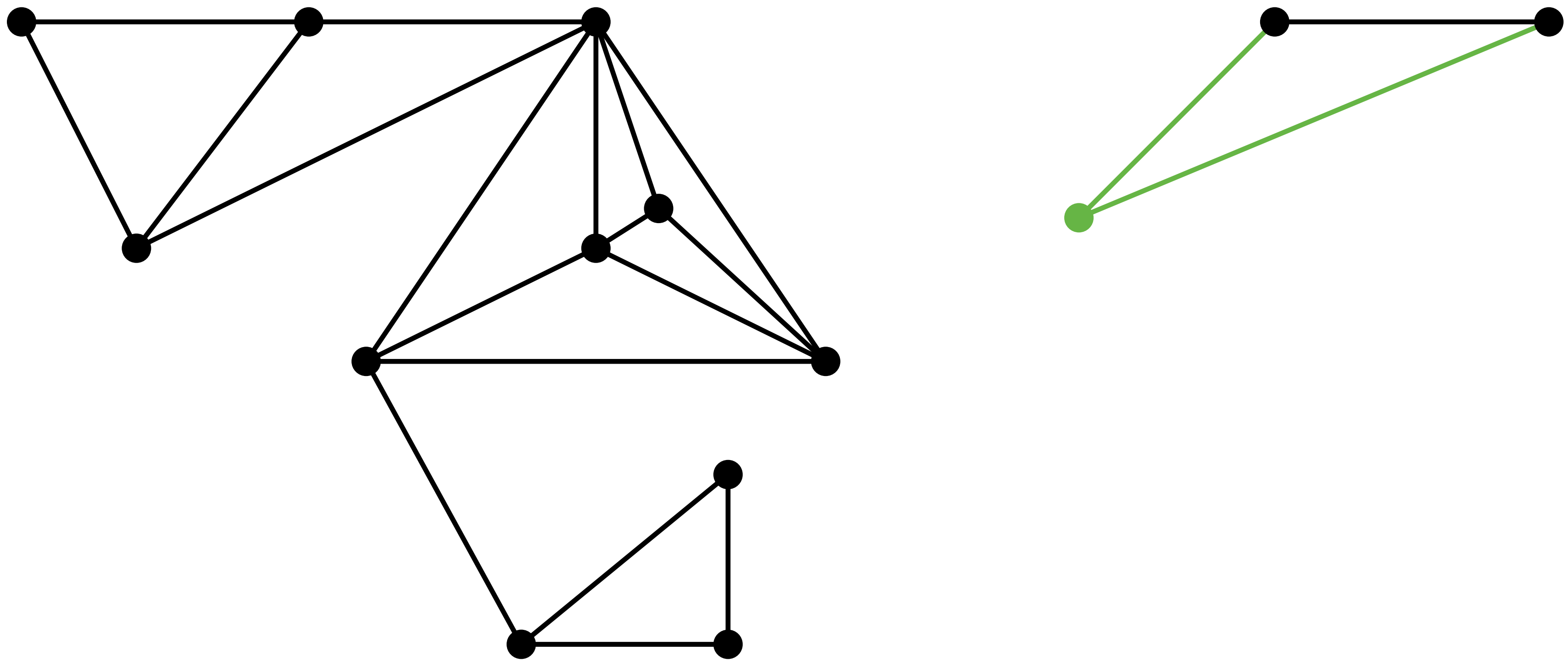

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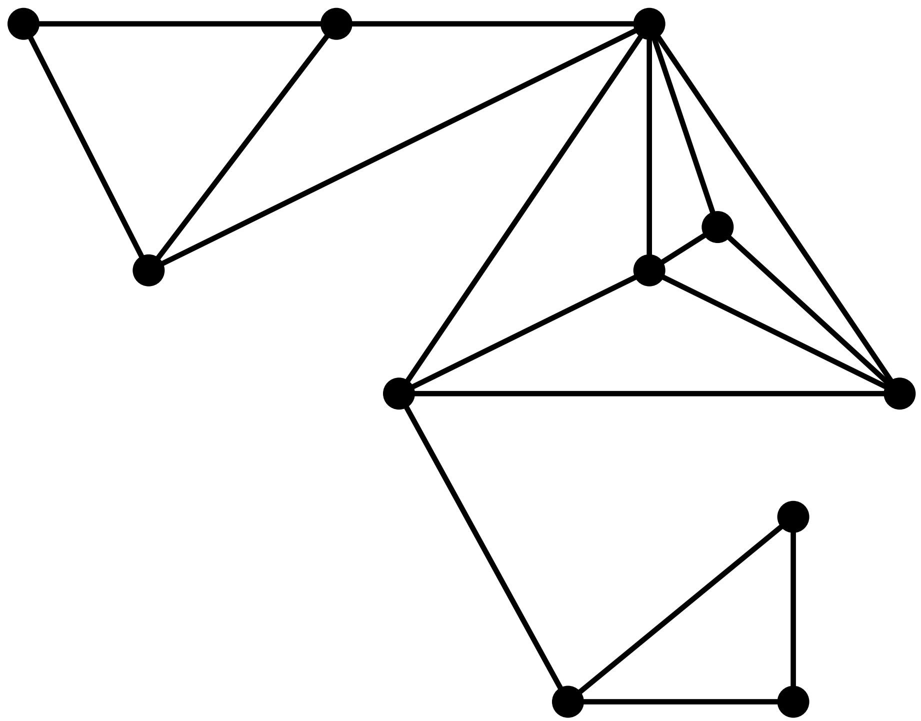

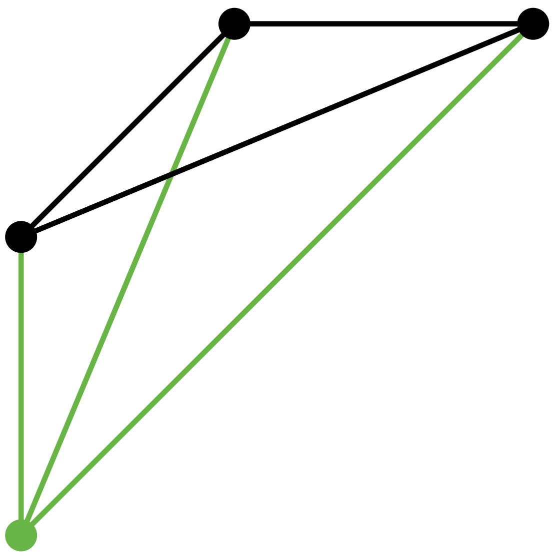

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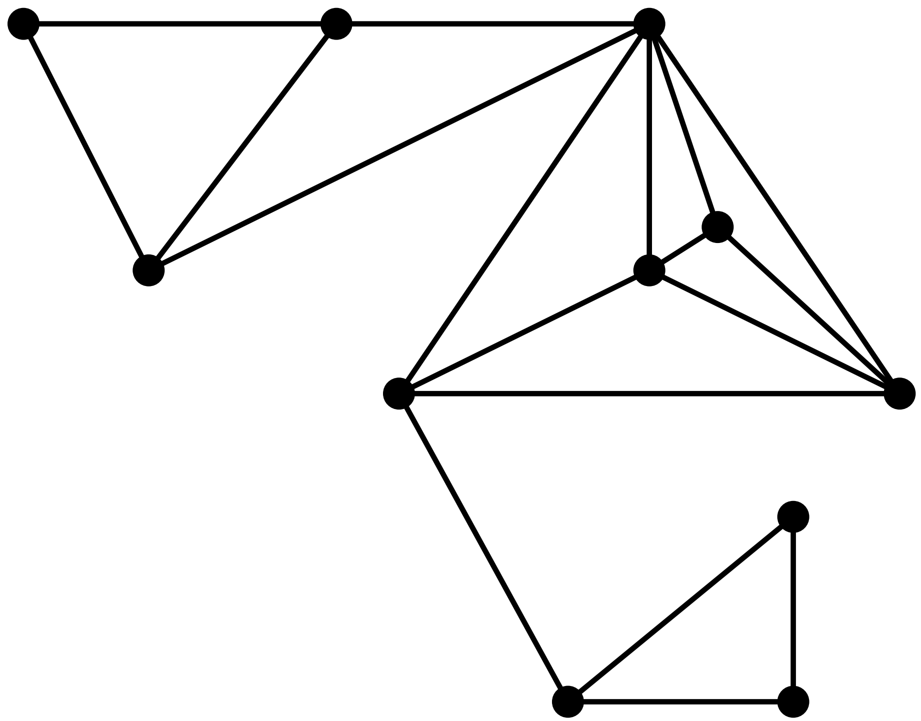

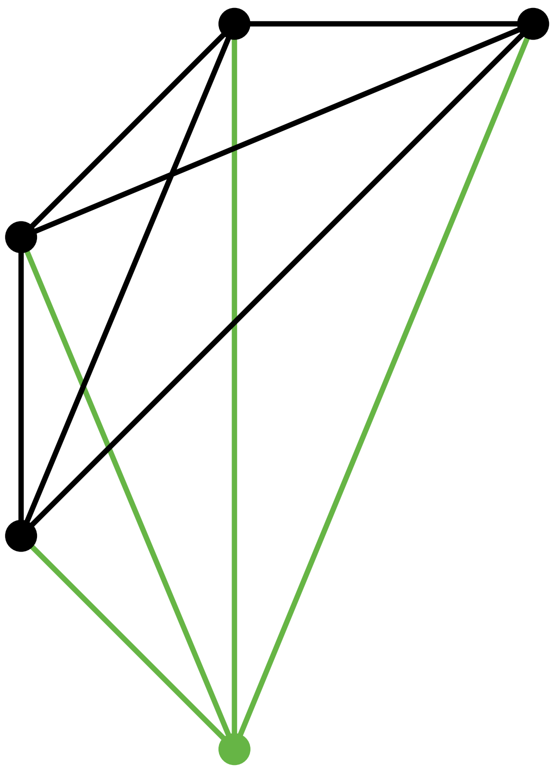

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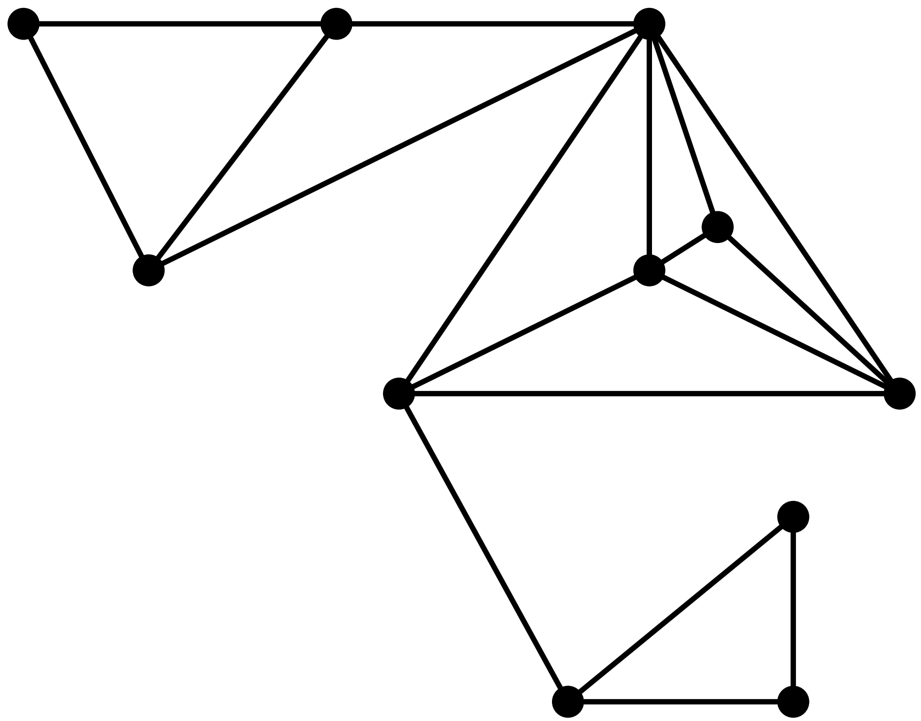

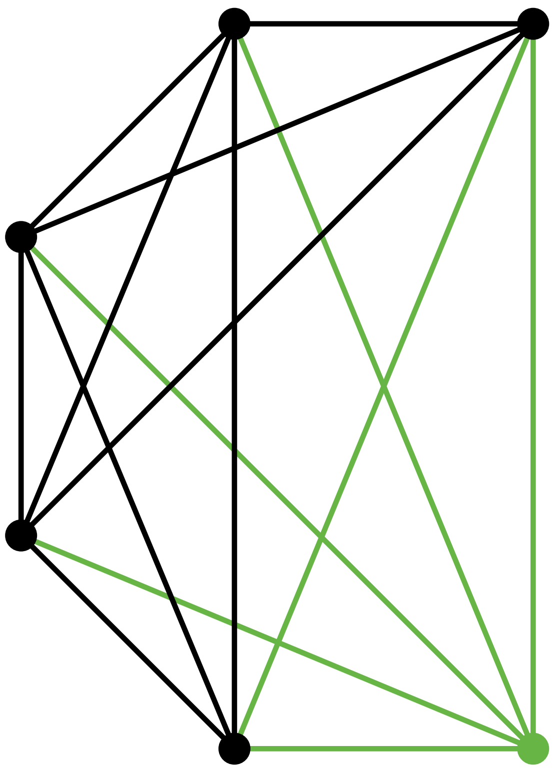

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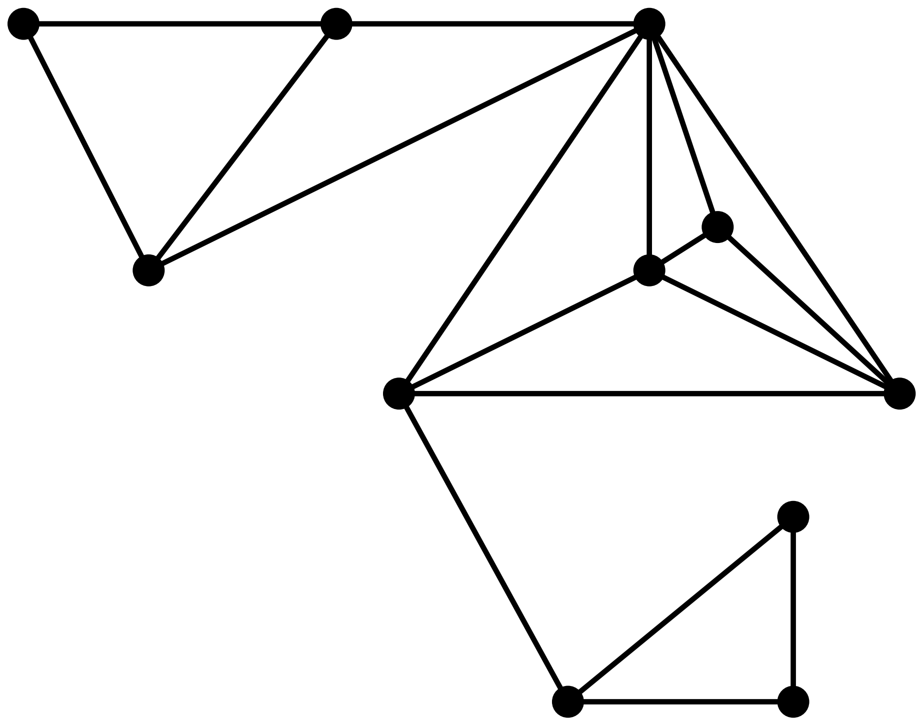

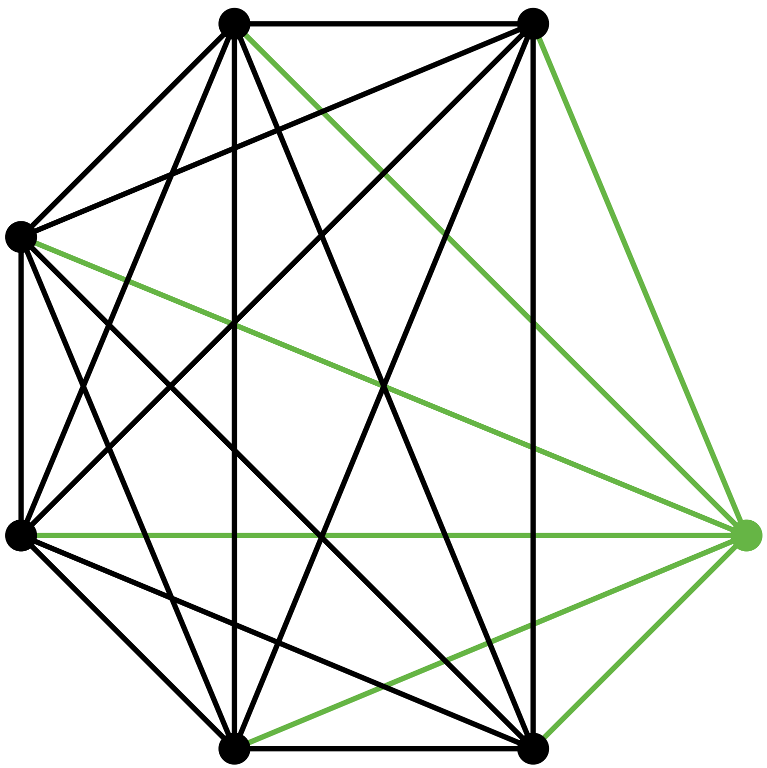

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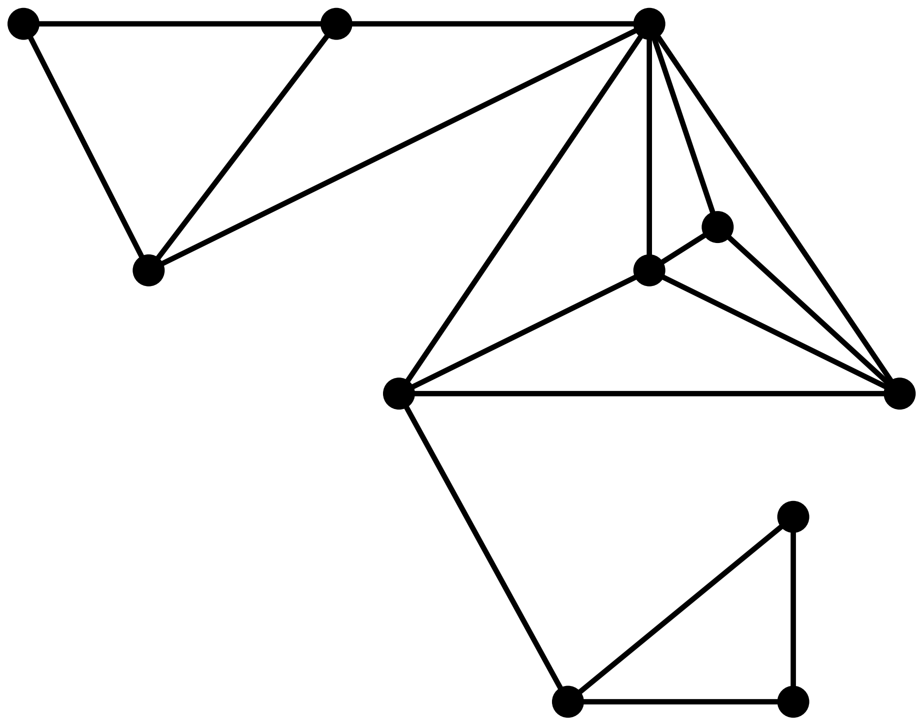

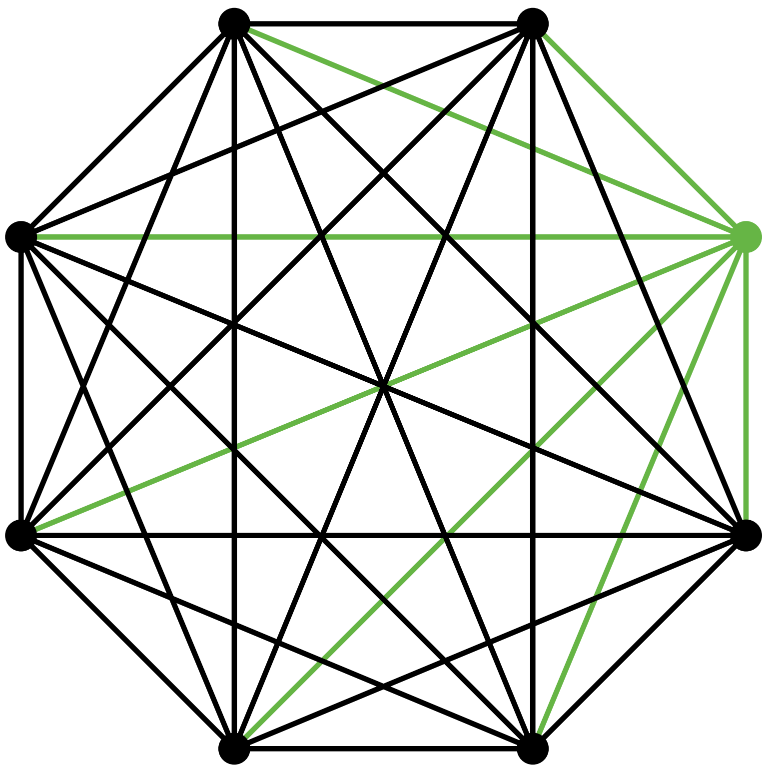

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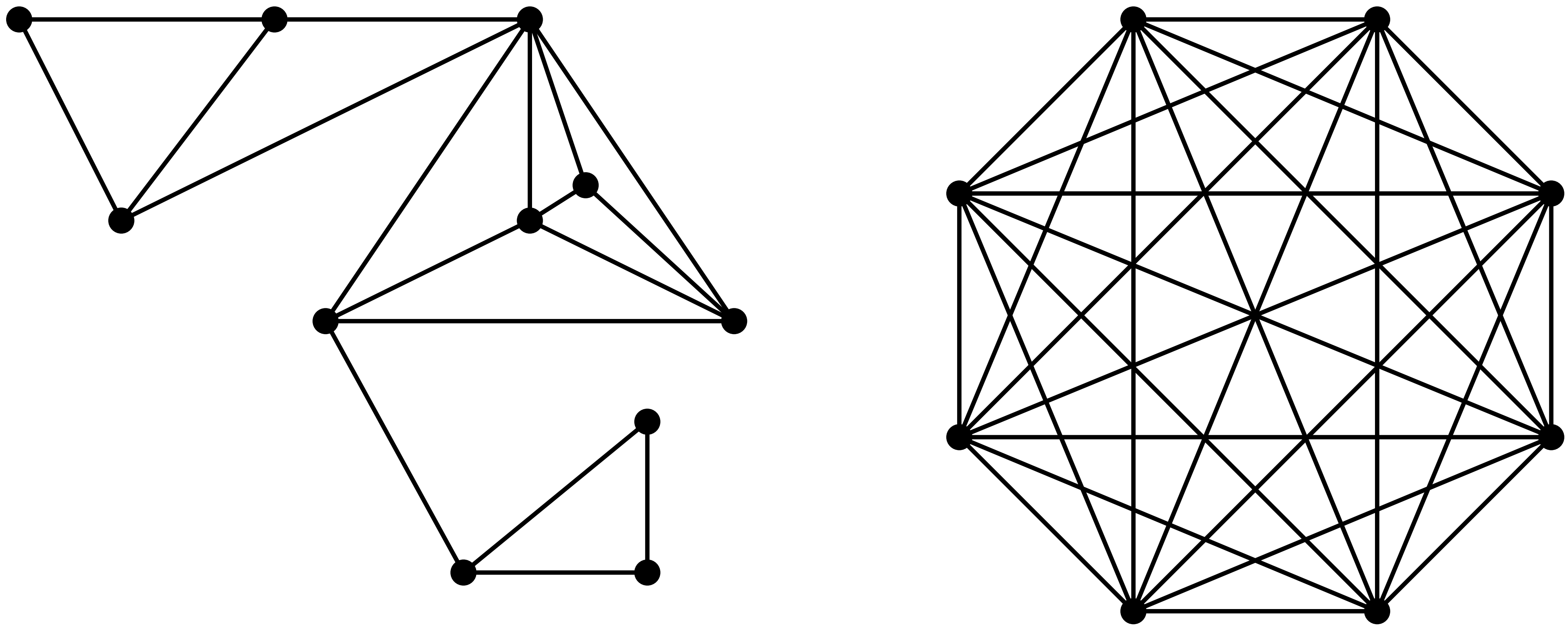

Chordal graphs

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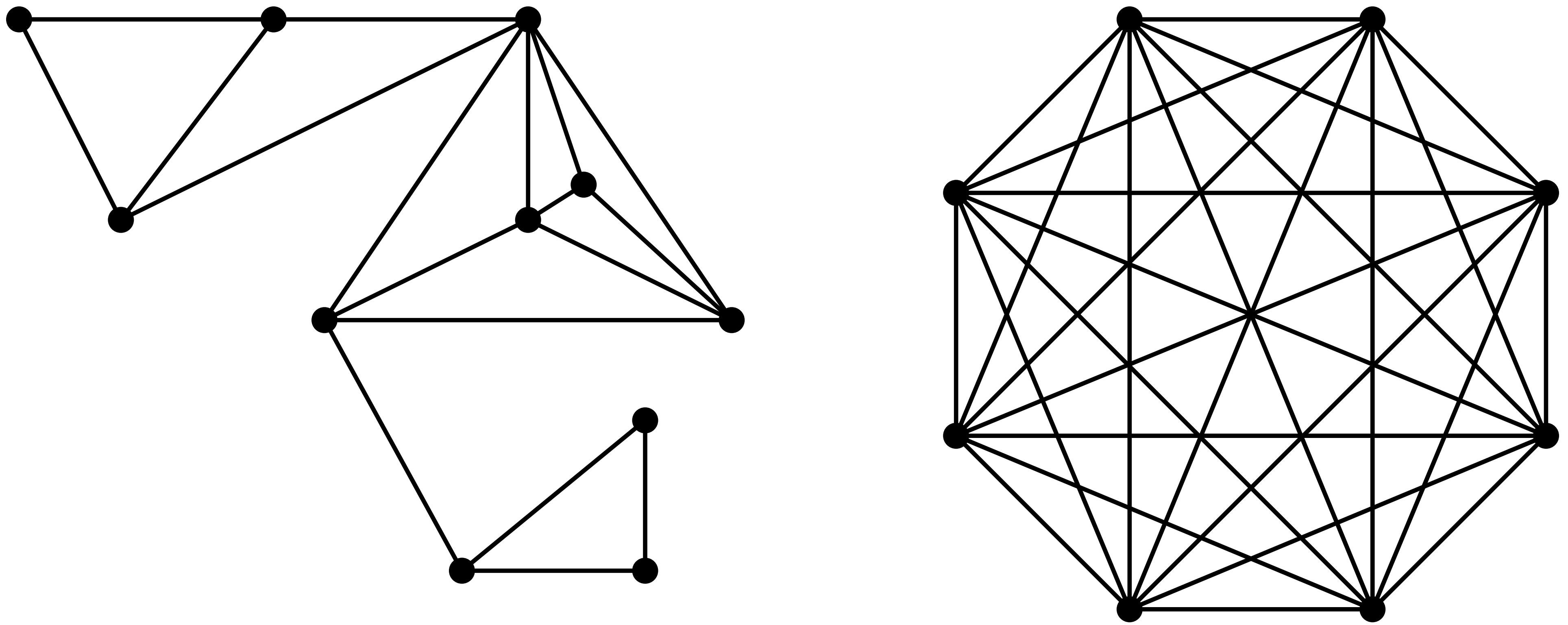

Chordal graphs

Definition. A graph is **chordal** if it has no induced cycle of lengh greater than 3.

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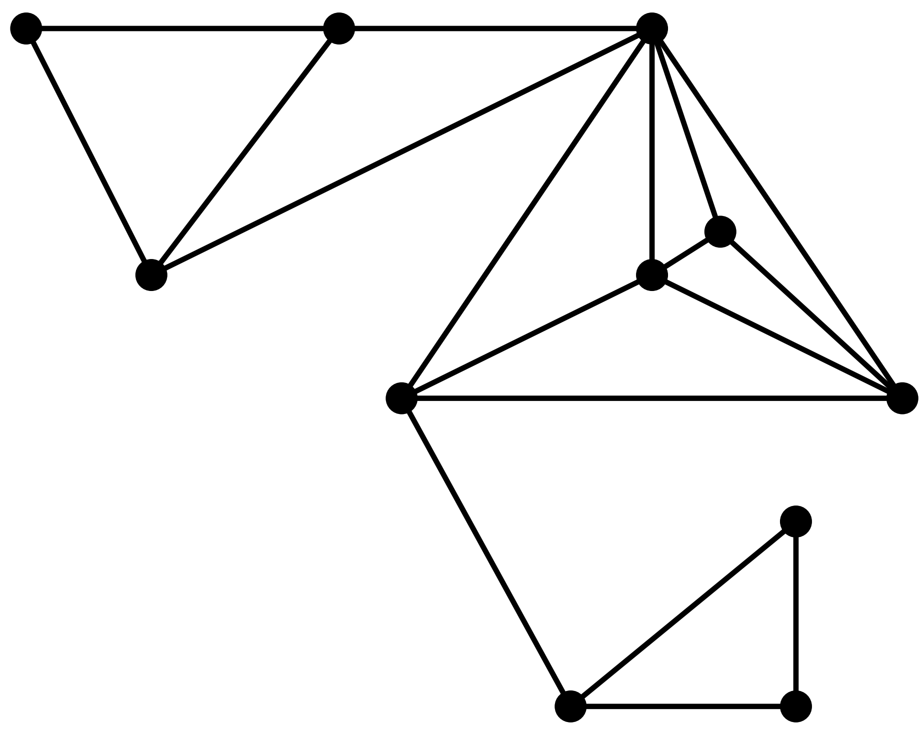

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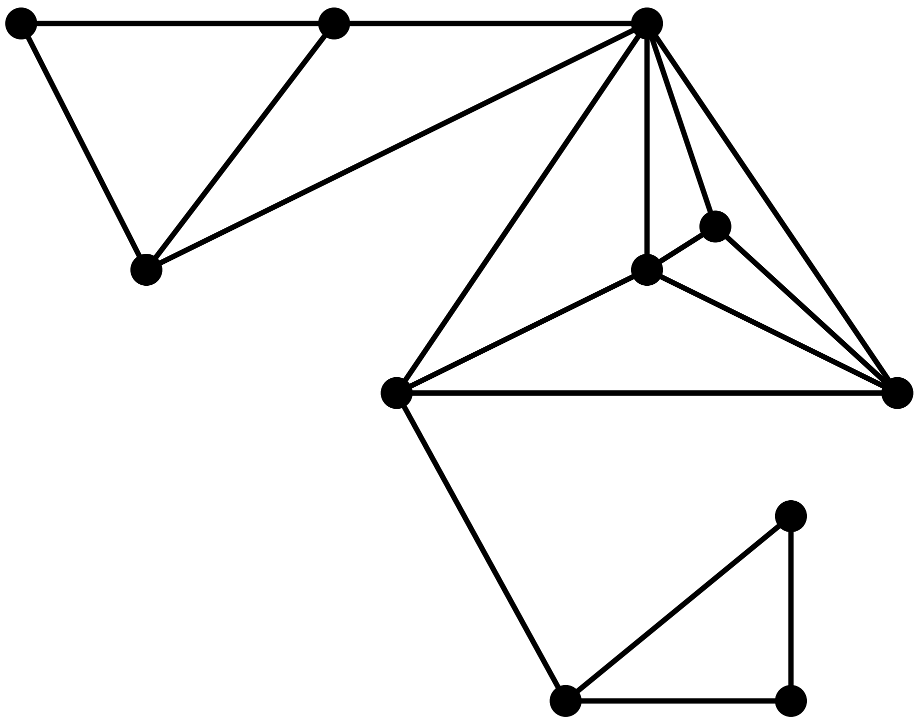

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Chordal graphs with tree-width at most t

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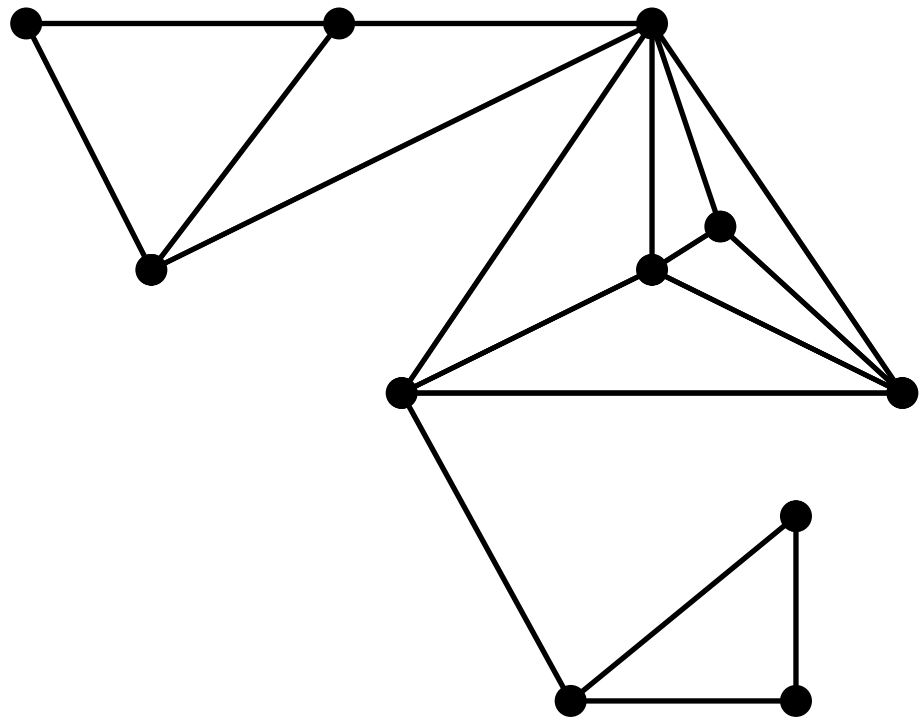

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Chordal graphs with tree-width at most t

Definition. The **tree-width** of a graph G is the minimum k such that G is the subgraph of a k-tree.

The symbolic method

Our goal is to determine the number of graphs in the family with size n.

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Definition. A **combinatorial class** is a pair $(\mathcal{A}, |\cdot|)$ where

- $\mathcal{A}$ is a family of combinatorial objects,
- $|\cdot| : \mathcal{A} \to \mathbb{N}$ is a size function,
- The number of objects with size n is $a_n < \infty$.

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$$A(x) = \sum_{n \geq 0} a_n x^n.$$

Suitable for unlabelled classes.

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Suitable for labelled classes.

Operations between **classes** translate into relations involving their **generating functions**. **The goal** is to obtain (a system of) equations that determine the GF of our class.

Decomposition of graphs into k-connected components

Definition. The **2-connected components** (or blocks) of a connected graph are its maximal 2-connected subgraphs.

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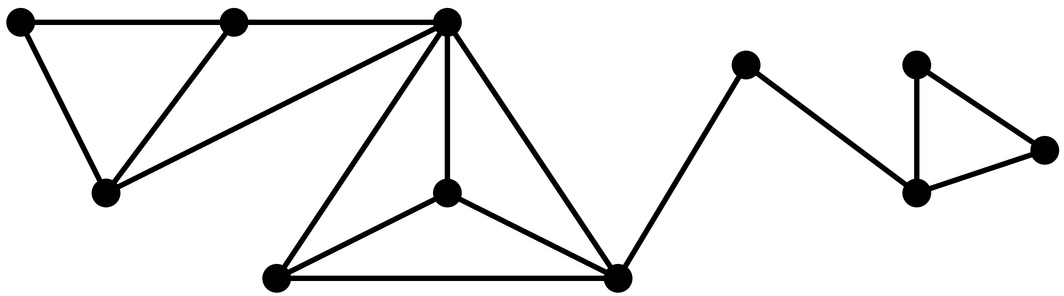

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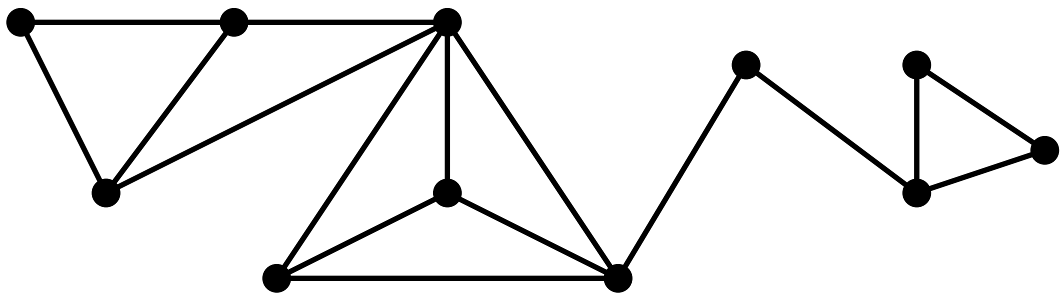

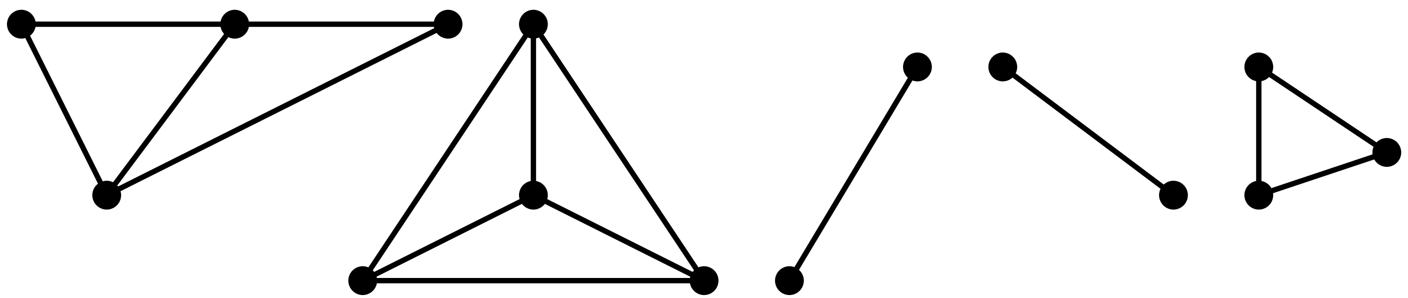

Decomposition of graphs into k-connected components

Let $\mathcal{B} \subset \mathcal{C}$ be the class of the 2-connected members of $\mathcal{G}$. Then,

$$C^\bullet(x) = x\exp(B'(C^\bullet(x))), \quad \text{where } C^\bullet(x) = xC'(x),$$

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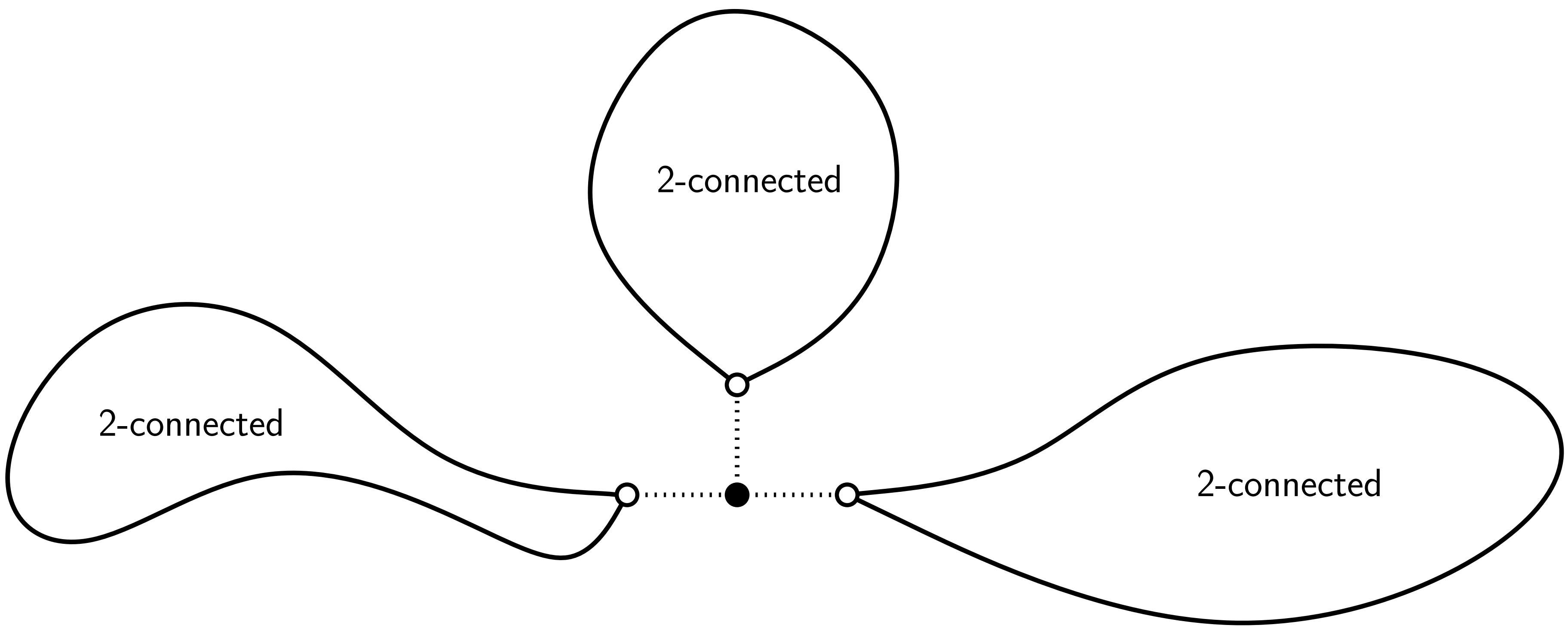

Decomposition of graphs into k-connected components

Let $\mathcal{B} \subset \mathcal{C}$ be the class of the 2-connected members of $\mathcal{G}$. Then,

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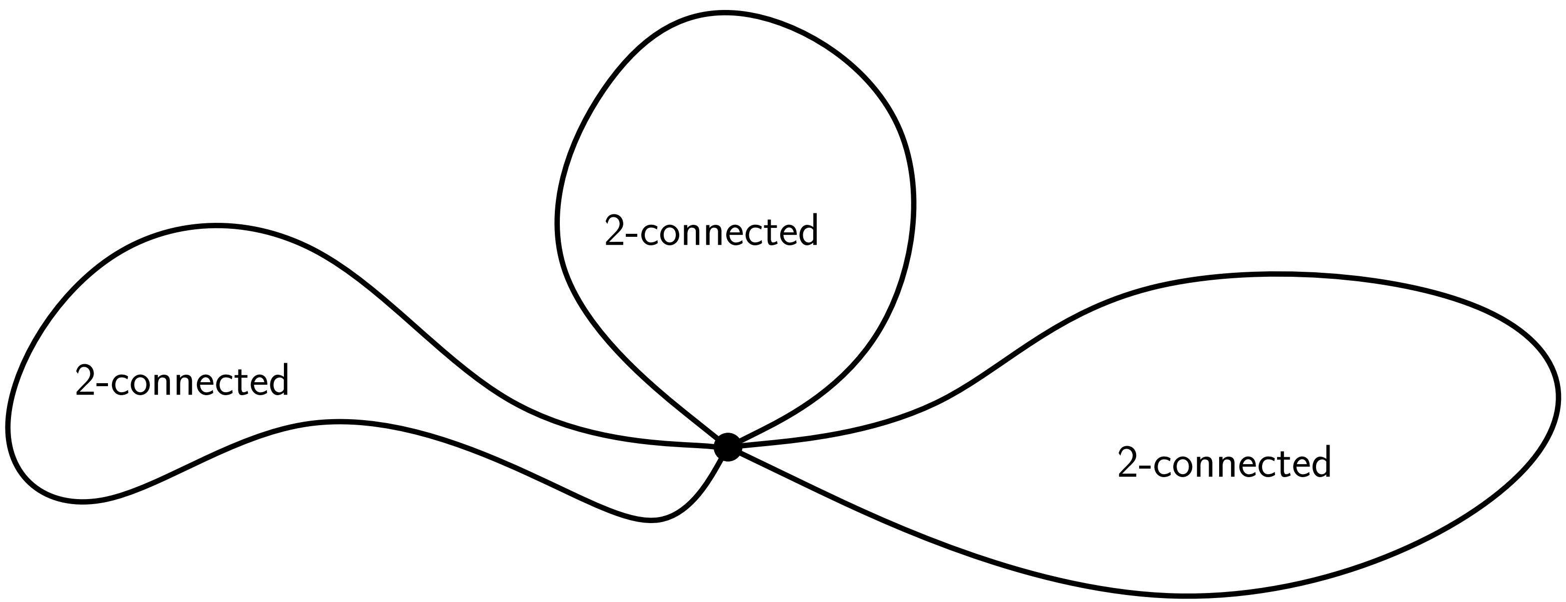

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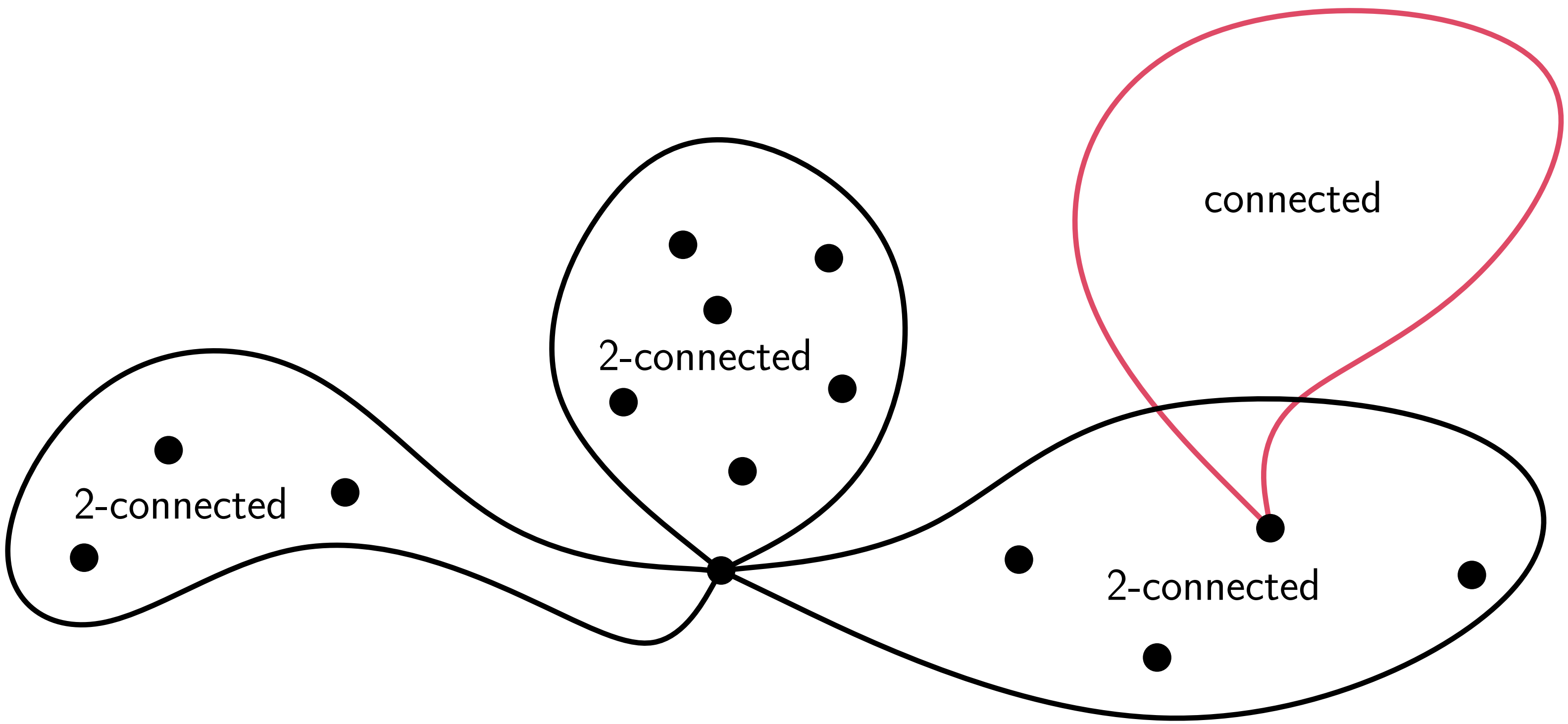

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Unfortunately, the 4-connected components of 3-connected graphs cannot be defined in general (in such a way that the decomposition is unambiguous and well-defined).

However, any k-connected **chordal** graph admits a decomposition into $(k+1)$-connected components! [Wormald, 1985]

Decomposition of chordal graphs into k-connected components

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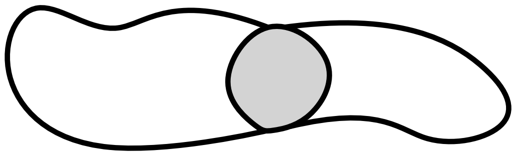

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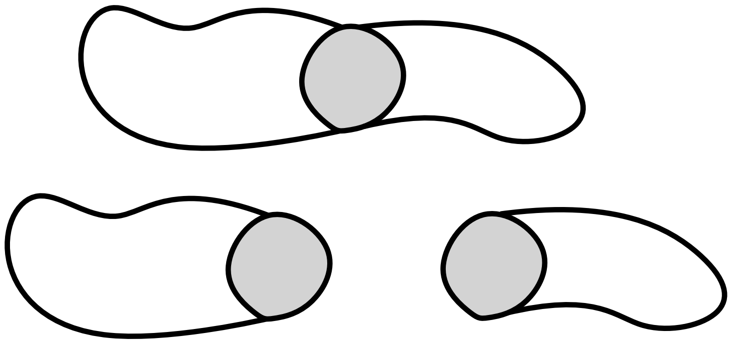

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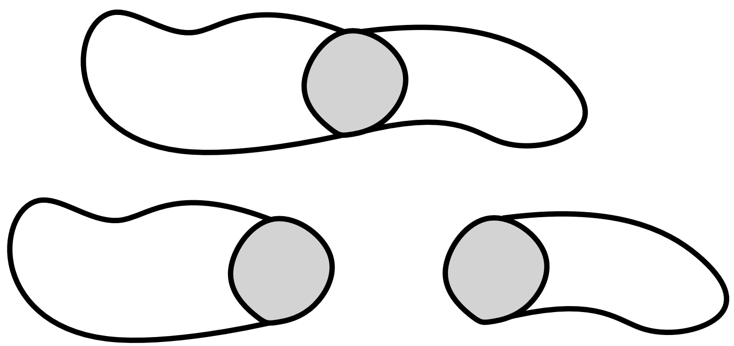

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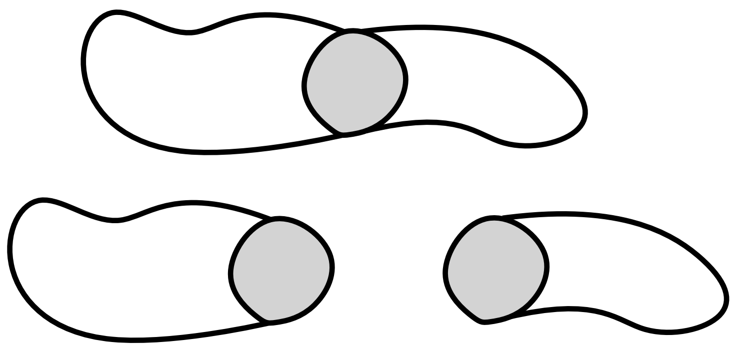

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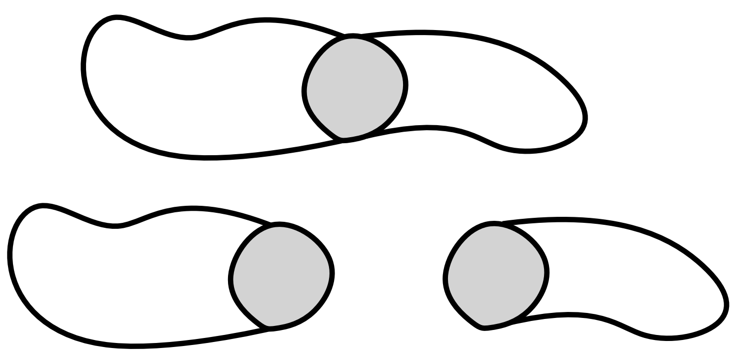

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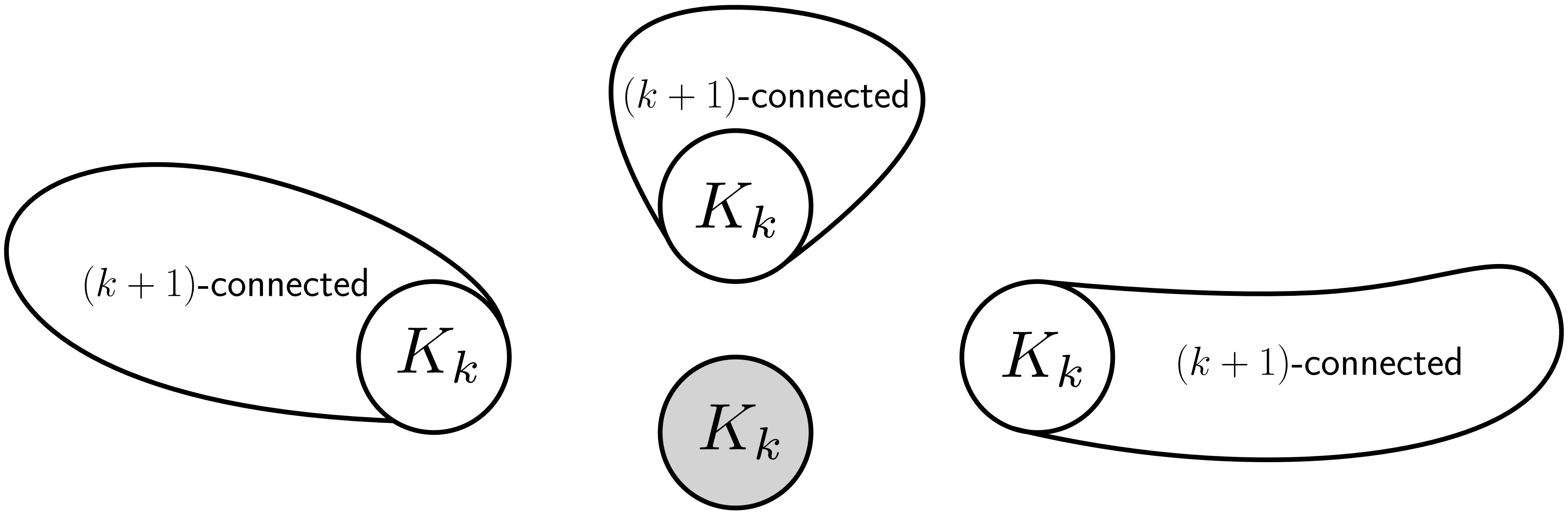
$\rightarrow$ Note that the $(k+1)$-connected components are the **maximal** $(k+1)$-**connected subgraphs**.

Decomposition of chordal graphs into k-connected components



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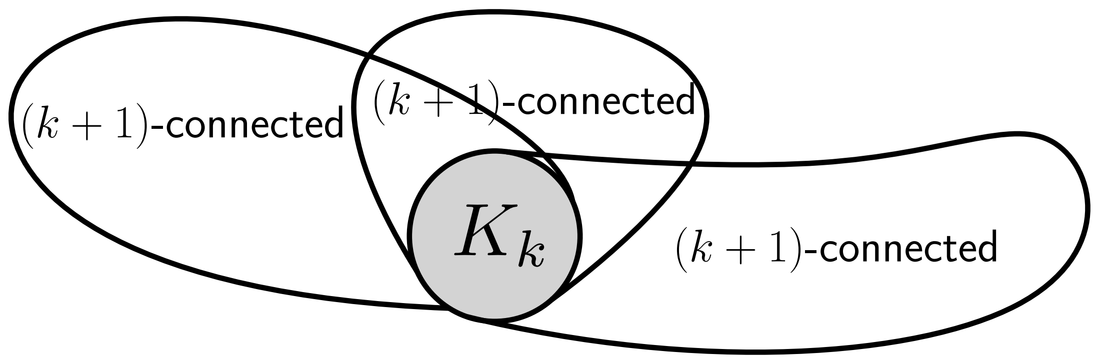

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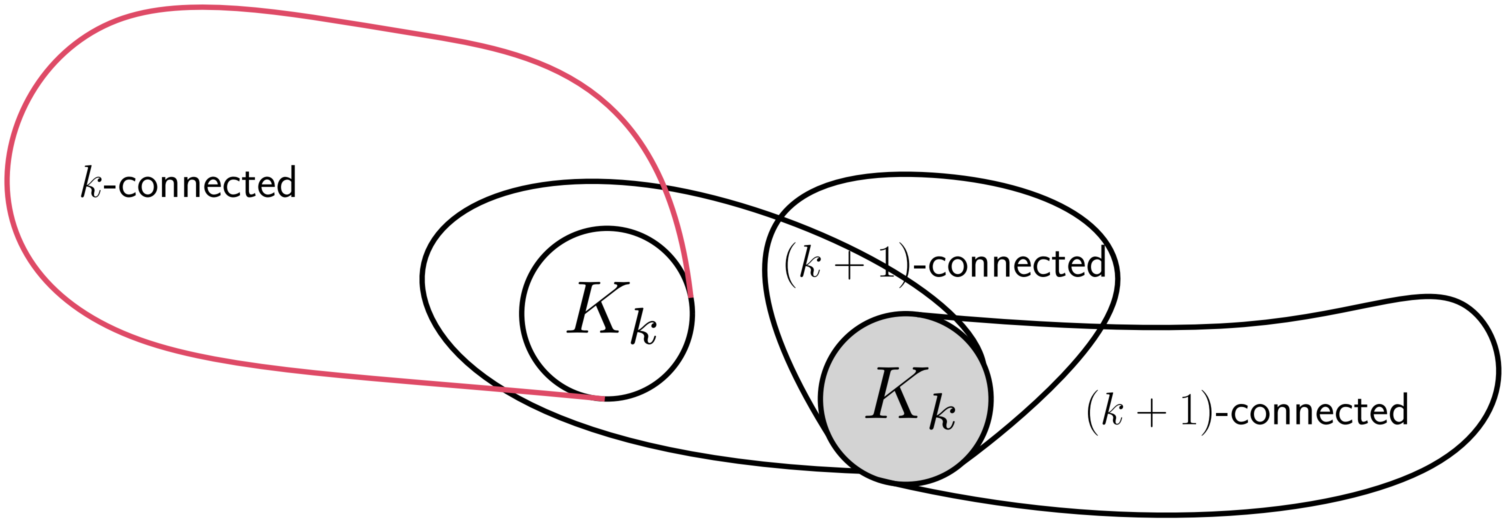

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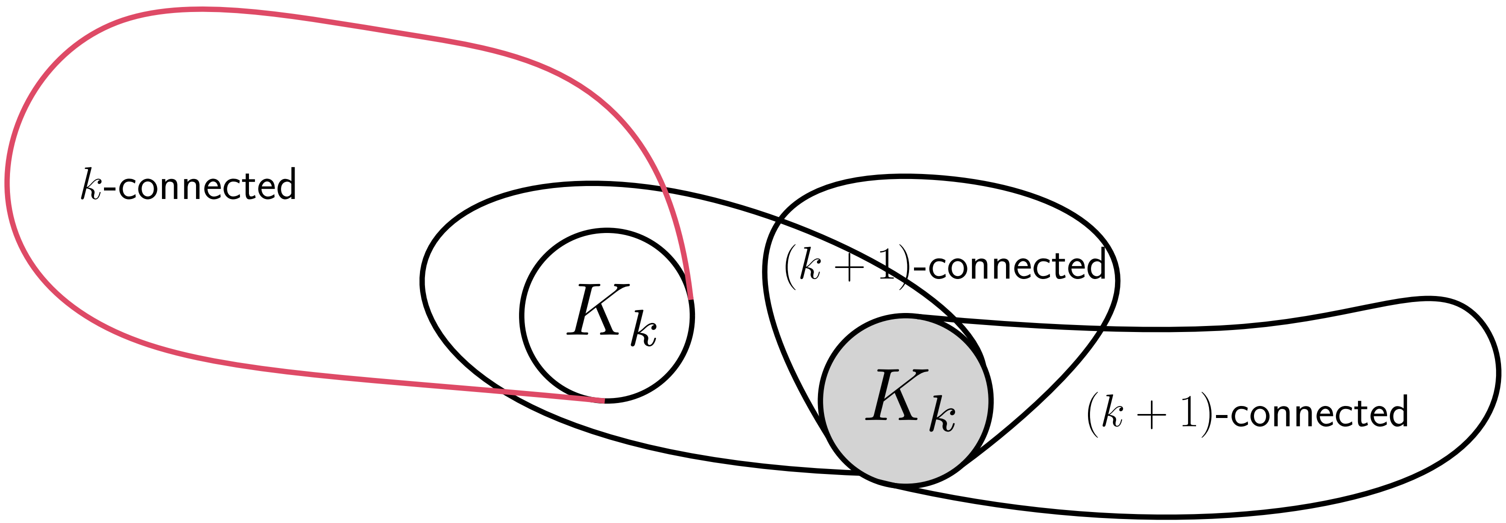

Let $\mathcal{G}_k^{(i)}$ be the class of k-connected chordal graphs rooted at an unlabelled, ordered i-clique.
Consider its multivariate exponential generating function $G_k^{(j)}(x, x_k)$, where the variable x_k marks the number of k-cliques. Then, we have that

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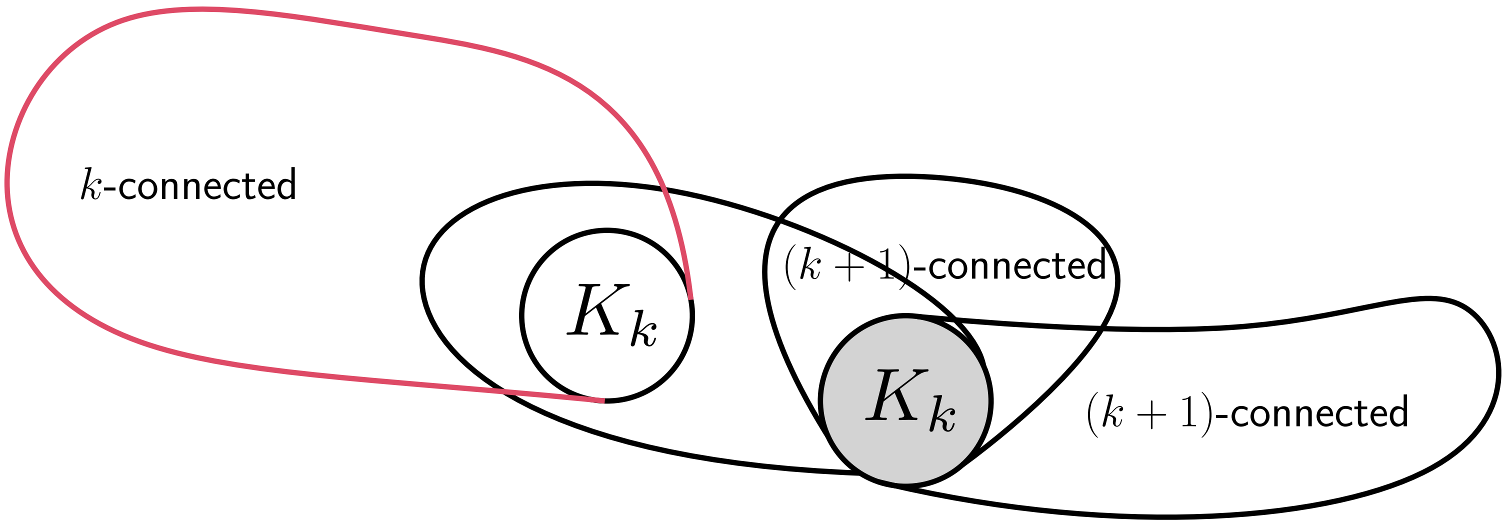

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This generalizes the classical decomposition of connected graphs into 2-connected components.

Labelled vs unlabelled

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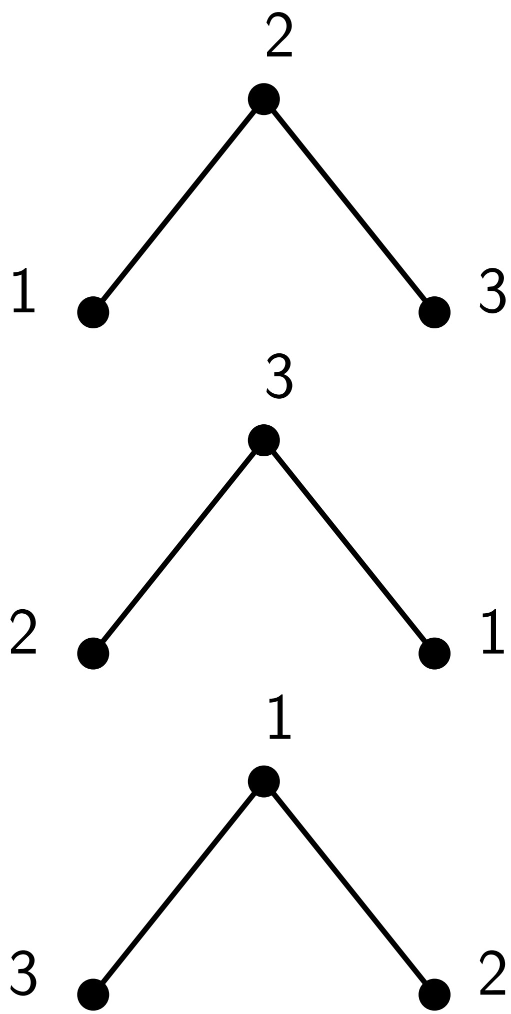

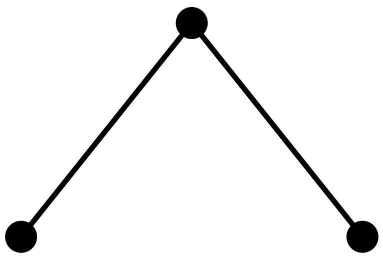

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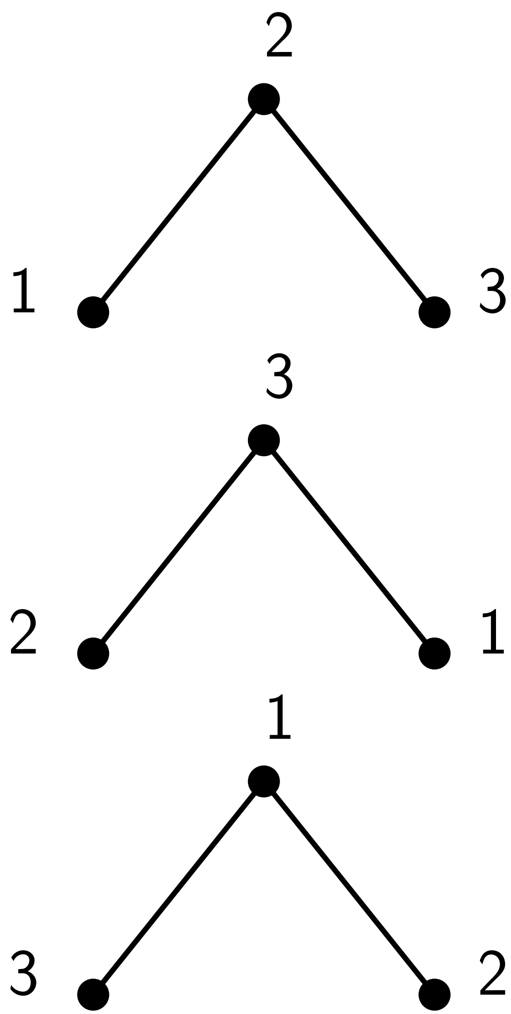
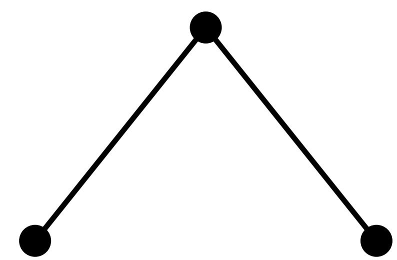
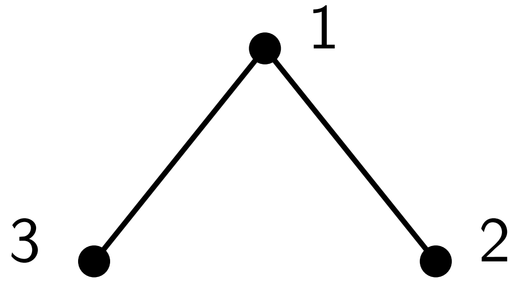




Unlabelled graphs are usually harder to count than labelled graphs.

- Labelled trees (Cayley trees) [Borchardt, 1860]
- Unlabelled trees (free trees) [Otter, 1948]

Counting unlabelled graphs - Pólya theory



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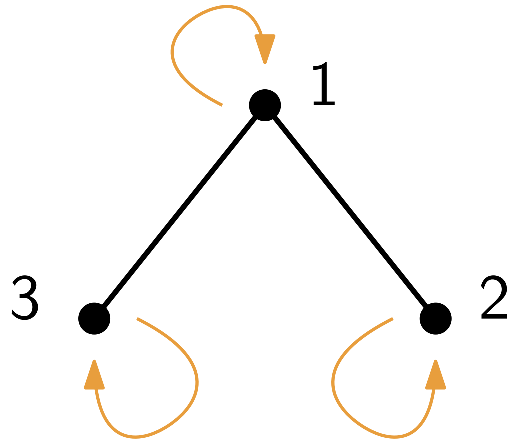

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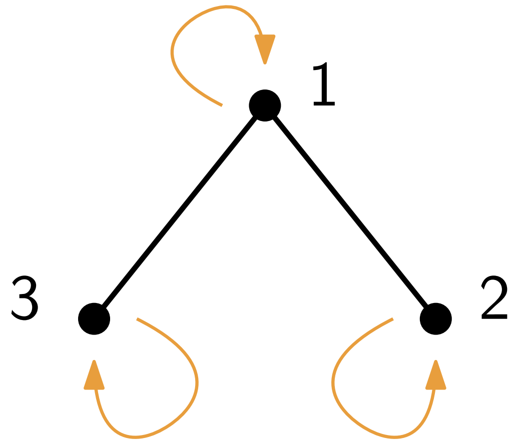

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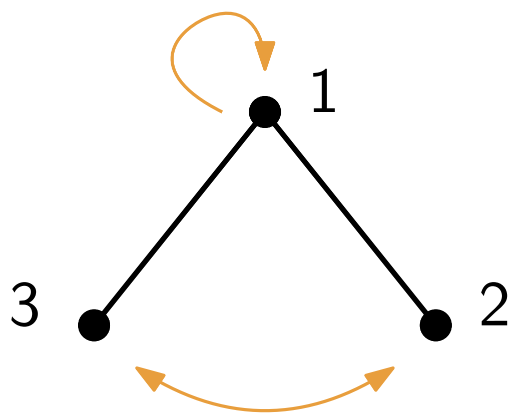

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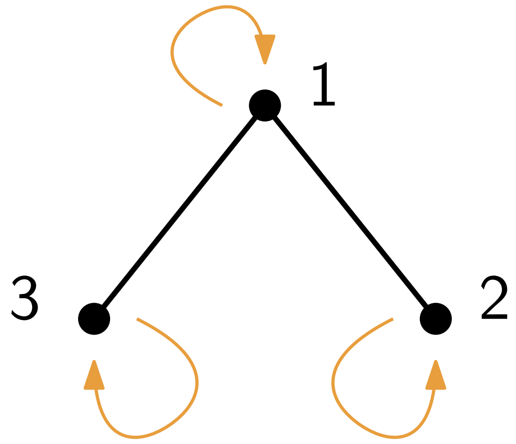

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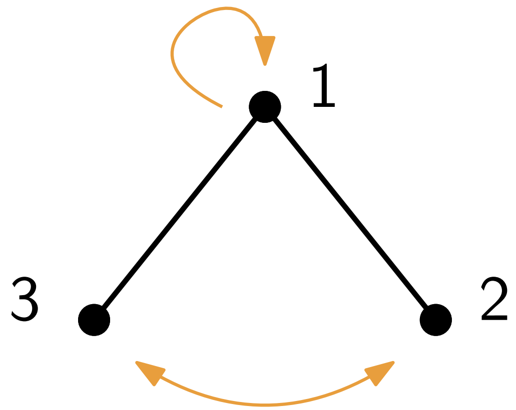

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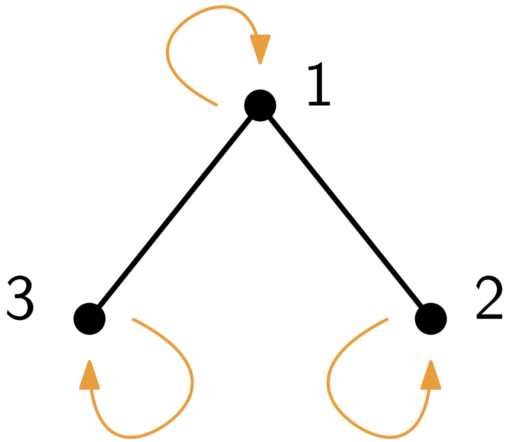

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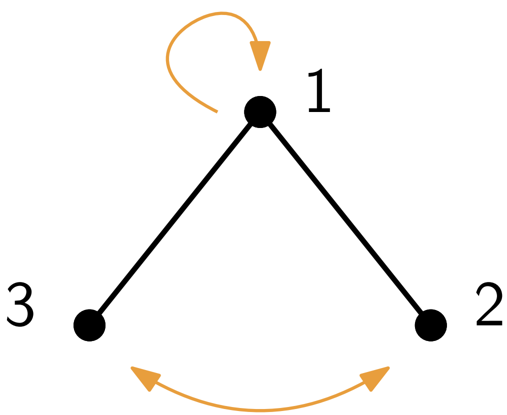

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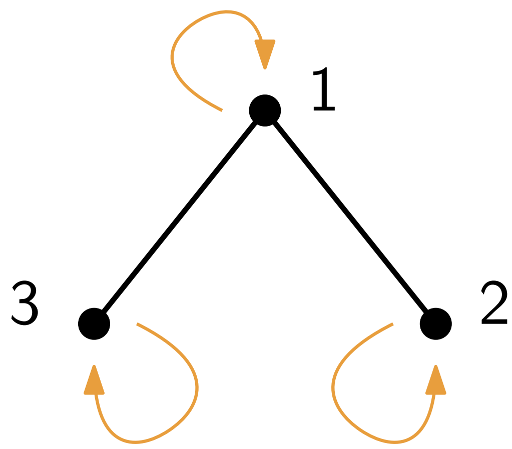

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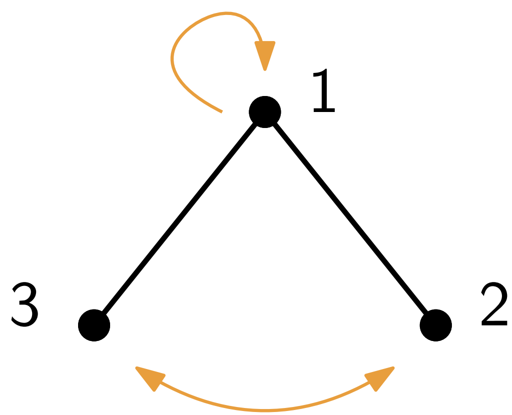

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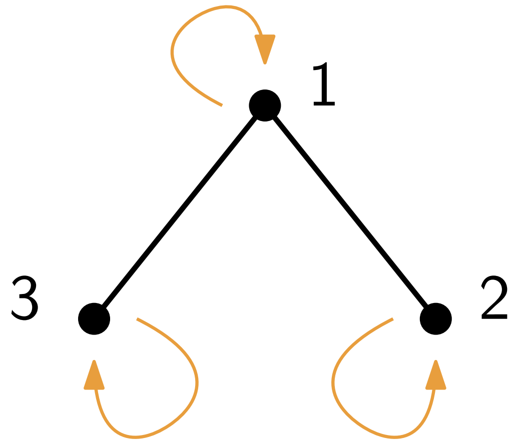

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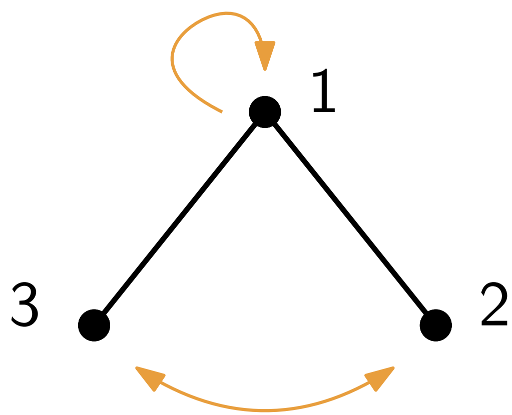

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The OGF of the unlabelled class $\tilde{\mathcal{G}}$ is given by

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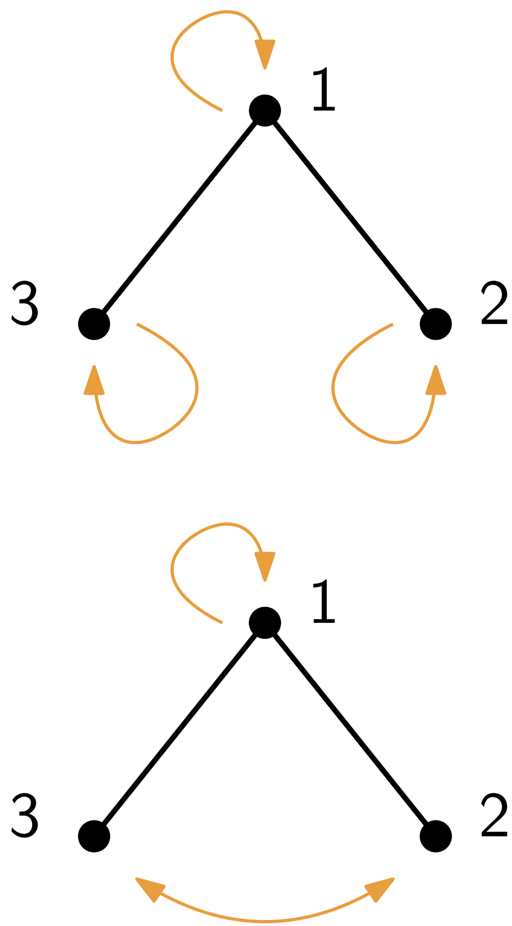

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In our case,

$$G(x) = \frac{3}{3!}(x^3 + x \cdot x^2) = x^3$$

Unlabelled trees

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Theorem. [Pólya, 1937]
The OGF $P(x)$ of Pólya trees is given by

$$P(x) = x \exp\left(P(x) + \frac{P(x^2)}{2} + \frac{P(x^3)}{3} + \dots\right).$$

As $n \to \infty$ we have

$$[x^n]P(x) \sim \frac{b\sqrt{\rho}}{2\sqrt{\pi}} \cdot n^{-3/2} \cdot \rho^{-n},$$

with $b \approx 2.681127$ and $\rho \approx 0.338219$.

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What about unrooted unlabelled trees?

Unlabelled trees

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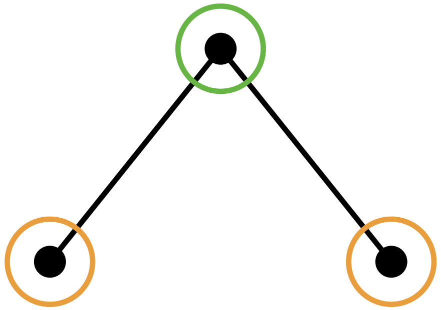
Not every unlabelled graph of size n gives rise to n rooted graphs.



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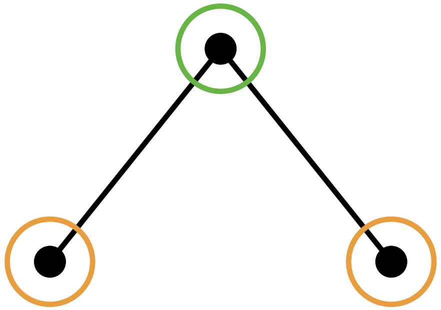

Theorem. [Otter, 1948]
The OGF $U(x)$ of unlabelled trees is given by

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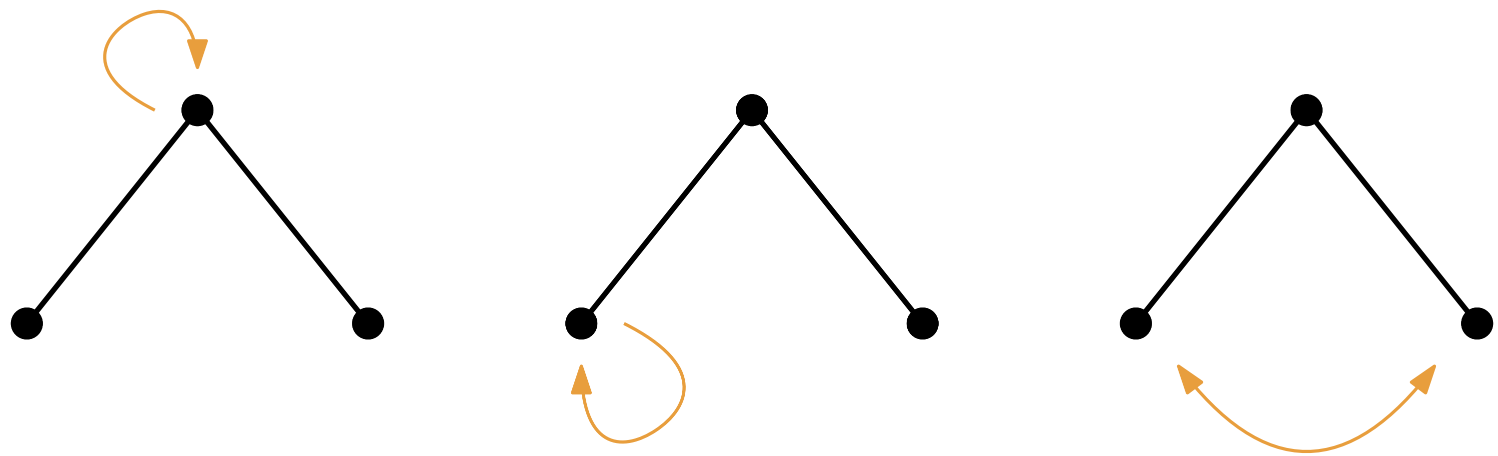
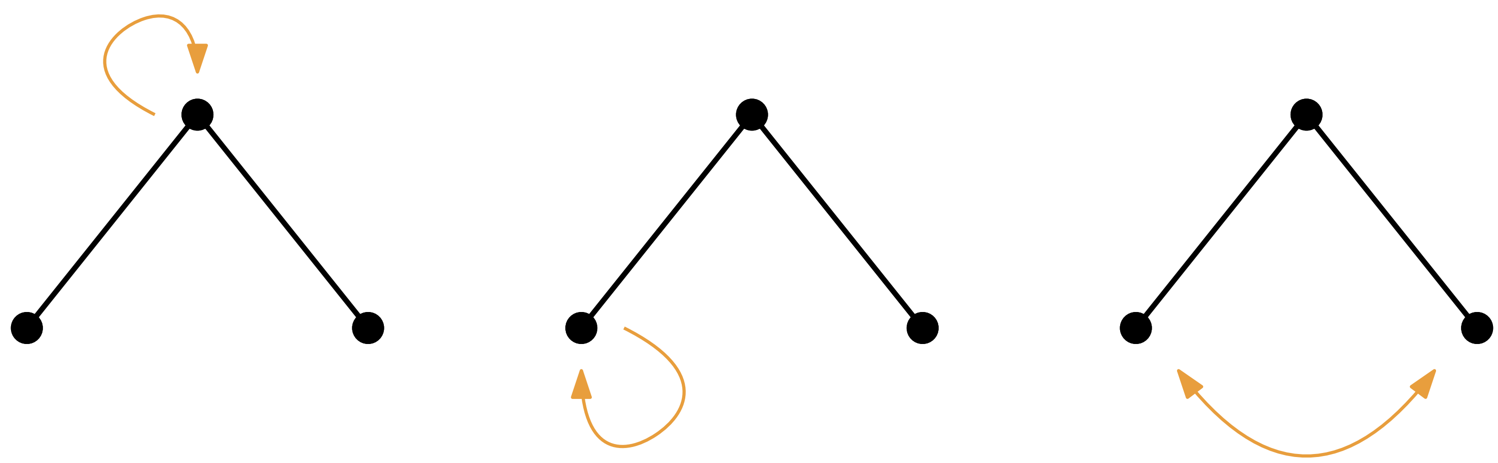
Proof. Using the **dissymmetry theorem**.

Cycle-pointing



Definition. A **cycle-pointed graph** is a pair (G, c) where $G \in \mathcal{G}$ is an unlabelled graph and c is a cycle of some automorphism of G.

Cycle-pointing



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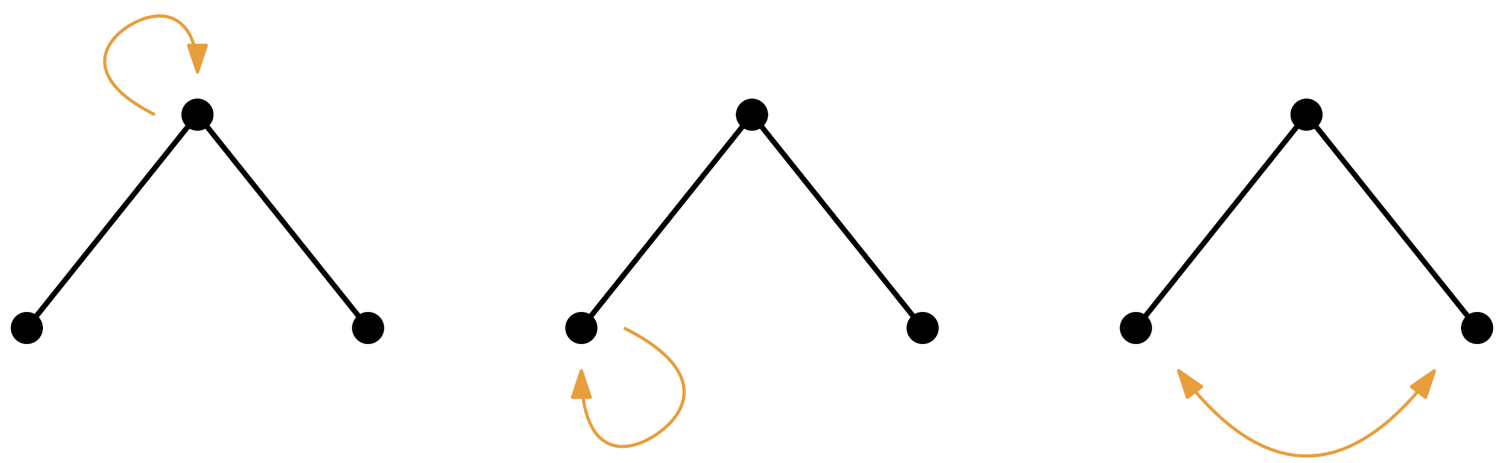

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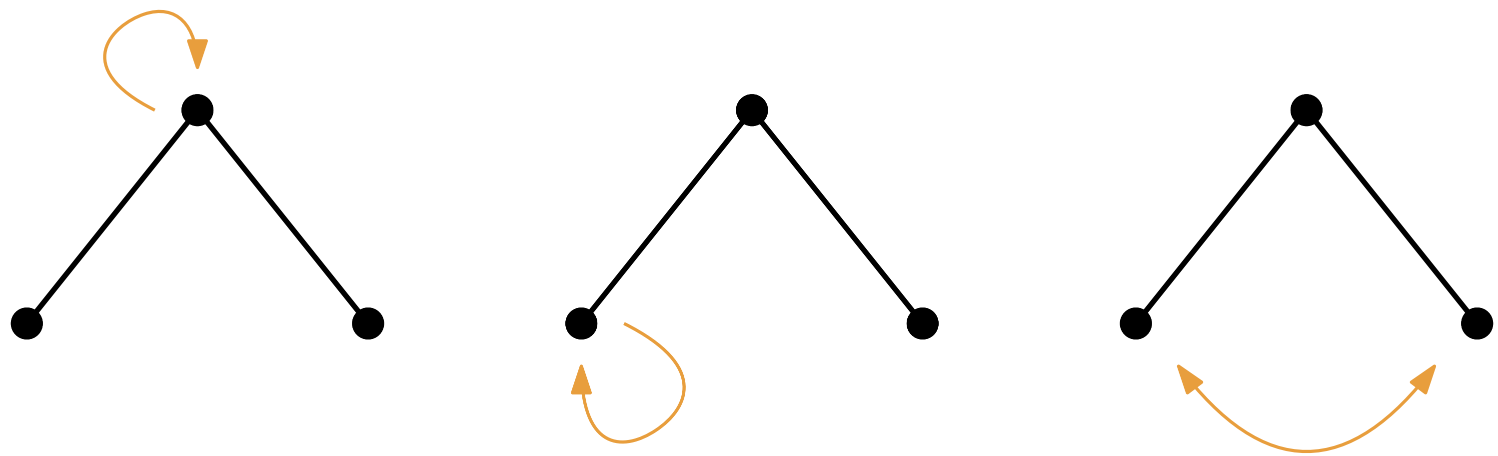

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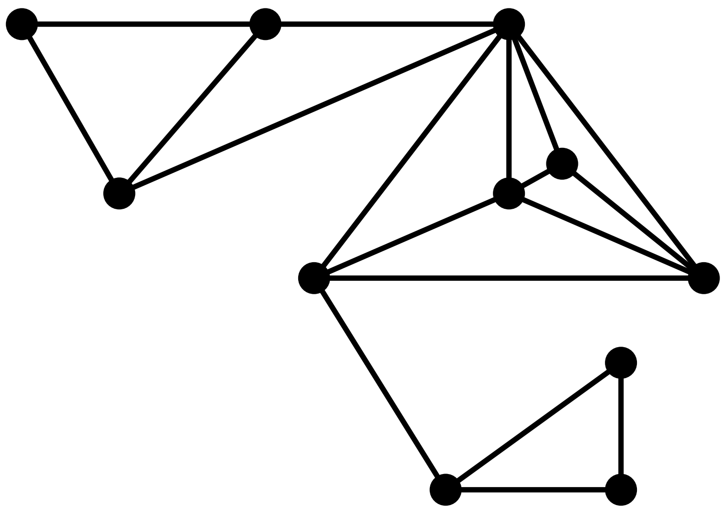
An unbiased rooting (pointing) operator!

They extend Pólya theory to cycle-pointed graphs. In particular, they manage to unroot Pólya trees via cycle-pointing and they recover Otter's formula.

Our class of graphs



Chordal graphs with tree-width at most t

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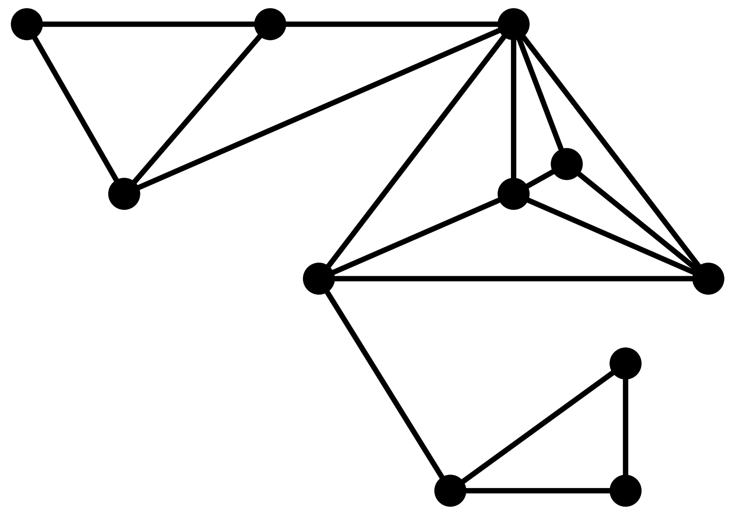

Chordal graphs with tree-width at most t

[C., Drmota, Noy & Requilé, 2023]: assymptotic enumeration of the labelled class.

$$|\mathcal{G}_{t,n}| \sim c_t \cdot n^{-5/2} \cdot \gamma_t^n \cdot n! \qquad \text{as } n \to \infty,$$

for some $c_t > 0$ and $\gamma_t > 1$

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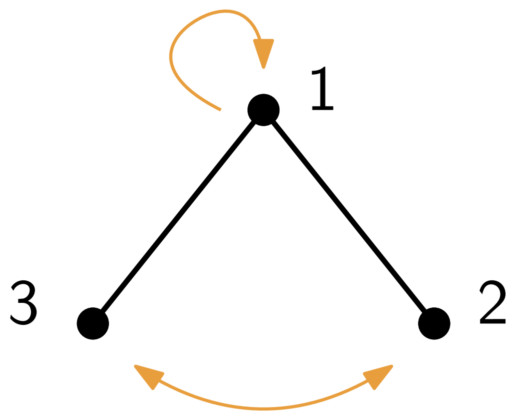

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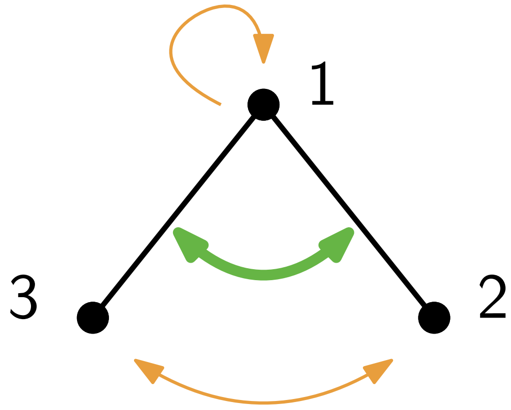

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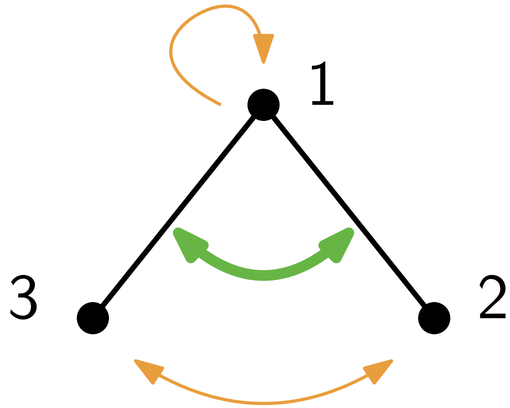

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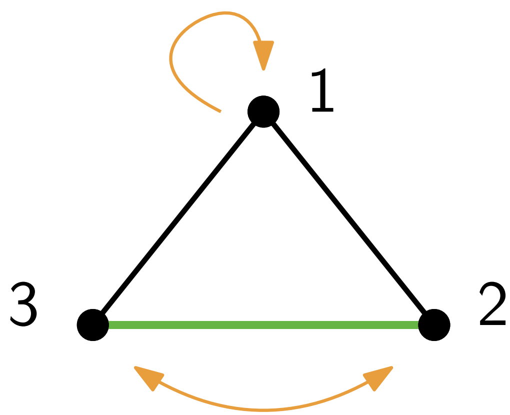

The new edge is in a cycle of length 1 but different type: it flips itself.

An extension of Pólya theory

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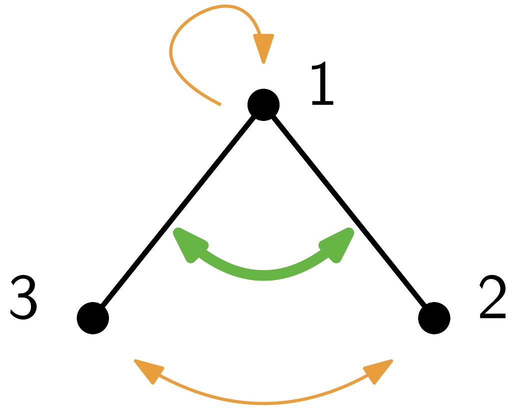
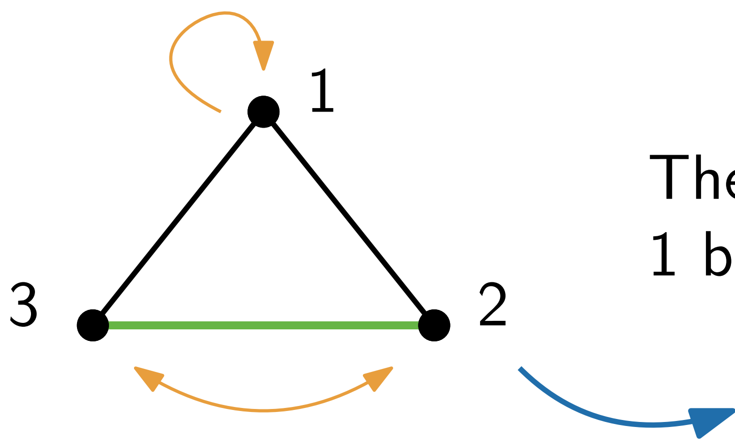


The **edges** are in a cycle of length 2.



The new edge is in a cycle of length 1 but different type: it flips itself.

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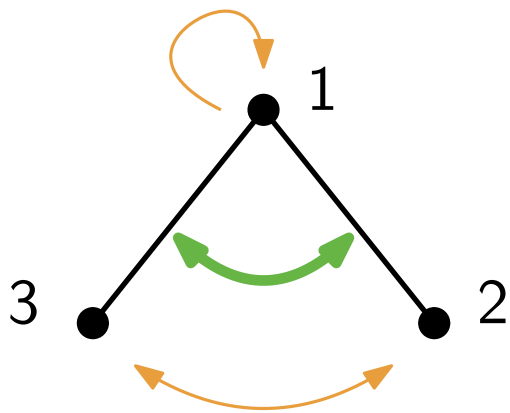

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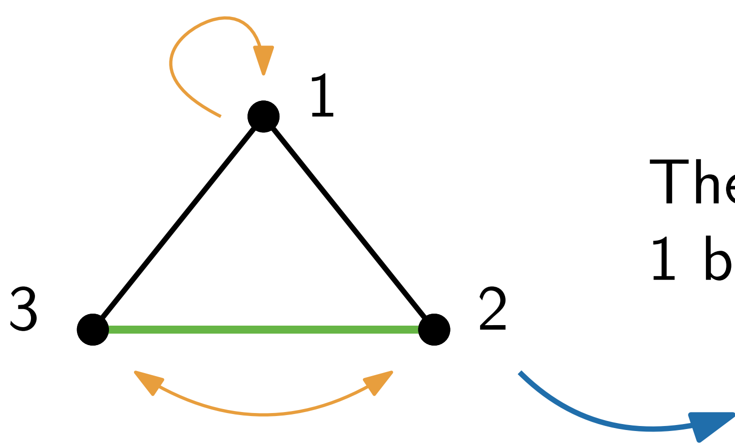

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What we do:

- Refinement of cycle index sums to encode cycles of cliques.
- Extend cycle-pointing to cycles of cliques.

Example: composition

Classic setting: EGF, labelled graphs, substitution of vertices. If $\mathcal{C} = \mathcal{A} \circ \mathcal{B}$

$$C(x) = A(B(X)).$$

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Pólya setting: Cycle index sum, unlabelled graphs, substitution of vertices. If $\mathcal{C} = \mathcal{A} \circ \mathcal{B}$,

$$\begin{aligned} Z_{\mathcal{C}} &= Z_{\mathcal{A}}(Z_{\mathcal{B}}(s_1, s_2, s_3, \dots), Z_{\mathcal{B}}(s_2, s_4, s_6, \dots), \dots) \\ &= Z_{\mathcal{A}}(s_j \to Z_{\mathcal{B}}(s_j, s_{2j}, s_{3j}, \dots))_{j \geq 1} \\ &= Z_{\mathcal{A}}(s_j \to Z_{\mathcal{B}}^{[j]})_{j \geq 1} \end{aligned}$$

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Our setting: Extended cycle index sum, unlabelled graphs, substitution of cliques. If $\mathcal{C} = \mathcal{A} \circ_k \mathcal{B}$,

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With cycle-pointing:

$$X_{\mathcal{P}} \odot_i (X_{\mathcal{A}}, X_{\mathcal{Q}}) := X_{\mathcal{P}}(s_{\lambda,j} \to (X_{\mathcal{A}}^{\lambda})^{[j]}, t_{\mu,l} \to (X_{\mathcal{Q}}^{\mu})^{[l]})$$

The system

$$
\begin{cases}
X^{\lambda}_{\mathcal{G}^{(k)}_{t,k+1}} = \frac{k!}{\alpha(\lambda)\kappa(\lambda)} \frac{\partial}{\partial s_{\lambda,1}} X_{\mathcal{G}_{t,k+1}}, \\
X^{\lambda}_{\mathcal{G}^{(k)}_{t,k}} = Z_{\mathrm{SET}}(s_j \to (X^{\lambda^j}_{\mathcal{G}^{(k)}_{t,k+1} \circ_k \mathcal{G}^{(k)}_{t,k}})^{[j]})_{j \geq 1}, \\
X^{\lambda}_{\mathcal{G}^{(k)}_{t,k+1} \circ_k \mathcal{G}^{(k)}_{t,k}} = X^{\lambda}_{\mathcal{G}^{(k)}_{t,k+1}} (s_{\mu,j} \to (X^{\mu}_{\mathcal{G}^{(k)}_{t,k}})^{[j]})_{\mu \vdash k, j \geq 1}, \\
X_{\mathcal{G}^{\bullet_k}_{t,k}} = \sum_{\mu \vdash k} \frac{\alpha(\mu)\kappa(\mu)}{k!} t_{\mu,1} X^{\mu}_{\mathcal{G}^{(k)}_{t,k}} + X_{(\mathcal{G}_{t,k})^{\bullet_k}_{\geq 2}}, \\
X_{(\mathcal{G}_{t,k})^{\bullet_k}_{\geq 2}} = X_{(\mathcal{G}_{t,k+1})^{\bullet_k}_{\geq 2}} (s_{\mu,j} \to (X^{\mu}_{\mathcal{G}^{(k)}_{t,k}})^{[j]}, t_{\mu,j} \to (X^{\mu}_{(\mathcal{G}^{(k)}_{t,k})^{\bullet_k}})^{[j]})_{\mu \vdash k, j \geq 1} \\
\qquad + \sum_{\mu \vdash k} \frac{\alpha(\mu)\kappa(\mu)}{k!} s_{\mu,1} Z_{\mathrm{SET}^{\bullet}_{\geq 2}} (s_j \to (X^{\mu^j}_{\mathcal{G}^{(k)}_{t,k+1} \circ_k \mathcal{G}^{(k)}_{t,k}})^{[j]}, \\
\qquad\qquad t_j \to (X^{\mu^j}_{(\mathcal{G}^{(k)}_{t,k+1} \circ_k \mathcal{G}^{(k)}_{t,k})^{\bullet_k}})^{[j]})_{j \geq 1}, \\
X_{\mathcal{G}_{t,k}} = \sum_{\lambda \vdash k} \sum_{1 \leq j \leq \binom{t+1}{k}} \int \frac{1}{j t_{\lambda,j}} X_{\mathcal{G}^{\bullet_k}_{t,k}} (S(\lambda,j) \to 0, T(\lambda,j) \to 0) ds_{\lambda,j}
\end{cases}
$$

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Future:

- [C.,Drmota & Requilé (soon?)]: asymptotic enumeration of unlabelled chordal graphs with bounded tree-width.

Concluding remarks

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Thank you!